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Characterization of topological spaces by some continuous functions.

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Introduction.

Let E be a Banach space and X a T_1-space. We shall consider a function $F(x)$ from X into the space 2^E of all subsets of E and assume it to be *lower semi-continuous* in the following sense: the set $\{x \in X \mid F(x) \cap U \neq \emptyset\}$ is open in X for every open subset U of E. Under these circumstances, E. A. Michael [5] has recently announced the following two theorems:

THEOREM I. *A necessary and sufficient condition for X to be paracompact and normal is the following:*

(A) *If E is an arbitrary Banach space and if a lower semi-continuous function $F: X \rightarrow 2^E$ is such that $F(x)$ is a non-empty convex closed subset of E for every $x \in X$, then there exists a continuous function $f: X \rightarrow E$ such that $f(x) \in F(x)$ for every $x \in X$.*

THEOREM II. *A necessary and sufficient condition for X to be a normal space is the following:*

(B) *If E is a separable Banach space and if a lower semi-continuous function $F: X \rightarrow 2^E$ is such that $F(x)$ is a non-empty convex compact subset of E for every $x \in X$, then there exists a continuous function $f: X \rightarrow E$ such that $f(x) \in F(x)$ for every $x \in X$.*

These two theorems suggest the following problem: what types of topological spaces will be characterized if we replace the conditions imposed upon the space E and upon the function F by other suitable ones? We shall give in this paper the characterization of the next two types of topological spaces: (1) a countably paracompact(1) normal space which is recently introduced by C. H. Dowker [3], and (2) a normal space in which every point-finite covering(2) has a locally finite refinement. (We shall call such a space to be *point-finitely paracom-*

pact.) Our main purpose is to prove the following two theorems:

THEOREM III. *A necessary and sufficient condition for X to be countably paracompact and normal is the following:*

(C) *If E is a separable Banach space and if a lower semi-continuous function $F(x): X \to 2^E$ is such that $F(x)$ is a non-empty convex closed subset of E for every $x \in X$, then there exists a continuous function $f: X \to E$ such that $f(x) \in F(x)$ for every $x \in X$.*

THEOREM IV. *A necessary and sufficient condition for X to be point-finitely paracompact and normal is the following:*

(D) *If E is an arbitrary Banach space and if a lower semi-continuous function $F: X \to 2^E$ is such that $F(x)$ is a non-empty convex compact subset of E for every $x \in X$, then there exists a continuous function $f: X \to E$ such that $f(x) \in F(x)$ for every $x \in X$.*

These theorems can be proved along the same line as Michael's theorems (I) and (II), whose proofs we shall include in this paper for the sake of completeness.(3)

1. Preliminary lemmas.

LEMMA 1. *Let $\varphi(x)$ be a continuous function from a topological space X into a Banach space E, and a function $F: X \to 2^E$ be lower semi-continuous. Then the function $G(x) = \overline{F(x) \cap S(\varphi(x); r)}$(4) is also lower semi-continuous, where r is any positive number.*

PROOF. Let U be any open subset of E, and denote by H the set $\{x \in X \mid G(x) \cap U \neq 0\}$. We must prove that the set H is open in X. Let x_0 be any point of H. Then $G(x_0) \cap U = \overline{F(x_0) \cap S(\varphi(x_0); r)} \cap U \neq 0$. Since U is open, we have $F(x_0) \cap S(\varphi(x_0); r) \cap U \neq 0$. Choose a point b in $F(x_0) \cap S(\varphi(x_0); r) \cap U$ and form a sphere $\sum$ about b so small as to have $\sum \subset S(\varphi(x_0); r) \cap U$. Then we have $\sum \subset S(\varphi(x); r)$ if $\varphi(x)$ is sufficiently near to $\varphi(x_0)$, hence by continuity of φ, there exists an open neighborhood V of x_0 such that $y \in V$ implies $S(\varphi(y); r) \supset \sum$. Then $W = V \cap \{y \mid F(y \cap \sum \neq 0\}$ is an open set containing x_0, and $y \in W$ implies

$$G(y) \cap U \supset F(y) \cap S(\varphi(y); r) \cap \sum = F(y) \cap \sum \neq 0$$

and $y \in H$. Therefore $W \subset H$ and H is an open set in X.

LEMMA 2. *A countable covering of a normal space is a normal* *covering if and only if it has a countable star-finite refinement. Any point-finite countable covering of a normal space has a countable star-finite refinement.*

See K. Morita [6; Theorem 6 and Corollary of Theorem 5].

LEMMA 3. *If a family of sets* $\{G_\alpha \mid \alpha \in \Omega\}$ *in* X *is locally finite, then we have* $\overline{\bigcup_{\alpha\in\Omega} G_\alpha} = \bigcup_{\alpha\in\Omega} \overline{G}_\alpha$.

LEMMA 4. *Let* $\{U_\alpha \mid \alpha \in \Omega\}$ *be a locally finite covering of a normal space* X. *Then there exists a family of continuous real-valued non-negative functions* $\{\varphi_\alpha \mid \alpha \in \Omega\}$ *such that*

(i) $\sum_{\alpha\in\Omega} \varphi_\alpha(x) = 1$ *for every* $x \in X$;

(ii) $x \notin U_\alpha$ *implies* $\varphi_\alpha(x) = 0$.

This family of functions $\{\varphi_\alpha\}$ is called "*partition of unity*" subordinated to the covering $\{U_\alpha\}$. See C. H. Dowker [2].

LEMMA 5. *Let* $\mathfrak{U} = \{U_\alpha \mid \alpha \in \Omega\}$ *be a covering of a space* X *and suppose that there exists a family* $\{\varphi_\alpha \mid \alpha \in \Omega\}$ *of real-valued non-negative continuous functions such that:*

(i) *the family* $\{\varphi_\alpha\}$ *is equicontinuous,* i. e., *given* $x_0 \in X$ *and* $\varepsilon > 0$, *there exists a neighborhood* V *of* x_0 *such that* $|\varphi_\alpha(x) - \varphi_\alpha(x_0)| < \varepsilon$ *for every* $x \in V$ *and for all* $\alpha \in \Omega$;

(ii) $\sum_{\alpha\in\Omega} \varphi_\alpha(x) = 1$ *for all* $x \in X$;

(iii) $\varphi_\alpha(x) > 0$ *implies* $x \in U_\alpha$.

Then the covering $\mathfrak{U}$ *has a locally finite refinement.*

PROOF. For every positive integer n we define the open sets $U_\alpha^n = \{x \in X \mid \varphi_\alpha(x) > 1/n\}$ and $X_n = \bigcup_{\alpha\in\Omega} U_\alpha^n$. Then clearly $\overline{U_\alpha^n} \subset U_\alpha^{n+1}$.

(1) *The family of sets* $\{U_\alpha^n \mid \alpha \in \Omega\}$ *is locally finite for a fixed n.* In fact, let x_0 be any point of X. Then there exist at most a finite number of indices $\alpha_1, \cdots, \alpha_s$ such that $\varphi_{\alpha_i}(x_0) > 1/(n+1)$ for $i = 1, \cdots, s$ by (ii). From (i) we can find a neighborhood V of x_0 such that $y \in V$ implies $|\varphi_\alpha(y) - \varphi_\alpha(x_0)| < 1/n - 1/(n+1)$ for every $\alpha \in \Omega$. If $\beta \neq \alpha_1, \cdots, \alpha_s$, then $\varphi_\beta(y) < \varphi_\beta(x_0) + (1/n - 1/(n+1)) \leqq 1/(n+1) + (1/n - 1/(n+1)) = 1/n$, therefore, $V \cap U_\beta^n = 0$.

By virtue of the property (1), we have at once

(2) $$\overline{X}_n = \bigcup_\alpha \overline{U_\alpha^n} \qquad \text{by Lemma 3;}$$

(3) $$X_n \subset \overline{X}_n \subset X_{n+1}.$$

The condition (ii) and the definition of X_n imply

(4) $$\bigcup_n X_n = X .$$

Now, let $Y_n = X_n - \overline{X}_{n-2}$ (where $X_{-1} = X_0 = \emptyset$). Each Y_n is clearly open and $\bigcup_n Y_n = X$; moreover,

(5) $$Y_m \cap Y_n = \emptyset \quad \text{if} \quad |m-n| \geqq 2 .$$

Finally, define a covering $\mathfrak{V} = \{V_{m,\alpha} \mid m=1, 2, \cdots ; \alpha \in \Omega\}$, where $V_{m,\alpha} = Y_m \cap U_\alpha^m$. Then by virtue of properties (1) and (5), we can conclude that V is a locally finite refinement of U. This completes the proof.

2. Proof of necessity.

The essential part of this section lies in the proof of the following

LEMMA 6. *Let F be a lower semi-continuous function from X into 2^E and ε a prescribed positive number. Then there exists a continuous function f from X into E such that $f(x) \in S(F(x); \varepsilon)$*[4] *for every $x \in X$, if any one of the following conditions is satisfied:*

(I) *X is a paracompact normal space, E is an arbitrary Banach space, and $F(x)$ is a non-empty convex closed subset of E for every $x \in X$.*

(II) *X is a normal space, E is a separable Banach space, and $F(x)$ is a non-empty convex compact subset of E for every $x \in X$.*

(III) *X is a countably paracompact normal space, E is a separable Banach space, and $F(x)$ is a non-empty convex closed subset of E for every $x \in X$.*

(IV) *X is a point-finitely paracompact normal space, E is an arbitrary Banach space, and $F(x)$ is a non-empty convex compact subset of E for every $x \in X$.*

PROOF. *Case* (I): Construct the covering $\mathfrak{S} = \{S_b \mid b \in E\}$ of E, where each S_b is an open sphere $S(b; \varepsilon)$[4]. Then by virtue of lower semi-continuity of F, the sets $U_b = \{x \in X \mid F(x) \cap S_b \neq \emptyset\}$ are all open in X and form a covering $\mathfrak{U}$ of X. X being paracompact, $\mathfrak{U}$ has a locally finite refinement $\mathfrak{V} = \{V_\alpha \mid \alpha \in \Omega\}$. Let $\{\varphi_\alpha \mid \alpha \in \Omega\}$ be the partition of unity subordinated to $\mathfrak{V}$ (Lemma 4). To each α, we choose an element b such that $V_\alpha \subset \{x \in X \mid F(x) \cap S_{b_\alpha} \neq \emptyset\}$ and define $f: X \to E$ by

(6) $$f(x) = \sum_\alpha \varphi_\alpha(x) b_\alpha .$$

Given any point $x \in X$, there exists a neighborbood W meeting only with a finite number of elements of $\mathfrak{V}$. Then (6) being a finite sum in W, $f(x)$ is continuous there, hence in X. Next, let $\alpha_1, \cdots, \alpha_s$ be the indices α such that $\varphi_\alpha(x) \neq 0$. Then $F(x) \frown S_{b_{\alpha_i}} \neq \emptyset$. Choose an element c_{α_i} from each $F(x) \frown S_{b_{\alpha_i}}$. Then $||b_{\alpha_i} - c_{\alpha_i}|| < \varepsilon$, and

$$||f(x) - \textstyle\sum_{i=1}^{s} \varphi_{\alpha_i}(x) c_{\alpha_i}|| = ||\sum \varphi_{\alpha_i}(x) b_{\alpha_i} - \sum \varphi_{\alpha_i}(x) c||$$

$$\leqq \sum \varphi_{\alpha_i}(x) ||b_{\alpha_i} - c_{\alpha_i}|| < \varepsilon \sum \varphi_\alpha(x) = \varepsilon$$

and $\sum \varphi_{\alpha_i}(x) c_{\alpha_i} \in F(x)$ by convexity of the set $F(x)$. Consequently, $f(x) \in S(F(x); \varepsilon)$.

Case (II): Let $B = \{b_n | n = 1, 2, \cdots\}$ be a countable dense subset of E and construct the open spheres $S_n = S(b_n; \varepsilon)$. Then $\mathfrak{S}_1 = \{S_n | n = 1, 2, \cdots\}$ is a countable covering of E. E being a metric space, it is paracompact (see A. H. Stone [7]), and so $\mathfrak{S}_1$ has a countable locally finite refinement $\mathfrak{T} = \{T_n | n = 1, 2, \cdots\}$. Let $U_n = \{x \in X | F(x) \frown T_n \neq \emptyset\}$. Then $\{U_n\}$ is a point-finite covering of X, for, given $x \in X$, $F(x)$ is compact, hence at most a finite number of T_n, say $T_{n_1}, \cdots, T_{n_s}$, intersect with $F(x)$. Then $x \in U_i$ only for $i = n_1, \cdots, n_s$. Since X is a normal space, the point-finite countable covering $\{U_n\}$ has a locally finite refinement $\mathfrak{V}$ by Lemma 2, and the proof may be accomplished precisely by the same way as above.

Case (III): Let $\mathfrak{S}_1$ be the same as in case (II). Then the open sets $U_n = \{x \in X | F(x) \frown S_n \neq \emptyset\}$ form a countable covering of X. By countable paracompactness of X, there exists a locally finite refinement $\mathfrak{V} = \{V_\alpha | \alpha \in \Omega\}$ of this covering. The desired function f may be constructed by the same way as above.

Case (IV): Since E is paracompact, the covering $\mathfrak{S}$ of E constructed in the proof of Case (I) has a locally finite refinement $\mathfrak{V} = \{V_\alpha | \alpha \in \Omega\}$. Let $U_\alpha = \{x \in X | F(x) \frown V_\alpha \neq \emptyset\}$. Then the covering $\mathfrak{U} = \{U_\alpha | \alpha \in \Omega\}$ is as above a point-finite covering of X. By virtue of point-finite paracompactness of X, this covering has a locally finite refinement. From this point, we can proceed as above.

PROOF OF NECESSITY. By virtue of the above lemma, we can inductively define a sequence $\{f_n\}$ of continuous functions from X to

E with the properties (i) $f_n(x) \in S(F(x); 1/2^n)$, and (ii) $\|f_n(x)-f_{n+1}(x)\| < 1/2^n$ for every $x \in X$. In fact, apply Lemma 6 to $F(x)$ for $\varepsilon=1/2$, and we have a function f_1 satisfying (i). Suppose that we have defined the function $f_n(x)$ with the properties (i) and (ii). Then the function $G(x)=\overline{F(x) \cap S(f_n(x); 1/2^n)}$ is lower semi-continuous by Lemma 1 and satisfies the same conditions as $F(x)$. Thus applying Lemma 6 to $G(x)$ instead of $F(x)$, we have a continuous function f_{n+1} such that $f_{n+1}(x) \in S(G(x); 1/2^{n+1})$. This f_{n+1} is easily seen to satisfy (i) and (ii), and our induction is completed.

The sequence $\{f_n(x)\}$ is obviously a Cauchy sequence and converges uniformly to a continuous function $f: X \to E$ by (ii). From (i) and the closedness of the set $F(x)$; we have readily $f(x) \in F(x)$, which completes the proof.

3. Proof of sufficiency.

LEMMA 7. *Let $\mathfrak{U}=\{U_\alpha \mid \alpha \in \Omega\}$ be any covering of a space X and $E=E(\mathfrak{U})$ be a Banach space formed of all $b=\sum \lambda_\alpha b_\alpha$ such that $\|b\| =\sum_\alpha |\lambda_\alpha| < \infty$.*[5] *Define a function $F=F_{\mathfrak{U}}$ as follows: $F(x)$ is formed of all $b=\sum \lambda_\alpha b_\alpha \in E$ such that $\|b\|=1$ and that $x \notin U_\alpha$ implies $\lambda_\alpha=0$. Then* (i) *each $F(x)$ is a non-empty convex closed subset of E, and* (ii) *the function $F: X \to 2^E$ is lower semi-continuous.*

PROOF. (i) is obvious. To prove (ii), it suffices to show that the set $G=\{x \in X \mid F(x) \cap S \neq 0\}$ is open in X for every open sphere $S =S(b_0; r)$. Let $x_0 \in G$. Then there exists an element $b=\sum \lambda_\alpha b_\alpha \in F(x)$ such that $\rho=\|b-b_0\| < r$. Since $\|b\|=\sum |\lambda_\alpha|=1$, at most countably infinite λ_α's, say $\lambda_{\alpha_1}, \lambda_{\alpha_2}, \cdots$, are not equal to zero. Choose n so large as to have $1-L_n < (r-\rho)/2$, where $L_n=\sum_{i=1}^{n} |\lambda_{\alpha_i}|$. Then $x_0 \in U_{\alpha_i}$ since $\lambda_{\alpha i} \neq 0$, and $V=\bigcap_{i=1}^{n} U_{\alpha_i}$ is a neighborhood of x_0. Let $y \in V$. Then $b'=1/L_n \sum_{i=1}^{n} \lambda_{\alpha_i} b_{\alpha_i} \in F(y)$, and

$$\begin{aligned}\|b'-b_0\| &\leqq \|b'-\sum_{i=1}^{n} \lambda_{\alpha_i} b_{\alpha_i}\| + \|\sum_{i=1}^{n} \lambda_{\alpha_i} b_{\alpha_i} - b\| + \|b-b_0\| \\ &=\sum_{i=1}^{n} |\lambda_{\alpha_i}/L_n - \lambda_{\alpha_i}| + \sum_{i=n+1}^{\infty} |\lambda_{\alpha_i}| + \rho \\ &=(1-L_n)+(1-L_n)+\rho < 2 \cdot (r-\rho)/2+\rho=r.\end{aligned}$$

Consequently $b' \in F(y) \cap S$ and $y \in G$, whence $V \subset G$. This completes the proof of the lemma.

Our proof is now divided into the following four cases:

(I) *In case the condition* (*B*) *holds.* In this case, we choose E as the space $\boldsymbol{R}$ of all real numbers. Let M and N be disjoint non-void closed subsets of X and define $F(x): X \rightarrow \boldsymbol{R}$ as follows: $F(x)=\{0\}$ if $x \in M$, $=\{1\}$ if $x \in N$, and $=I$ the closed interval $[0, 1]$, otherwise. Then the function $F(x)$ is easily seen to satisfy the assumption in (B), and therefore a continuous function $f: X \rightarrow \boldsymbol{R}$ is obtained so that $f(x) \in F(x)$, i. e., $f(x)=0$ if $x \in M$, and $=1$ if $x \in N$, and always $0 \leqq f(x) \leqq 1$. This shows that X is a normal space.

(II) *In case the condition* (*A*) *holds.* In this case the condition (B) is also satisfied, which implies that X is normal. It suffices therefore to show that every covering $\mathfrak{U}=\{U_\alpha | \alpha \in \Omega\}$ of X has a locally finite refinement. To this end, let us construct a Banach space $E=E(\mathfrak{U})$ and a function $F=F_{\mathfrak{U}}$ as in Lemma 7. The condition (A) implies the existence of a continuous function $f: X \rightarrow E$ such that $f(x)=\sum_\alpha \lambda_\alpha(x) b_\alpha \in F(x)$ for every $x \in X$. Define $\varphi_\alpha(x)=|\lambda_\alpha(x)|$. Then $\{\varphi_\alpha\}$ is easily seen to satisfy the hypotheses of Lemma 5 and we can conclude the existence of a locally finite refinement of $\mathfrak{U}$.

(III) *In case the condition* (*C*) *holds.* The condition (B) implies as above that X is normal. Let $\mathfrak{U}=\{U_n\}$ be any countable covering of X and construct a Banach space $E=E(\mathfrak{U})$ and a function $F=F_{\mathfrak{U}}$ as in Lemma 7. In this case the space E is clearly separable and the condition (C) implies the existence of a locally finite refinement of $\mathfrak{U}$ as above.

(IV) *In case the condition* (*D*) *holds.* Let $\mathfrak{U}=\{U_\alpha | \alpha \in \Omega\}$ be any point-finite covering of X and construct $E=E(\mathfrak{U})$ and $F(x)=F_{\mathfrak{U}}(x)$ as in Lemma 7. Then each set $F(x)$ is compact. Indeed, given any point $x \in X$, let U_{α_i} $(1 \leqq i \leqq n)$ be the elements of $\mathfrak{U}$ containing x. By point-finiteness of the covering $\mathfrak{U}$, n is finite and the set $F(x)$ is homeomorphic to the compact set in the Euclidean n-space defined by the equation $|x_1|+|x_2|+\cdots+|x_n|=1$, hence $F(x)$ is compact. From this point, we can proceed in the same way as above.

4. Some applications.

As is stated in E. A. Michael [5], the above theorems may be applied to obtain Arens' theorem [1] which asserts the extensibility of continuous Banach space-valued function f defined on a closed subset A of a paracompact normal space X to the whole space, or to obtain the inserting theorem of a continuous function between lower and upper semi-continuous real-valued functions defined on a normal space (cf. H. Tong [8]). Some similar result may be obtained for other class of topological spaces.

COROLLARY 1. *Let A be a closed subset of a countably paracompact normal space X and E be a separable Banach space. Then every continuous function $f: A \to E$ may be extended to a continuous function $X \to E$.*

PROOF. Let $F(x)$ be defined as follows: $F(x)=\{f(x)\}$ if $x \in A$ and $=E$ otherwise. Then $F(x)$ is easily seen to be semi-continuous and Theorem III is applied to obtain the desired extension of f.

COROLLARY 2. *Let A be a compact subset of a point-finitely paracompact normal space X. Then every continuous function from A into any Banach space E may be extended to a continuous function $X \to E$.*

COROLLARY 3. *Let A be a compact subset of a normal space X. Then every continuous function from A into a separable Banach space E may be extended continuously all over the space X.*

Corollary 2 and 3 may be proved analogously as Corollary 1.

5. An example of a point-finitely paracompact space.

Finally, we shall give an example of a point-finitely paracompact space which is not paracompact. Let X be the set of all countable transfinite ordinals and the neighborhood of its point $x_0 \in X$ be the set of the form $\{x \mid y < x \leqq x_0\}$. Then the space X is well known to be completely normal and locally compact but not compact.

LEMMA 8. *Let $f: X \to X$ be a mapping such that $f(x) < x$ for sufficiently large $x \in X$. Then there is an element $c \in X$ such that, for every $x \in X$, there exists an element $y \geqq x$ satisfying $f(y) < c$.* See N. Bourbaki [10, Chap. I, § 10, ex. 21)].

PROOF. If the lemma were not true, we could inductively define a sequence $\{z_n\}$ such that $y \geqq z_{n+1}$ implies $f(y) \geqq z_n$. Clearly $z_1 < z_2 < \cdots$.

Let z_0 be the ordinal which follows immediately after $\{z_n\}$. Then $z_0 \in X$ and $z_0 \geqq z_{n+1}$, hence $z_0 > f(z_0) \geqq z_n$ for every n, which contradicts the definition of z_0.

From this lemma, we see that X is point-finitely paracompact. More precisely, we have the following

THEOREM. *Every point-finite covering of the space X has a finite subcovering.*

PROOF. Let $\mathfrak{U} = \{U_a \mid a \in A\}$ be a point-finite covering of X. Define a mapping $f: X \rightarrow X$ as follows: $f(\alpha) = 1$ if $\alpha = 1$. In case $\alpha > 1$, let $\{U_b \mid b \in B\}$ be the collection of all U_a's ($a \in A$) containing α. By virtue of point-finiteness of $\mathfrak{U}$, B is a finite set and the set $V = \bigcap \{U_b \mid b \in B\}$ is an open set containing α. Put $f(\alpha) = \beta$, where β is the smallest β such that $\{\gamma \mid \beta < \gamma \leqq \alpha\} \subset V$. Let c be the ordinal obtained by the above lemma and $U_1, U_2, \cdots, U_s$ be the elements of $\mathfrak{U}$ which contain c. For any $\alpha > c$, there exists an ordinal $\beta \geqq c$ such that $f(\beta) < c < \alpha \leqq \beta$. Then the neighborhood $\{\gamma \mid f(\beta) < \gamma \leqq \beta\}$ is contained in some $U_a \in \mathfrak{U}$. Since c is contained in U_a, U_a must be one of the U_i's, $1 \leqq i \leqq s$, and $\alpha \in U_1 \cup U_2 \cup \cdots \cup U_s$. The segment $\{\alpha \mid \alpha \leqq c\}$ of X is easily seen to be compact and is covered by some finite number of U_a. Consequently, X is covered by finite number of U_a, and the proof is complete.

COROLLARY. *The space X is not paracompact.*

In fact, let $\mathfrak{U}$ be a covering of X which has no finite subcoverings (such a covering may be easily constructed.) Then the above theorem shows that $\mathfrak{U}$ has no locally finite refinements.

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Notes.

(1) According to C. H. Dowker [3], a topological space is called countably paracompact, if every countable covering has a locally finite refinement. In this paper we use the term "covering" as "open covering", and locally finite = neighborhood-finite in the sense of S. Lefschetz [4].

(2) See S. Lefschetz [4, p. 13].

(3) Michael's original proof is not yet known to us.

(4) We denote by $S(b_0; r)$ the open sphere of radius r about b_0 in the Banach space E i.e., the set $\{b \mid \|b-b_0\| < r,\ b \in E\}$ and by $S(A; \varepsilon)$ the open ε-neighborhood of $A \subset E$, i.e., the set $\{b \mid \|b-a\| < \varepsilon$ for some $a \in A\}$.

(5) We assume that the vectors $\{b_\alpha\}$ are linearly indepenpent; therefore the space E is an (l^1)-space not necessarily of countable dimension.

Added in proof: Corollary 3 in § 4 is valid for an arbitrary Banach space E. In fact, let H be the closed linear subspace of E spanned by $f(A)$. $f(A)$ being compact metric, hence separable, H is a separable Banach space, and the given function f may be considered as a function from A into H.