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On ϕ-congruences.

By M.D. UPADHYAY

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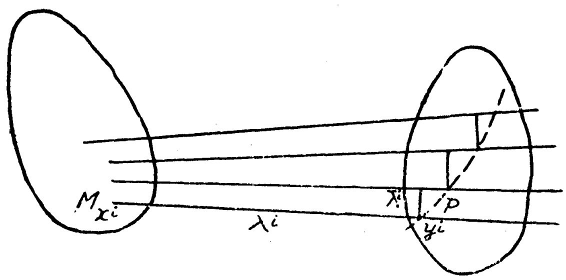
1. Let x^i $(i=1,2,3)$ be the co-ordinates of a point M, on the surface of reference, and λ^i $(i=1,2,3)$ the direction cosines of a line of congruence passing through M. Also, let $\bar{\lambda}^i$ $(i=1,2,3)$ be the direction cosines of a line of another congruence, intersecting the consecutive lines of the given congruence at a constant angle φ. I shall call this the φ-congruence. The line of striction of a ruled surface passing through the original congruence will lie on a surface. This ruled surface will be taken fixed. Ranga Chariar (1945) has shown that the feet of the rays of φ-congruence lie on the line of striction of the given ruled surface. Hence the surface on which this line of striction lies will be taken as the surface of reference of the φ-congruence.

The object of this paper is to find expressions for the parameter of distribution, and the distance of the central point from the surface of reference of the φ-congruence and the equation of ruled surfaces of φ-congruence whose spherical representations are minimal lines. Some particular cases yielding interesting results have been studied.



Surface of reference of the original congruence.

Surface of reference of φ-congruence.

2. Suppose a line of φ-congruence with direction cosines $\bar{\lambda}^i$, $(i=1,2,3)$ intersects its surface of reference at a point P, whose coordinates are y^i $(i=1,2,3)$. Then,

$$y^i = x^i + t\lambda^i, \tag{2.1}$$

where t is the distance of the central point of the given ruled surface of the original congruence from the point M. x^i, y^i, λ^i and $\bar{\lambda}^i$ are functions of $u^\alpha(\alpha=1,2)$* and

$$(2.2) \qquad \bar{\lambda}^i \cdot \bar{\lambda}^i = 1 .$$

If, for convenience, the notation of covariant derivative $\bar{\lambda}^i_{,\alpha}$ of $\bar{\lambda}^i$ with respect to the first fundamental tensor $\bar{G}_{\alpha\beta}$ of the spherical representations of φ-congruence is used instead of $\frac{\partial \bar{\lambda}^i}{\partial u^\alpha}$; then the two quadratic forms used by Kummer (1860) are

$$(2.3) \qquad \bar{G}_{\alpha\beta}\, du^\alpha\, du^\beta ,$$

and

$$(2.4) \qquad \bar{\mu}_{\alpha\beta}\, du^\alpha\, du^\beta ,$$

where

$$(2.5) \qquad \bar{G}_{\alpha\beta} = \bar{\lambda}^i_{,\alpha} \cdot \bar{\lambda}^i_{,\beta} ,$$

and

$$(2.6) \qquad \bar{\mu}_{\alpha\beta} = \bar{\lambda}^i_{,\alpha} \cdot y^i_{,\beta} .$$

The quadratic forms used by Sannia (Bianchi, 1927) are,

$$(2.7) \qquad \bar{G}_{\alpha\beta}\, du^\alpha\, du^\beta ,$$

and

$$(2.8) \qquad \xi_{\alpha\beta}\, du^\alpha\, du^\beta ,$$

where

$$(2.9) \qquad \xi_{\alpha\beta} = (y^i_{,\alpha} \quad \bar{\lambda}^i \quad \bar{\lambda}^i_{,\beta}) .$$

$\bar{\lambda}^i$ can be expressed as (Ranga Chariar, 1945)

$$(2.10) \qquad \bar{\lambda}^i = \lambda^i \cos\varphi + \lambda^i \times \frac{d\lambda^i}{d\sigma} \sin\varphi ,$$

where $d\sigma$ is the small element of length on the spherical representation of the original congruence.

* In what follows Latin indices take the values (1, 2, 3) and Greek indices the values (1, 2).

Now, let $d\bar{\sigma}$ be the corresponding element of length on the spherical representation of the φ-congruence. Then,

$$\left(\frac{d\bar{\sigma}}{d\sigma}\right)^2=\frac{d\bar{\lambda}^i}{d\sigma}\cdot\frac{d\bar{\lambda}^i}{d\sigma}. \tag{2.11}$$

$$\left(\frac{d\bar{\sigma}}{d\sigma}\right)^2=\left(\frac{d\lambda^i}{d\sigma}\cos\varphi+\lambda^i\times\frac{d^2\lambda^i}{d\sigma^2}\sin\varphi\right)\cdot\left(\frac{d\lambda^i}{d\sigma}\cos\varphi+\lambda^i\times\frac{d^2\lambda^i}{d\sigma^2}\sin\varphi\right),$$

by virtue of (2.10).

$$=\cos^2\varphi+\left(\frac{d\lambda^i}{d\sigma}\quad\lambda^i\quad\frac{d^2\lambda^i}{d\sigma^2}\right)\sin\varphi\cos\varphi \tag{2.12}$$

$$+\left(\lambda^i\quad\frac{d^2\lambda^i}{d\sigma^2}\quad\frac{d\lambda^i}{d\sigma}\right)\sin\varphi\cos\varphi+\left(\lambda^i\times\frac{d^2\lambda^i}{d\sigma^2}\right)\cdot\left(\lambda^i\times\frac{d^2\lambda^i}{d\sigma^2}\right)\sin^2\varphi,$$

since

$$\frac{d\lambda^i}{d\sigma}\cdot\frac{d\lambda^i}{d\sigma}=1.$$

Now

$$\frac{d\lambda^i}{d\sigma}=\frac{du^\alpha}{d\sigma}\cdot\frac{\partial\lambda^i}{\partial u^\alpha},$$

$$\frac{d^2\lambda^i}{d\sigma^2}=\frac{\partial^2\lambda^i}{\partial u^\alpha\,\partial u^\beta}\frac{du^\alpha}{d\sigma}\frac{du^\beta}{d\sigma}+\lambda^i_{,\alpha}\frac{d^2u^\alpha}{d\sigma^2}.$$

But

$$\frac{\partial^2\lambda^i}{\partial u^\alpha\,\partial u^\beta}=\left\{\begin{matrix}\gamma\\ \beta\ \alpha\end{matrix}\right\}\lambda^i_{,\gamma}-G_{\alpha\beta}\lambda^i,$$

where $\left\{\begin{matrix}\gamma\\ \beta\ \alpha\end{matrix}\right\}$ are Christoffel symbols of the second kind, for the unit sphere on which the original congruence is represented, and

$$G_{\alpha\beta}=\lambda^i_{,\sigma}\cdot\lambda^i_{,\beta}.$$

Therefore

$$\frac{d^2\lambda^i}{d\sigma^2}=\left[\left(\left\{\begin{matrix}\gamma\\ \beta\ \alpha\end{matrix}\right\}\lambda^i_{,\gamma}-G_{\alpha\beta}\lambda^i\right)\frac{du^\alpha}{d\sigma}\frac{du^\beta}{d\sigma}+\lambda^i_{,\gamma}\frac{d^2u^\gamma}{du\sigma^2}\right],$$

$$=\left[\frac{d^2 u^\gamma}{d\sigma^2}+\begin{Bmatrix}\gamma\\ \beta\ \alpha\end{Bmatrix}\frac{du^\alpha}{d\sigma}\frac{du^\beta}{d\sigma}\right]\lambda^i_{,\gamma}-\lambda^i\,,$$

(2.13) $$=\rho^\gamma\lambda^i_{,\gamma}-\lambda^i\,,$$

where (Eisenhart, 1940) $\rho^\gamma\equiv\frac{d^2 u^\gamma}{d\sigma^2}+\begin{Bmatrix}\gamma\\ \beta\ \alpha\end{Bmatrix}\frac{du^\alpha}{d\sigma}\frac{du^\beta}{d\sigma}$, is the curvature vector of the spherical representations of the original congruence. By virtue of (2.13), the equation (2.12) becomes

$$\left(\frac{d\bar{\sigma}}{d\sigma}\right)^2=\cos^2\varphi-2E_{\beta\gamma}\rho^\gamma u'^\beta\sin\varphi\cos\varphi+G_{\gamma\beta}\rho^\gamma\rho^\beta\sin^2\varphi\,,$$

where dashes denote differentiation with respect to the arc length of the spherical representations of the original congruence; and

$$E_{\beta\gamma}=(\lambda^i\quad\lambda^i_{,\beta}\quad\lambda^i_{,\gamma})\,.$$

Therefore

(2.14) $$\left(\frac{d\bar{\sigma}}{d\sigma}\right)^2=(\cos\varphi-k_g\sin\varphi)^2\,,$$

since (Eisenhart, 1940)

$$E_{\alpha\beta}\,\rho^\beta u'^\alpha=k_g\,,$$

where

$$k_g^2=E_{\alpha\beta}E_{\gamma\delta}u'^\alpha u'^\gamma\rho^\beta\rho^\delta\,,$$
$$=(G_{\alpha\gamma}G_{\beta\delta}-G_{\alpha\delta}G_{\beta\gamma})\,u'^\alpha u'^\gamma\rho^\beta\rho^\delta\,,$$
$$=G_{\beta\delta}\,\rho^\beta\rho^\delta\,,$$

k_g, being the geodesic curvature of the spherical indicatrix of the lines of the original congruence. Mishra, (1951, 1) has shown that the geodesic curvature of the spherical indicatrix of the generators of a ruled surface is equal to the skewness of distribution μ, (a term defined by Ranga Chariar, 1945) of the ruled surface. Therefore

(2.15) $$\frac{d\bar{\sigma}}{d\sigma}=(\cos\varphi-\mu\sin\varphi)\,,$$

which is the same as found by Ranga Chariar (1945).

The equation of the ruled surfaces of φ-congruence, whose spherical representations are minimal lines is given by

$$\bar{G}_{\alpha\beta}\, du^{\alpha}\, du^{\beta}=0\,,$$

which by virtue of (2.15) assumes the form

$$(\cos\varphi-\mu\sin\varphi)=0\,,$$

or

$$\mu=\cot\varphi\,. \tag{2.16}$$

Hence *the skewness of distribution of the ruled surfaces of φ-congruence, whose spherical representations are minimal lines is a constant and equal to the cotangent of the constant angle φ.*

3. The distance of the central point of a line of φ-congruence from its surface of reference is given by (Weatherburn, 1931)

$$\bar{t}=-\left(\frac{dy^i}{d\bar{\sigma}}\cdot\frac{d\bar{\lambda}^i}{d\bar{\sigma}}\right), \tag{3.1}$$

$$=-\left(\frac{dy^i}{d\sigma}\cdot\frac{d\bar{\lambda}^i}{d\sigma}\right)\Big/\left(\frac{d\bar{\sigma}}{d\sigma}\right)^2,$$

which by virtue of (2.1), (2.10) and (2.15) assumes the form

$$\bar{t}=-\frac{\left[\left(\frac{dx^i}{d\sigma}+t\frac{d\lambda^i}{d\sigma}\right)\cdot\left(\frac{d\lambda^i}{d\sigma}\cos\varphi+\lambda^i\times\frac{d^2\lambda^i}{d\sigma^2}\sin\varphi\right)\right]}{(\cos\varphi-\mu\sin\varphi)^2},$$

$$=-\left[\frac{dx^i}{d\sigma}\cdot\frac{d\lambda^i}{d\sigma}\cos\varphi+\left(\frac{dx^i}{d\sigma}\quad\lambda^i\quad\frac{d^2\lambda^i}{d\sigma^2}\right)\sin\varphi+t\cos\varphi+\right.$$

$$\left.+t\left(\frac{d\lambda^i}{d\sigma}\quad\lambda^i\quad\frac{d^2\lambda^i}{d\sigma^2}\right)\sin\varphi\right]\Big/(\cos\varphi-\mu\sin\varphi)^2,$$

$$=-\frac{[(x^i_{,\alpha}\quad\lambda^i\quad\lambda^i_{,\gamma})\rho^{\gamma}u'^{\alpha}\sin\varphi+t(\lambda^i_{,\beta}\quad\lambda^i\quad\lambda^i_{,\gamma})\rho^{\gamma}u'^{\beta}\sin\varphi]}{(\cos\varphi-\mu\sin\varphi)^2}.$$

But (Mishra 1951, 2) $x^i_{,\alpha}=p_{\alpha}\lambda^i+q^{\mu}_{\alpha}\lambda^i_{,\mu}$,

whence,

$$q^{\mu}_{\alpha}=\mu_{\nu\alpha}\, G^{\nu\mu},$$

where $\mu_{\nu\alpha}=\lambda^{i}_{,\nu}\cdot x^{i}_{,\alpha}$, is the fundamental tensor of Kummer's second quadratic form and $G^{\nu\mu}$ is a tensor conjugate to the tensor $G_{\nu\mu}$; and $p_{\alpha}=\lambda^{i}\cdot x^{i}_{,\alpha}$. Therefore

$$\bar{t}=\frac{[q^{\mu}_{\alpha}\, E_{\mu\gamma}\, \rho^{\gamma}\, u'^{\alpha}\sin\varphi+t\, E_{\beta\gamma}\, \rho^{\gamma}\, u'^{\beta}\sin\varphi]}{(\cos\varphi-\mu\sin\varphi)^{2}},$$

$$=\frac{\rho^{\gamma}\, u'^{\alpha}\sin\varphi\,(\xi_{\alpha\gamma}+t\, E_{\alpha\gamma})}{(\cos\varphi-\mu\sin\varphi)^{2}}, \tag{3.2}$$

since (Ram Behari and Mishra, 1949)

$$q^{\mu}_{\alpha}\, E_{\mu\gamma}=\xi_{\alpha\gamma}.$$

Hence

$$\bar{t}=\frac{\mu\sin\varphi(\xi_{\alpha\gamma}\,\mu^{\gamma}\, u'^{\alpha}+t)}{(\cos\varphi-\mu\sin\varphi)^{2}}. \tag{3.3}$$

where $\rho^{\gamma}=\mu\,\mu^{\gamma}$; μ^{γ} being a unit vector in the direction of the curvature vector of the spherical representation of the original congruence.
In particular,

1) when $\varphi=0$, the lines of the φ-congruence coincide with the lines of the original congruence and

$$\bar{t}=0. \tag{3.4}$$

2) when the congruence is formed by the lines perpendicular to the lines of the original congruence, then, $\varphi=\frac{\pi}{2}$, and (3.3) yields

$$\bar{t}=\frac{\left(\xi_{\alpha\gamma}\,\mu^{\gamma}\,\frac{du^{\alpha}}{d\sigma}+t\right)}{\mu}. \tag{3.5}$$

3) when the original congruence is isotropic (Mishra, 1945)

$$\xi_{\alpha\beta}=\lambda\, G_{\alpha\beta},$$

where λ is proportionality factor between the coefficients of Sannia's quadratic forms, then,

(3.6)
$$\bar{t}=\frac{\mu \sin\varphi(\lambda G_{\alpha\gamma}\,\mu^{\gamma}\,u'^{\alpha}+t)}{(\cos\varphi-\mu\sin\varphi)^2}$$

$$=\frac{\mu\sin\varphi\, t}{(\cos\varphi-\mu\sin\varphi)^2},$$

since (Eisenhart, 1940)

$$G_{\alpha\gamma}\,\rho^{\gamma}\,u'^{\alpha}=0\,.$$

In this case, when

$$\varphi=0;\quad \bar{t}=0\,.$$

and, when

$$\varphi=\frac{\pi}{2};\ \bar{t}=\frac{t}{\mu}\,.$$

Hence *the ratio between the distances of the central points from the surfaces of reference, of an isotropic congruence and a congruence formed by the common perpendiculars to the consecutive rays of the isotropic congruence is equal to the skewness of distribution of the isotropic congruence.*

4) when the skewness of distribution μ, of the φ-congruence vanishes, (3.4) reduces to $\bar{t}=0$.

Hence, *when the rays of the original congruence are parallel to a plane, the lines of striction of φ congruence lie on its surface of reference.*

4. The parameter of distribution for a ruled surface of φ-congruence is given by (Weatherburn, 1931)

(4.1)
$$\bar{d}=\left(\frac{dy^i}{d\bar{\sigma}}\ \ \bar{\lambda}^i\ \ \frac{d\bar{\lambda}^i}{d\bar{\sigma}}\right),$$

$$=\left(\frac{dy^i}{d\sigma}\ \ \bar{\lambda}^i\ \ \frac{d\bar{\lambda}^i}{d\sigma}\right)\ \left(\frac{d\sigma}{d\bar{\sigma}}\right)^2,$$

$$= \frac{\left[\frac{dx^i}{d\sigma} + t\frac{d\lambda^i}{d\sigma} \quad \lambda^i \cos\varphi + \lambda^i \times \frac{d\lambda^i}{d\sigma}\sin\varphi \quad \frac{d\lambda^i}{d\sigma}\cos\varphi + \lambda^i \times \frac{d^2\lambda^i}{d\sigma^2}\sin\varphi\right]}{(\cos\varphi - \mu\sin\varphi)^2},$$

$$\begin{aligned}
=\Bigg[&\left(\frac{dx^i}{d\sigma} \quad \lambda^i \quad \frac{d\lambda^i}{d\sigma}\right)\cos^2\varphi + \left(\frac{dx^i}{d\sigma} \quad \lambda^i \quad \lambda^i \times \frac{d^2\lambda^i}{d\sigma^2}\right)\sin\varphi\cos\varphi \\
&+\left(\frac{dx^i}{d\sigma} \quad \lambda^i \times \frac{d\lambda^i}{d\sigma} \quad \frac{d\lambda^i}{d\sigma}\right)\sin\varphi\cos\varphi \\
&+\left(\frac{dx^i}{d\sigma} \quad \lambda^i \times \frac{d\lambda^i}{d\sigma} \quad \lambda^i \times \frac{d^2\lambda^i}{d\sigma^2}\right)\sin^2\varphi \\
&+t\left(\frac{d\lambda^i}{d\sigma} \quad \lambda^i \quad \lambda^i \times \frac{d^2\lambda^i}{d\sigma^2}\right)\sin\varphi\cos\varphi \\
&+t\left(\frac{d\lambda^i}{d\sigma} \quad \lambda^i \times \frac{d\lambda^i}{d\sigma} \quad \lambda^i \times \frac{d^2\lambda^i}{d\sigma^2}\right)\sin^2\varphi\Bigg] \Big/ \left(\cos\varphi - \mu\sin\varphi\right)^2,
\end{aligned}$$

$$\begin{aligned}
=[&(q^\mu_\alpha \lambda^i_{,\mu} + p_\alpha \lambda^i \quad \lambda^i \quad \lambda^i_{,\beta})\, u'^\alpha u'^\beta \cos^2\varphi \\
&+(q^\mu_\alpha \lambda^i_{,\mu} + p_\alpha \lambda^i \quad \lambda^i \quad \lambda^i \times \lambda^i_{,\gamma})\, u'^\alpha \rho^\gamma \sin\varphi\cos\varphi \\
&+(q^\mu_\alpha \lambda^i_{,\mu} + p_\alpha \lambda^i \quad \lambda^i \times \lambda^i_{,\delta} \quad \lambda^i_{,\beta})\, u'^\alpha u'^\beta u'^\delta \sin\varphi\cos\varphi \\
&+(q^\mu_\alpha \lambda^i_{,\mu} + p_\alpha \lambda^i \quad \lambda^i \times \lambda^i_{,\delta} \quad \lambda^i \times \lambda^i_{,\gamma})\, u'^\alpha u'^\delta \rho^\gamma \sin^2\varphi \\
&+t(\lambda^i_{,\beta} \quad \lambda^i \quad \lambda^i \times \lambda^i_{,\gamma})\, u'^\beta \rho^\gamma \sin\varphi\cos\varphi \\
&+t(\lambda^i_{,\beta} \quad \lambda^i \times \lambda^i_{,\delta} \quad \lambda^i \times \lambda^i_{,\gamma})\, u'^\beta u'^\delta \rho^\gamma \sin^2\varphi] / (\cos\varphi - \mu\sin\varphi)^2,
\end{aligned}$$

$$\begin{aligned}
\bar{d} = [&d\cos^2\varphi - q^\mu_\alpha G_{\mu\gamma} u'^\alpha \rho^\gamma \sin\varphi\cos\varphi + p_\alpha E_{\delta\gamma} u'^\alpha u'^\delta \rho^\gamma \sin^2\varphi \\
&+ p_\alpha G_{\delta\beta} u'^\alpha u'^\beta u'^\delta \sin\varphi\cos\varphi - t\, G_{\beta\gamma} u'^\beta \rho^\gamma \sin^2\varphi] / (\cos\varphi - \mu\sin\varphi)^2,
\end{aligned}$$

(4.2)
$$\begin{aligned}
&= (d\cos^2\varphi - \mu_{\gamma\alpha} \rho^\gamma u'^\alpha \sin\varphi\cos\varphi + k_g\, p_\alpha u'^\alpha \sin^2\varphi \\
&\quad + (p_\alpha u'^\alpha \sin\varphi\cos\varphi) / (\cos\varphi - \mu\sin\varphi)^2,
\end{aligned}$$

since

$$q^\mu_\alpha G_{\mu\gamma} = \mu_{\gamma\alpha},$$

$$G_{\beta\delta} u'^\beta u'^\delta = 1.$$

Hence (4.2) *shows that the parameter of distribution of φ congruence is independent of the distance of the central point of the original congruence from its surface of reference.*

1) when a congruence is formed by the perpendiculars to the lines of the original congruence

$$\bar{d}=\frac{\left(p_\alpha \frac{du^\alpha}{d\sigma}\right)}{\mu}. \tag{4.3}$$

Hence *for a congruence formed by the common perpendiculars to the lines of another congruence which are parallel to a plane, the parameter of distribution becomes infinite. Therefore, the ruled surface through a line of former congruence is represented on the unit sphere by minimal lines.*

Ranga Chariar (1945) has taken the line of striction of a ruled surface of φ-congruence to be the line of striction of given ruled surface and on this supposition he has given the parameter of distribution of the ruled surface of φ-congruence as

$$B=-\frac{\sin(\theta-\varphi)}{d\sigma(\cos\varphi-\mu\sin\varphi)}. \tag{4.4}$$

In § 3, I have proved that the distance of the central point of the ruled surface of φ-congruence from its surface of reference is not zero in general and hence the expression for its parameter of distribution given by Ranga Chariar holds good only when $\bar{t}$ is zero.

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Department of Mathematics.
Lucknow University,
Lucknow, INDIA.

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