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On the coefficients of multivalent functions.

By Toshio UMEZAWA

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1. Introduction.

It has recently been conjectured by A. W. Goodman [1] that if

$$f(z)=b_1 z+b_2 z^2+\cdots+b_n z^n+\cdots \tag{1.1}$$

is regular and p-valent for $|z|<1$, then for $n>p$

$$|b_n|\leqq\sum_{k=1}^{p}\frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|b_k|. \tag{1.2}$$

When $p=1$, this becomes the Bieberbach conjecture

$$|b_n|\leqq n|b_1|, \qquad n=2,3,\cdots \tag{1.3}$$

which has been proved for some special cases and has a long history [2]. When $p=2$ and $n=3$ (1.2) becomes

$$|b_3|\leqq 5|b_1|+4|b_2| \tag{1.4}$$

an inequality which has been proved valid, if $f(z)$ is regular 2-valent in $|z|<1$, starlike with respect to the origin, and in addition, if all b_i's are real [3].

Quite recently, by A. W. Goodman and M. S. Robertson [4], the inequality (1.2) has been proved to be valid for the class of functions called typically-real of order p, i. e. for functions with real coefficients such that $\Im f(z)$ changes its sign $2p$ times on $|z|=r$ for some range $0<\rho<r<1$.

In attempting to generalize the above results to the case where the coefficients b_n are complex, the present author was unable to obtain (1.2) for a certain class of functions to be defined in § 2, but was able to prove

$$|b_{p+1}| \leqq \sum_{k=1}^{p} \frac{2(2p)!\,(p^2+p+k^2)}{(p+k+1)!\,(p-k+1)!}\,|b_k|\,. \tag{1.5}$$

Results similar to the above one will be given in the present paper.

2. Preliminaries.

LEMMA 1. *Let*

$$w=f(z)=\sum_{n=0}^{\infty} a_n z^n \tag{2.1}$$

be regular for $|z|\leqq 1$ *and have* $p\,(\geqq 0)$ *zeros in* $|z|<1$, *no zeros lying on* $|z|=1$. *Then there exists a point* ζ $(|\zeta|=1)$ *for which the following equality holds*

$$\arg f(-\zeta)=\arg f(\zeta)+p\,\pi\,. \tag{2.2}$$

This lemma was used in [5].

The special cases of Lemma 1 and the following Definition 1 we owe to N. G. DeBruijn [6] and S. Ozaki [7].

DEFINITION 1. Let us call the diametral line of $f(z)$ for the straight line $\overline{f(\zeta)\,0\,f(-\zeta)}$ when ζ satisfies Lemma 1.

Accordingly we have the following:

LEMMA 1′. Let (2.1) be a function regular for $|z|\leqq 1$. Then there exists at least one diametral line of $f(z)$ in the w-plane.

Hereafter we shall suppose $f(z)\neq 0$ on $|z|=1$ without loss of generality.

DEFINITION 2. Let $f(z)$ be regular for $|z|\leqq 1$ and let C be the image curve of $|z|=1$. If C is cut by a straight line passing through the origin in $2p$, and not more than $2p$ points, then $f(z)$ is said to be starlike of order p in the direction of the straight line. In particular, if a direction of starlikeness of order p coincides with that of a diametral line of $f(z)$, i. e. if a diametral line in the w-plane cuts C in precisely $2p$ points, $f(z)$ is said to be a member of $D(p)$.

The class $D(p)$ was studied in [5]. The idea of being starlike in one direction was introduced by M. S. Robertson [8] and also extended to general p by him [9; 10]. And $D(1)$ was studied in [6; 7].

3. The main theorems.

THEOREM 1. *Let*

(3.1) $$f(z)=a_1 z+a_2 z^2+\cdots+a_n z^n+\cdots$$

be a function of the class $D(p)$. Then

(3.2) $$|a_{p+1}|\leqq\sum_{k=1}^{p}\frac{2(2p)!(p^2+p+k^2)}{(p+k+1)!(p-k+1)!}|a_k|.$$

To prove the theorem we need the following lemma due to M. S. Robertson [10].

LEMMA 2. *If*

$$F(z)=\sum_{k=-p}^{\infty}A_k z^k$$

be regular for $0<|z|\leqq 1$ with a pole of order not exceeding p at the origin, and if merely $\Re F(e^{i\theta})\geqq 0$ for all real θ, then

(3.3) $$|A_n+\bar{A}_{-n}|\leqq 2\Re A_0, \qquad n=1,2,\cdots,$$

where $\bar{A}_{-n}$ denotes the complex conjugate of A_{-n}.

PROOF OF THEOREM 1.

For our purpose it will be sufficient to assume that the diametral line in whose direction $f(z)$ is starlike of order p is $\overline{f(1)0f(-1)}$ since otherwise we may consider $f(\zeta z)=g(z)$ for which $g(1)0g(-1)$ is the diametral line.

Let $f(1)=\omega=|\omega|e^{-i\alpha}$; then by our hypothesis

(3.4) $$\begin{aligned}&\Im e^{i\alpha}f(e^{i\theta})>0 \quad \text{for} \quad \theta_{2s-1}<\theta<\theta_{2s},\\ &\Im e^{i\alpha}f(e^{i\theta})<0 \quad \text{for} \quad \theta_{2s}<\theta<\theta_{2s+1},\end{aligned}$$

$$s=1,2,\cdots,p,\ \theta_{2p+1}=\theta_1+2\pi,\ \theta_1=0,\ \theta_j=\pi,\ 1<j\leqq 2p.$$

Let

(3.5) $$g(z)=(-1)^{p-1}\exp\left(-\frac{i}{2}\sum_{s=1}^{2p}\theta_s\right)\cdot\frac{1}{z^p}\prod_{s=1}^{2p}(e^{i\theta_s}-z),$$

then

(3.6) $$g(e^{i\theta})=-2^{2p}\prod_{s=1}^{2p}\sin\frac{\theta_s-\theta}{2}.$$

Hence we obtain

$$(3.7)\qquad \begin{aligned} &g(e^{i\theta})>0 \quad \text{for} \quad \theta_{2s-1}<\theta<\theta_{2s}, \\ &g(e^{i\theta})<0 \quad \text{for} \quad \theta_{2s}<\theta<\theta_{2s+1}, \qquad s=1,2,\cdots,p. \end{aligned}$$

Let

$$(3.8)\qquad G(z)=-i\,e^{i\alpha}f(z)\,g(z),$$

then $G(z)$ is regular for $0<|z|\leqq 1$ and $\Re\, G(e^{i\theta})\geqq 0$. Moreover $G(z)$ has a pole of order not exceeding $p-1$ at $z=0$, depending upon whether some of the first coefficients $a_1, a_2, \cdots, a_p$ of $f(z)$ are different from zero or not. Hence, if we put

$$(3.9)\qquad G(z)=\sum_{k=-p+1}^{\infty} A_k z^k$$

then by Lemma 2

$$(3.10)\qquad |A_n+\bar{A}_{-n}|\leqq 2\,\Re\, A_0, \qquad n=1,2,\cdots.$$

On the other hand from (3.8) and (3.9) we have the identity

$$(3.11)\qquad f(z)=ie^{-i\alpha}G(z)/g(z)=\frac{(-)^p e^{-i\alpha} z^p \sum_{n=-p+1}^{\infty} A_n z^n}{\exp\left(-\frac{i}{2}\sum_{s=1}^{2p}\theta_s\right)(1-z^2)\prod_{\substack{s=2\\ s\neq j}}^{2p}(e^{i\theta_s}-z)}.$$

For the simplicity, let us put

$$(-1)^p \exp\left(i\alpha-\frac{i}{2}\sum_{s=1}^{2p}\theta_s\right)\prod_{\substack{s=2\\ s\neq j}}^{2p}(e^{i\theta_s}-z)=\sum_{s=0}^{2p-2} c_s z^s,$$

where $|c_s|\leqq\binom{2p-2}{s}$, since $(1+z)^{2p-2}$ dominates $\sum_{s=0}^{2p-2} c_s z^s$.

Then we have

$$(3.12)\qquad (1-z^2)\sum_{n=1}^{\infty} a_n z^n \sum_{s=0}^{2p-2} c_s z^s=\sum_{k=-p+1}^{\infty} A_k z^{k+p},$$

whence we have on equating coefficients of z^n from both sides of (3.12),

$$(3.13)\qquad A_{-p+n}=\sum_{k=1}^{n}(a_k-a_{k-2})\,c_{n-k}, \qquad n=1,2,\cdots,$$

where $\qquad a_{-1}=0, \qquad a_0=0.$

Hence we have from (3.10) and (3.13)

$$\left|\sum_{k=1}^{p+1}(a_k-a_{k-2})\,c_{p+1-k}+\overline{\sum_{k=1}^{p-1}(a_k-a_{k-2})\,c_{p-1-k}}\right|\leqq 2\,\Re\left[\sum_{k=1}^{p}(a_k-a_{k-2})\,c_{p-k}\right]$$

and hence

$$\begin{aligned}|a_{p+1}|&\leqq\sum_{k=1}^{p}[|c_{p-k+1}|+2(|c_{p-k}|+|c_{p-k-1}|+|c_{p-k-2}|)+|c_{p-k-3}|]\,|a_k|^{(*)}\\&\leqq\sum_{k=1}^{p}\left[\binom{2p-1}{p-k+1}+\binom{2p-1}{p-k}+\binom{2p-1}{p-k-1}+\binom{2p-1}{p-k-2}\right]|a_k|\\&=\sum_{k=1}^{p}\left[\binom{2p}{p-k+1}+\binom{2p}{p-k-1}\right]|a_k|\\&=\sum_{k=1}^{p}\frac{2\,(2p)!\,(p^2+p+k^2)}{(p+k+1)!\,(p-k+1)!}\,|a_k|\,.\end{aligned}$$

q. e. d.

REMARK. Repeated use of Lemma 2 enables us to obtain estimates for $|a_n|$ with general n. But the author could not obtain satisfactorily elegant inequality for general n and p.

COROLLARY 1. *Let $f(z)$ of the form* (3.1) *be regular for $|z|\leqq 1$. Suppose that $f(z)$ satisfies one of the following conditions:*

(i) $\Re[zf'(z)/f(z)]>0$ *for $|z|=1$ and $f(z)$ has p zeros in $|z|<1$.*

(ii) $f(1)=$ *real*, $f(-1)=$ *real and* $\Im f(e^{i\theta})$ *changes its sign $2p$ times on $|z|=1$.*

Then (3.2) *holds.*

PROOF. With one of the above conditions, $f(z)\in D(p)$. Cf. [5].

The same method of proof yields the following:

THEOREM 2. *Let* $f(z)=a_1 z+a_2 z^2+\cdots+a_n z^n+\cdots$ *be a member of the class $D(p)$. If the diametral line in whose direction $f(z)$ is starlike of order p cuts the image curve of $|z|=1$ under $f(z)$ at $f(\zeta e^{i\frac{k}{p}\pi})$ $(k=0,1,\cdots,2p-1)$, where ζ is a certain point on $|z|=1$, then*

$$(3.14)\qquad \begin{aligned}&|a_{mp}|\leqq m\,|a_p|\,,\\&|a_{np\pm q}|\leqq n\,|a_p|+|a_q|\,,\qquad q=1,2,\cdots,p-1\,,\end{aligned}$$

where m is an arbitrary integer and n is an arbitrary even integer.

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*) $C_i\equiv 0$ if i is negative.

COROLLARY 2. *Let* $w=f(z)=a_1 z+a_2 z^2+\cdots+a_n z^n+\cdots$ *be regular for* $|z|\leqq 1$. *If there exists a straight line passing through the origin in the w-plane such that the image curve of* $|z|=1$ *under* $f(z)$ *is cut by the straight line at* $f(+1)$, $f(i)$, $f(-1)$ *and* $f(-i)$, *then*

$$(3.15)\qquad \begin{aligned} |a_{2m}| &\leqq m|a_2|, \\ |a_{2n\pm 1}| &\leqq n|a_2|+|a_1|, \end{aligned}$$

where m is an arbitrary integer and n is an arbitrary even integer.

THEOREM 3. *Let* $f(z)=a_1 z+a_2 z^2+\cdots+a_n z^n+\cdots$ *be regular for* $|z|\leqq 1$ *and let* $z_k\,(k=1,2,\cdots,m)$ *be certain m points on the unit circle* $|z|=1$, *for which the centre of gravity of* $z_k^{2p}\,(k=1,2,\cdots,m)$ *coincides with the origin* 0. *If there exists m diametral lines in whose directions* $f(z)$ *is starlike of order p and if the m diametral lines cut the image curve of* $|z|=1$ *under* $f(z)$ *at* $f(z_k e^{\frac{is\pi}{p}})\,(s=0,1,2,\cdots,2p-1)$, $(k=1,2,\cdots,m)$ *respectively, then*

$$(3.16)\qquad \begin{aligned} |a_{n-p}|^2+|a_{n+p}|^2 &\leqq 4|a_p|^2 \qquad (n=p+1,\ p+2,\cdots), \\ |a_{n-p}|+|a_{n+p}| &\leqq 2\sqrt{2}\,|a_p| \qquad (n=p+1,p+2,\cdots). \end{aligned}$$

PROOF. In the elementary geometry it is known that if G is the centre of gravity of m points $A_1, A_2,\cdots, A_m$, then for any point P in this plane there exists the following relations:

$$m\overline{PG}^2+\sum_{k=1}^{m}\overline{GA}_k^2=\sum_{k=1}^{m}\overline{PA}_k^2.$$

If we put

$$A_k\equiv a_{n+p}\, z_k^{2p}\ (k=1,2,\cdots,m) \text{ and } P\equiv a_{n-p},$$

then

$$G\equiv 0$$

and we obtain the relation

$$m\,(|a_{n-p}|^2+|a_{n+p}|^2)=\sum_{k=1}^{m}|a_{n+p}\, z_k^{2p}-a_{n-p}|^2.$$

Now, since $f(z)$ is starlike of order p in the direction of these m diametral lines which cut the image curve of $|z|=1$ under $f(z)$ at $f(z_k e^{\frac{is\pi}{p}})$ $(s=0,1,2,\cdots,2p-1;\ k=1,2,\cdots,m)$ respectively, we obtain

$$|a_{n+p} z_k^{2p} - a_{n-p}| \leqq 2|a_p| \qquad (k=1, 2, \cdots, m)$$

by the method of proof used in Theorem 1.

Hence

$$|a_{n-p}|^2 + |a_{n+p}|^2 \leqq 4|a_p|^2 \qquad (n>p)$$

and

$$|a_{n-p}| + |a_{n+p}| \leqq \sqrt{|a_{n-p}|^2 + |a_{n+p}|^2}\ \sqrt{1+1}$$

$$\leqq 2\sqrt{2}\ |a_p|. \qquad \text{q. e. d.}$$

COROLLARY 3. *Let* $w=f(z)=a_1 z + a_2 z + \cdots + a_n z^n + \cdots$ *be regular for* $|z| \leqq 1$. *If there exist two straight lines passing through the origin in the w-plane such that the image curve of* $|z|=1$ *under* $f(z)$ *are cut by the straight lines at* $f(1)$, $f(i)$, $f(-1)$, $f(-i)$ *and* $f(e^{\frac{1}{4}\pi i})$, $f(e^{\frac{3}{4}\pi i})$, $f(e^{\frac{5}{4}\pi i})$, $f(e^{\frac{7}{4}\pi i})$ *respectively, then, for* $n>2$,

$$(3.17) \qquad \begin{aligned} &|a_{n-2}|^2 + |a_{n+2}|^2 \leqq 4|a_2|^2, \\ &|a_{n-2}| + |a_{n+2}| \leqq 2\sqrt{2}\ |a_2|. \end{aligned}$$

REMARK. If we put $p=1$ in Theorems 1, 2 and 3, then we have Ozaki's theorems. [7]

Gumma University.

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