Journal of the Mathematical Society of Japan Vol. 5, Nos. 3~4, November, 1953.

Generalized evolute in Klein spaces.

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(Received Feb. 11, 1953)

We investigate in this paper the generalization of the enveloping theorem of an evolute of a curve on the euclidean plane to the case of figures in Klein spaces by the method of moving frame of E. Cartan [1]. The idea of this paper is the same with that of [3]. In addition we state the process of obtaining theorems in Klein spaces analogous to Euler-Savary's theorem on the euclidean plane ([2] pp. 28–29).

1. Generalized evolute

1.1 Let $\mathfrak{G}$ be a fundamental Lie group of the Klein space and $\mathfrak{H}$ a closed subgroup of $\mathfrak{G}$. We consider the figure F consisting of one-parametric set of points of the homogeneous space $\mathfrak{G}/\mathfrak{H}$ and attach to each element of F a Frenet's frame defined in [1] pp. 131–132. Let the Frenet's frame defined at the point A on F be S_aR, where R is a fundamental frame and S_a is an element of $\mathfrak{G}$, and let the Frenet's frame at a consecutive point of F be $S_{a+da}R$. The frames whose relative displacements are each given by S_t with respect to S_aR and $S_{a+da}R$ are S_aS_tR and $S_{a+da}S_tR$. The infinitesimal relative displacement between S_aS_tR and $S_{a+da}S_tR$ is given by $(S_aS_t)^{-1}(S_{a+da}S_t)=S_t^{-1}(S_a^{-1}S_{a+da})S_t$. We take S_t which depends on the parameter a, so that $S_t^{-1}(S_a^{-1}S_{a+da})S_t$ is an infinitesimal element of a certain fixed subgroup $\mathfrak{K}$ of $\mathfrak{G}$ for all a. $\mathfrak{K}$ is not in general unique. We call the elements of the homogeneous space $\mathfrak{G}/\mathfrak{K}$ belonging to S_aS_tR a central figure. To each point of F a central figure is defined and we call a set of central figures an evolute of F, which we denote by E. The infinitesimal relative displacement of the frames S_aS_tR attached to E can be given by

$$(S_aS_t)^{-1}(S_{a+da}S_{t+dt})=S_t^{-1}(S_a^{-1}S_{a+da})S_t\cdot S_t^{-1}S_{t+dt}.$$

Let the relative components of the relative displacement of S_aS_t, S_a, S_t be $\omega_p, \omega_p^{(1)}, \omega_p^{(0)}$ $(p=1,2,\cdots,r)$ respectively. Then we have the relations

$$\omega_p = \sum_{q=1}^{r} \tau_{pq}\, \omega_q^{(1)} + \omega_p^{(0)} \qquad (p=1,2,\cdots r) \tag{1}$$

by virtue of [4] p. 4. If ω_p's are so chosen that $\omega_1, \cdots, \omega_n$ are principal relative components of $\mathfrak{G}/\mathfrak{K}$, we have $\sum \tau_{iq}\, \omega_q^{(1)} = 0$ $(i=1,\cdots,n)$ because $S_t^{-1}(S_a^{-1}S_{a+da})S_t$ is an infinitesimal element of $\mathfrak{K}$. Hence we get the relations

$$\omega_i = \omega_i^{(0)} \qquad (i=1,\cdots,n). \tag{2}$$

These are the fundamental relations, which can be stated as follows.

THEOREM. *The principal relative components of an evolute E of a one-parametric figure F in the homogeneous space are equal to those of the figure generated by the positions of central figures relative to the corresponding Frenet's frame of F.*

1.2 We verify that the above process gives an enveloping theorem in the classical case. Let the origin of Frenet's frame of a curve on the euclidean plane be A and the axes be e_1, e_2. The relative displacement of Frenet's frame is given by $dA = ds\, e_1$, $de_1 = \rho^{-1} ds\, e_2$, $de_2 = -\rho^{-1} ds\, e_1$. If we take $\bar{A} = A + he_2$ with constant h, we have $d\bar{A} = \rho^{-1}(\rho - h)\, ds\, e_1$. So we have $d\bar{A} = 0$, when h is a constant equal to the value of ρ at a point on the curve. The center of curvature $\bar{A} = A + \rho e_2$ is a central figure in our sense. As $d\bar{A} = d\rho\, e_2$, the frame (A, e_2, e_1) is a Frenet's frame for the evolute and $d\rho$ is equal to an arc-element of the evolute. This is a special case of (2).

1.3 Next we treat the case of the curve in the euclidean 3-space. Let a vertex of a Frenet's frame of the curve be A and the axes be e_1, e_2, e_3. The relative displacement of the Frenet's frame of the curve is given by

$$\begin{aligned} &dA = ds\, e_1, && de_1 = k ds\, e_2 \\ &de_2 = -k ds\, e_1 + t ds\, e_3, && de_3 = -t ds\, e_2 \end{aligned} \tag{3}$$

and this infinitesimal displacement can be realized by the screw motion around the axis which passes through $\bar{A} = A + kc^2 e_2$ and has the direction $\bar{e}_1 = -tc e_1 + kc e_3$, where $c = (k^2 + t^2)^{-1/2}$. This can be verified in the following. We take a rectangular frame $(\bar{A}, \bar{e}_1, \bar{e}_2, \bar{e}_3)$ such that

$$\begin{aligned} &\bar{A} = A + kc^2 e_2, && \bar{e}_1 = -tc e_1 + kc e_3 \\ &\bar{e}_2 = e_2, && \bar{e}_3 = -kc e_1 - tc e_3. \end{aligned} \tag{4}$$

Then putting $d\bar{A}=\sum\omega_i\bar{e}_i$, $d\bar{e}_i=\sum\omega_{ij}\bar{e}_j$ we have

$$\text{(5)}\qquad \begin{aligned} &\omega_2=d(kc^2), &&\omega_3=0\\ &\omega_{12}=0, &&\omega_{13}=d\tan^{-1}(k/t). \end{aligned}$$

Hence if k and t in (4) are constants which are equal to the values of curvature and torsion at a certain point of the curve we have $\omega_2=\omega_3=\omega_{12}=\omega_{13}=0$ at the point. As $\omega_2, \omega_3, \omega_{12}, \omega_{13}$ are principal relative components of the line through $\bar{A}$ with the direction $\bar{e}_1$ the line is an instantaneous axis of rotation of the displacement (3). Hence this is a central figure of our curve F, and a set of these lines is an evolute E of F. The relation (5) indicates that the frame $(\bar{A}, \bar{e}_1, \bar{e}_2, \bar{e}_3)$ is a Frenet's one of the ruled surface E (cf. [1] p. 50). If we denote the parameter of distribution of E by κ and the angle between two consecutive generating lines of E by $d\sigma$ we have

$$\text{(6)}\qquad \begin{aligned} &d\bar{A}=d\sigma(\alpha\bar{e}_1+\kappa\bar{e}_2), &&d\bar{e}_1=d\sigma\bar{e}_2\\ &d\bar{e}_2=-\beta\, d\sigma\bar{e}_3, &&d\bar{e}_3=d\sigma(-\bar{e}_1+\beta\bar{e}_2). \end{aligned}$$

Hence comparing (5) and (6) we get

$$\text{(7)}\qquad \kappa\, d\sigma=d(kc^2), \qquad d\sigma=d\tan^{-1}(t/k).$$

We put $h=kc^2$, $\theta=\tan^{-1}(t/k)$. h is a distance between the tangent at a point of the curve F and the line which is a central figure at the point, while θ is an angle between the central figure and the binormal of the curve F. Integrating (7) from a point 1 to a point 2 we get

$$\int_{12}\kappa\, d\sigma=h_2-h_1, \qquad \sigma_2-\sigma_1=\theta_2-\theta_1.$$

1.4 Next we take a ruled surface F in the euclidean 3-space. Let the relative displacement of the Frenet's frame (A, e_1, e_2, e_3) along F be

$$\begin{aligned} &dA=d\sigma(\alpha e_1+\kappa e_3), &&de_1=d\sigma e_3\\ &de_2=d\sigma(-e_1+\beta e_3), &&de_3=-\beta\, d\sigma e_2. \end{aligned}$$

The line through $\bar{A}=A+(\alpha-\beta\kappa)\rho^2 e_2$ with the direction $\bar{e}_1=\beta\rho e_1+\rho e_3$, where $\rho=(1+\beta^2)^{-1/2}$, is a central figure and the locus of these lines is an evolute E of F. When we put

$$\bar{A}=A+(\alpha-\beta\kappa)\rho^2 e_2\,, \qquad \bar{e}_1=\beta\rho\, e_1+\rho\, e_3$$
$$\bar{e}_2=e_2\,, \qquad \bar{e}_3=-\rho\, e_1+\beta\rho\, e_3$$

the frame (A, e_1, e_2, e_3) is a Frenet's one of E, and if we denote the parameter of distribution by K and the angle between consecutive lines of E by $d\Sigma$ we have

$$(8) \qquad K d\Sigma=dh\,, \qquad d\Sigma=d\theta\,,$$

where $h=(\alpha-\beta\kappa)\rho^2$ is a distance between the generating line of F and the corresponding central figure and $\theta=\tan^{-1}(1/\beta)$ is an angle between these two lines.

2. Curve on the affine plane

2.1 In this section we treat a curve on the affine plane. As a preliminary we consider a pair of intersecting straight lines on the affine plane. Let A be the point of intersection and e_1, e_2 be the vectors which are parallel to these lines and are two sides of parallelogram of area 1. For a set of such frames we put

$$dA=\omega_1 e_1+\omega_2 e_2\,, \qquad de_1=\omega_{11} e_1+\omega_{12} e_2\,, \qquad de_2=\omega_{21} e_1-\omega_{11} e_2\,,$$

When we take $\bar{e}_1=\lambda e_1$, $\bar{e}_2=\lambda^{-1}e_2$ with variable λ in the place of e_1, e_2 and put

$$dA=\bar{\omega}_1\bar{e}_1+\bar{\omega}_2\bar{e}_2\,, \qquad d\bar{e}_1=\bar{\omega}_{11}\bar{e}_1+\bar{\omega}_{12}\bar{e}_2\,, \qquad d\bar{e}_2=\bar{\omega}_{21}\bar{e}_1-\bar{\omega}_{11}\bar{e}_2$$

we get $\bar{\omega}_1=\lambda^{-1}\omega_1$, $\bar{\omega}_2=\lambda\omega_2$, $\bar{\omega}_{12}=\lambda^2\omega_{12}$, $\bar{\omega}_{21}=\lambda^{-2}\omega_{21}$. These do not contain $d\lambda$, hence $\omega_1, \omega_2, \omega_{12}, \omega_{21}$ are the principal relative components of a homogeneous space with pairs of intersecting lines as its elements. The invariants of this space are $\omega_1\omega_2$, $\omega_1^2\omega_{12}$, $\omega_2^2\omega_{21}$.

2.2 Let the frame (A, e_1, e_2) be Frenet's one of the curve F on the affine plane. Then according to [1] p. 160 we have

$$(9) \qquad dA=d\sigma\, e_1\,, \qquad de_1=d\sigma\, e_2\,, \qquad de_2=k d\sigma\, e_1\,.$$

Now we take a frame consisting of

$$(10) \qquad \bar{A}=A-k^{-1}e_2\,, \qquad \bar{e}_1=k^{1/2}e_1+e_2\,, \qquad \bar{e}_2=-2^{-1}e_1+(2k^{1/2})^{-1}e_2$$

which is imaginary if the point on the curve F is elliptic. Then putting $d\bar{A}=\omega_1\bar{e}_1+\omega_2\bar{e}_2$, $d\bar{e}_1=\omega_{11}\bar{e}_1+\omega_{12}\bar{e}_2$, $d\bar{e}_2=\omega_{21}\bar{e}_1-\omega_{11}\bar{e}_{22}$ we get

(11) $\omega_1=(2k^2)^{-1}dk\,,\quad \omega_2=k^{-3/2}dk\,,\quad \omega_{12}=-(2k^{1/2})^{-1}dk,\quad \omega_{21}=-(8k^{3/2})^{-1}dk.$

These correspond to (2). If we take k in (10) as a constant equal to the affine curvature at the point, we have $\omega_1=\omega_2=\omega_{12}=\omega_{21}=0$. Hence the pair of lines passing through A and having the directions $\bar{e}_1, \bar{e}_2$ is invariant for the displacement (7), and this pair is a central figure of F in our sense. The set of these pairs is an evolute E and the invariants of E can be expressed by k as follows

(12) $$\omega_1\omega_2=(2k^{7/2})^{-1}(dk)^2\,,\qquad \omega_1^2\omega_{12}=\omega_2^2\omega_{21}=-(8k^{9/2})^{-1}(dk)^3\,.$$

2.3 Next we take as a central figure the center of affine curvature $\bar{A}=A-k^{-1}e_2$ and the evolute which is the locus of this point, as is usually done. The frame $(\bar{A}, e_1, e_2)$ is not a Frenet's one of the curve F, as is seen from the relations

$$d\bar{A}=k^2dk\,e_2\,,\qquad de_1=d\sigma\,e_1\,,\qquad de_2=kd\sigma\,e_1\,.$$

We denote by a prime the differentiation with respect to the variable σ and put $\lambda=k(k')^{-1/3}$, $\mu=k^{-1}\lambda'$. If we take a frame determined by $\bar{A}$, $\bar{e}_1=\lambda e_1-\mu e_2$, $\bar{e}_2=\lambda^{-1}e_2$, we get by calculation

$$d\bar{A}=d\Sigma\cdot\bar{e}_2\,,\qquad d\bar{e}_1=K\,d\Sigma\cdot\bar{e}_2\,,\qquad d\bar{e}_2=d\Sigma\cdot\bar{e}_1\,,$$

where we have put

(13) $$\begin{aligned} d\Sigma&=k^{-1}(k')^{2/3}d\sigma\\ K&=k^2(k')^{-4/3}\left(k+(k^{-1}k')^2-2/3\,k^{-1}k''-4/9(k^{-1}k'')^2+1/3(k')^{-1}k'''\right)\,. \end{aligned}$$

3. Curve on the projective plane

3.1 We consider the homogeneous space with conics as its elements. Let the fundamental frame on the projective plane be three analytic points A_1^0, A_2^0, A_3^0 (cf. [1] p. 75). We take a conic Q_0 represented by $x_1^2+x_2^2+x_3^2=0$ with respect to this frame and consider the conics obtained from Q_0 by the frame transformation from (A_1^0, A_2^0, A_3^0) to (A_1, A_2, A_3). If we put $dA_i=\sum\omega_{ij}A_j$ $(\sum\omega_{ii}=0)$, the principal relative components of our homogeneous space are $\pi_{ij}=\omega_{ij}+\omega_{ji}$ $(i,j=1,2,3)$ (cf. [4] p. 26) and det. $|\pi_{ij}-\delta_{ij}\lambda|=L-K\lambda-\lambda^3$ is an invariant polynomial in λ, where δ_{ij} is the Kronecker delta. Hence the coefficients

$$K = -1/2 \sum_{ij} \pi_{ij}^2, \qquad L = \det. |\pi_{ij}| \tag{14}$$

are invariants of the homogeneous space with conics as its elements. If we take a one-parametric family of conics and a suitable frame $(\bar{A}_1, \bar{A}_2, \bar{A}_3)$ obtained from (A_1, A_2, A_3) by an orthogonal transformation we have $\pi_{ij} = \varepsilon_i \delta_{ij}$ $(i, j = 1, 2, 3)$ and $K = -1/2(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$, $L = \varepsilon_1 \varepsilon_2 \varepsilon_3$. If the point whose coordinates are (x_1, x_2, x_3) for the frame (A_1, A_2, A_3) has coordinates (x_1', x_2', x_3') for the frame $(A_1 + dA_1, A_2 + dA_2, A_3 + dA_3)$, then we have $x_i = x_i' + \sum \omega_{ji} x_j'$, hence $x_i' = x_i - \sum \omega_{ji} x_j$ but for a term of higher order. The conic, which has an equation $\sum x_i'^2 = \sum (\delta_{ij} - \pi_{ij}) x_i x_j = 0$ for (A_1, A_2, A_3) has an equation $\sum (1 - \varepsilon_i) x_i^2 = 0$ for $(\bar{A}_1, \bar{A}_2, \bar{A}_3)$. Thus ε_i's are the coefficients appearing in the equation of the conic when we take a frame which is selfconjugate with respect to two consecutive conics and for which they are represented by $\sum x_i^2 = 0$ and $\sum (1 - \varepsilon_i) x_i^2 = 0$. This is a geometric interpretation of ε_i.

3.2 We consider a curve on the projective plane and let a Frenet's frame be (A_1, A_2, A_3). Then we have (cf. [1] p. 75)

$$dA_1 = d\sigma A_2, \quad dA_2 = d\sigma(-kA_1 + A_3), \quad dA_3 = d\sigma(-A_1 - kA_2). \tag{15}$$

If the point $\sum x_i A_i$, where x_i's are constant, coinsides with the point $\sum x_i (A_i + dA_i)$, we get $\sum x_i dA_i = \lambda \sum x_i A_i$. Hence by (15) we get

$$\lambda x_1 - kx_2 - x_3 = 0, \qquad x_1 + \lambda x_2 - kx_3 = 0, \qquad x_2 + \lambda x_3 = 0. \tag{16}$$

Eliminating x_1, x_2, x_3 we get

$$\lambda^3 + 2k\lambda - 1 = 0. \tag{17}$$

If k is not equal to $-3(32)^{-1/3}$, three roots of (17) are different, though they may not be real. Following the consideration of **1.1** a set of three points (x_1, x_2, x_3) determined by (16) for each root is a central figure and a one-parametric set of such figures is an evolute. The invariants of the evolute can be expressed by k, as in the affine plane (12), but the calculation is complicated and the result is not interesting. We take another way and derive some formulas in the following.

3.3 Let the three roots of (17) be λ_i $(i = 1, 2, 3)$. Then coordinates (x_1, x_2, x_3) of three points P_i $(i = 1, 2, 3)$ defined by (16) for each λ_i are $(\lambda_i^2 + k, -\lambda_i, 1)$. These three points are conjugate with respect to a conic $x_2^2 + 2x_1 x_3 = 0$. Now we take conics which have equations with respect to the frame (A_1, A_2, A_3)

$$Q_1:\ -x_1x_3+x_2^2+kx_3^2=0\,, \qquad Q_2:\ x_1x_2+kx_2x_3+x_3^2=0\,.$$

These pass through A and P_i. Moreover Q_1 passes through $B(k,0,1)$ and touches the line A_2B at B and the curve F at A_1, while Q_2 touches A_1A_3 at A_1. We calculate invariants K, L of a set of conics Q_1. When we take a frame $(\bar{A}_1, \bar{A}_2, \bar{A}_3)$ determined by $\bar{A}_1=(k-1)A_1+A_3$, $\bar{A}_2=A_2$, $\bar{A}_3=-\sqrt{-1}\,(k+1)A_1-\sqrt{-1}\,A_3$, the point which has the coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ for the frame $(\bar{A}_1, \bar{A}_2, \bar{A}_3)$ has the following coordinates for (A_1, A_2, A_3),

$$x_1=(k-1)\bar{x}_1-\sqrt{-1}\,(k+1)x_3\,, \qquad x_2=\bar{x}_2\,, \qquad x_3=\bar{x}_1-\sqrt{-1}\,\bar{x}_3\,.$$

Then Q_1 has the equation $\bar{x}_1^2+\bar{x}_2^2+\bar{x}_3^2=0$ with respect to $(\bar{A}_1, \bar{A}_2, \bar{A}_3)$. Putting $d\bar{A}_i=\sum_j \omega_{ij}\bar{A}_j$, $\omega_{ij}+\omega_{ji}=\pi_{ij}$ we get

$$\pi_{11}=d\sigma-dk\,, \qquad \pi_{22}=0\,, \qquad \pi_{33}=-(d\sigma-dk)$$

$$\pi_{23}=-\sqrt{-1}\left(k+\frac{1}{2}\right)d\sigma\,, \quad \pi_{31}=-\sqrt{-1}\,(d\sigma-dk)\,, \quad \pi_{12}=\left(k-\frac{1}{2}\right)d\sigma\,.$$

Hence by (12) we get

(18) $$K=2kd\sigma^2\,, \qquad L=(d\sigma-dk)\,d\sigma^2\,.$$

The same calculation with respect to Q_2 leads to

(19) $$K=2k(d\sigma-dk)\,d\sigma\,, \qquad L=d\sigma(d\sigma-dk)^2\,.$$

4. Generalization of Euler-Savary's theorem

4.1 When a curve C_1 rolls without slipping on another curve C_2 on the euclidean plane, a point P which is relatively fixed to C_1 describes a curve C_0 and according to Euler-Savary's theorem the curvature of C_0 at a point P can be represented by those of C_1 and C_2 and the polar coordinates of P with respect to the Frenet's frame at the point of contact (cf. [2] p. 28–29).

We consider in the homogeneous space $\mathfrak{G}/\mathfrak{H}$ two one-parametric sets of points C_1 and C_2, of which the latter is fixed in the space and the former rolls on it. Here the rolling shall be meant as follows. We take Frenet's frames $S_a^{(1)}R$ and $S_b^{(2)}R$ along C_1 and C_2, where b is a function of a. This function determines the way by which the rolling is defined. We take a frame $S_a^{(1)}S_tR$ whose relative position to $S_a^{(1)}R$

is S_t and let it belong to an element P of a certain homogeneous space $\mathfrak{G}/\mathfrak{K}$ which is relatively fixed with respect to C_1. As C_1 rolls on C_2, the point P describes a figure C_0 consisting of elements of $\mathfrak{G}/\mathfrak{K}$. This is a generalized roulette. We can express the invariants of C_0 by those of C_1, C_2 and the parameters which indicate the position of P relative to the frame $S_a^{(1)}R$.

Let the relative components of the fundamental Lie group be ω_p $(p=1,\cdots,r)$, while the principal relative components of the homogeneous space $\mathfrak{G}/\mathfrak{K}$ be ω_i $(i=1,\cdots,n)$. P is relatively fixed with respect to C_1, and the frame $S_a^{(1)}S_tR$ is attached to P. The relative displacement of $S_a^{(1)}S_t$ is given by $(S_a^{(1)}S_t)^{-1}(S_{a+da}^{(1)}S_{t+dt})=S_t^{-1}\left((S_a^{(1)})^{-1}S_{a+da}^{(1)}\right)S_t\cdot S_t^{-1}S_{t+dt}$. Hence we get by (1)

$$\omega_i^{(0)}+\sum_{p=1}^{r}\tau_{ip}\,\omega_p^{(1)}=0 \qquad (i=1,\cdots,n) \tag{20}$$

where $\omega_p^{(0)}, \omega_p^{(1)}$ are relative components corresponding to $S_t, S_a^{(1)}$ and (τ_{pq}) is an element of a linear adjoint group corresponding to S_t. When we superpose $S_a^{(1)}R$ on $S_b^{(2)}R$, the frame $S_a^{(1)}S_tR$ attached to P has the position $S_b^{(2)}S_tR$, and if we denote relative components of $S_b^{(2)}S_t$, $S_b^{(2)}$ by $\omega_p, \omega_p^{(2)}$, we get by (1)

$$\omega_p=\omega_p^{(0)}+\sum_{q=1}^{r}\tau_{pq}\,\omega_q^{(2)} \qquad (p=1,\cdots,r)\,. \tag{21}$$

The invariants of the generalized roulette C_0 can be calculated by ω_p. In the calculation the differentials of the parameters of S_t may appear repeatedly, which we eliminate by the use of (20).

4.2 Applying the above process we can get the classical case [2] p. 28–29. Here we apply this process to the case of an affine plane. Let C_1 and C_2 be curves on an affine plane, and let C_1 roll on C_2 in such a way that affine lengths of the corresponding arcs are the same. Thus a roulette motion can be realized. One-parametric motion on an affine plane is not in general a roulette motion. We do not enter into the details here. A roulette is a curve described by a point which is relatively fixed to C_1. Let the Frenet's frame along C_i $(i=1,2)$ be

$$dA=d\sigma\, e_1\,, \qquad de_1=d\sigma\, e_2\,, \qquad de_2=k_i\, d\sigma\, e_1\,,$$

and let a translation from (A, e_1, e_2) to $(\bar{A}, e_1, e_2)$ be given by $\bar{A}=A+$

$x_1 e_1 + x_2 e_2$. If (A, e_1, e_2) is a frame of C_1 the conditions under which $\bar{A}$ is relatively fixed to C_1 are

$$dx_1 + (1 + k_1 x_2)\, d\sigma = 0, \qquad dx_2 + x_1\, d\sigma = 0. \tag{22}$$

We superpose a Frenet's frame of C_1 on that of C_2. Let the relative displacement of the frame $(\bar{A}, e_1, e_2)$ determined by $\bar{A} = A + x_1 e_1 + x_2 e_2$, where (A, e_1, e_2) is a Frenet's frame of C_2, be $d\bar{A} = \omega_1 e_1 + \omega_2 e_2$, $de_1 = \omega_{11} e_1 + \omega_{12} e_2$, $de_2 = \omega_{21} e_1 - \omega_{11} e_1$, then we get for relative components of frames attached to a roulette

$$\omega_1 = dx_1 + (1 + k_2 x_2)\, d\sigma, \qquad \omega_2 = dx_2 + x_1\, d\sigma. \tag{23}$$

$$\omega_{11} = 0, \qquad \omega_{12} = d\sigma, \qquad \omega_{21} = k_2\, d\sigma. \tag{24}$$

This corresponds to (20). From (22) and (23) we get

$$\omega_1 = (k_2 - k_1)\, x_2\, d\sigma, \qquad \omega_2 = 0. \tag{25}$$

We take vectors $\bar{e}_1 = p e_1$, $\bar{e}_2 = q e_1 + 1/p\, e_2$, where p and q are defined by $p^3 = (k_2 - k_1) x_2$, $q = \dfrac{dp}{d\sigma} \Big/ p^2$. Then $(\bar{A}, \bar{e}_1, \bar{e}_2)$ is a Frenet's frame as is seen by

$$d\bar{A} = d\Sigma \cdot \bar{e}_1, \qquad d\bar{e}_1 = d\Sigma \cdot \bar{e}_2, \qquad d\bar{e}_2 = K d\Sigma \cdot \bar{e}_1.$$

Here K and $d\Sigma$ are an affine curvature and an affine arc length and they are

$$d\Sigma = \big((k_2 - k_1) x_2\big)^{2/3}\, d\sigma$$

$$K = \big((k_2 - k_1) x_2\big)^{-4/3} \Big(k_2 + \frac{1}{3} k_1 - \frac{1}{9} \Big(\frac{d}{d\sigma} \log (k_2 - k_1) - \frac{x_1}{x_2}\Big)^2 + \frac{1}{3} \frac{d^2}{d\sigma^2} \log (k_2 - k_1) + \frac{1}{3} \Big(\frac{1}{x_2}\Big)^2 - \frac{1}{3} \Big(\frac{x_1}{x_2}\Big)^2\Big). \tag{26}$$

This is Euler-Savary's theorem on the affine plane.

4.3 We take two ruled surfaces C_1 and C_2 in the euclidean 3-space and we assume by a suitable correspondence of the generating lines of both surfaces the angles between corresponding consecutive lines are equal and the parameters of distribution coinside. We take Frenet's frames of both surfaces. Then we have

$$dA = d\sigma(a_i e_1 + k e_3)\,, \qquad de_1 = d\sigma\, e_2 \qquad\qquad (i=1,2)$$
$$de_2 = d\sigma(-e_1 + b_i e_3)\,, \qquad de_3 = -b_i\, d\sigma\, e_2$$

When C_1 rolls on C_2 in such a way that the corresponding lines coinside, a line L_1 which is relatively fixed to C_1 generates a ruled surface C_0. Let the relative displacement of the Frenet's frame of C_0 be

$$dA = d\tau(\alpha e_1 + \kappa e_3)\,, \qquad de_1 = d\tau\, e_2$$
$$de_2 = d\tau(-e_1 + \beta e_3)\,, \qquad de_3 = -\beta\, d\tau\, e_2\,,$$

and let the distance of L_1 and the line of contact L of C_1 and C_2 be h, and l be a distance from the origin of the Frenet's frame of C_2 to the point where common perpendicular of L_1 and L intersect L. Moreover let (p_1, p_2, p_3) be direction cosines of L with respect to Frenet's frame. Then we get by calculation

$$d\tau = q^{-1}(b_2 - b_1)\, d\sigma$$
$$\kappa = -(a_2 - a_1)(b_2 - b_1)^{-1} - p_1 q h$$
$$\alpha = \bigl(p_1 q(a_2 - a_1) - p_1 p_2 q^4 h + p_3 q^3 k + p_2 q^3 l\bigr)(b_2 - b_1)^{-1} - h$$
$$\beta = p_1 q + p_3 q^3 (b_2 - b_1)^{-1}\,,$$

where we have put $q = (1 - p_1^2)^{-1/2}$.

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