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On the converse of Abel's theorem.

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Concerning the converse of Abel's theorem, Hardy and Littlewood proved the following two theorems.

THEOREM 1.[1] *Let* $f(x)=\sum_{n=0}^{\infty} a_n x^n$ *be regular for* $|x|<1$ *and* $f(x)\to s$, *when* x *tends to* $x=1$ *along the real axis. If* a_n *are real and* $n\,a_n \leq K$ $(n=1, 2, \cdots)$, *then* $\sum_{n=0}^{\infty} a_n = s$.

THEOREM 2.[2] *Let* $f(x)=\sum_{n=0}^{\infty} a_n x^n$ *be regular for* $|x|<1$ *and* $f(x)\to s$, *when* x *tends to* $x=1$ *along a curve* C *in* $|x|<1$, *which ends at* $x=1$. *If* $n\,|a_n| \leq K\,(n=1, 2, \cdots,)$, *then* $\sum_{n=0}^{\infty} a_n = s$.

The original proof of theorem 1 is very complicated. Recently Wielandt[3] gave a remarkably simple proof of it. In this paper, I shall simplify somewhat the original proof of Theorem 2.

First we shall prove a lemma.

LEMMA. *Let* D *be a simply connected domain on the* z*-plane, which is contained in an angular domain* Δ: $0<\arg z<\alpha$. *The boundary of* D *consists of a segment* AB *on the positive real axis and a Jordan arc* C *in* Δ, *which connects* A *and* B. *Let* $f(z)$ *be regular in the closed domain* $\bar{D}$ *and* $|f(z)|\leq M$ *on* AB, $|f(z)|\leq m$ $(m\leq M)$ *on* C. *Let* L: $\arg z=\theta$ $(0<\theta<\alpha)$ *be a half-line. If* L *penetrates into* D, *then at any point* $z\in D$ *on* L,

$$|f(z)|\leq M^{1-\frac{\theta}{\alpha}}\, m^{\frac{\theta}{\alpha}}\,.$$

1) Hardy and Littlewood: Tauberian theorem concerning power series and Dirichlet's series whose coefficient are positive. Proc. London Math. Soc. 13 (1914).

2) Hardy and Littlewood: Abel's theorem and its converse. (II). Proc. London Math. Soc. 22 (1923).

3) Wielandt: Zur Umkehrung des Abelschen Stetigkeitssatzes. Math. Zeits. 56 (1952).

Proof. Let $u(z)$ be the harmonic measure of AB with respect to D, such that $u(z)$ is harmonic in D and $u(z)=1$ on AB, $u(z)=0$ on C. Then $\log|f(z)|\leqq u(z)\log M+(1-u(z))\log m=u(z)(\log M-\log m)+\log m$. Since $u(z)\leqq 1-\frac{\arg z}{\alpha}$ and $\log M-\log m\geqq 0$, we have $\log|f(z)|\leqq\left(1-\frac{\arg z}{\alpha}\right)(\log M-\log m)+\log m=\left(1-\frac{\arg z}{\alpha}\right)\log M+\frac{\arg z}{\alpha}\log m$.
Hence on L,

$$|f(z)|\leqq M^{1-\frac{\theta}{\alpha}}m^{\frac{\theta}{\alpha}}.$$

Proof of Theorem 2. We may assume that C is an analytic Jordan arc and $s=0$, $n|a_n|\leqq 1\ (n=1,2,\cdots)$. We put $x=e^{-z}$, $f(x)=F(z)$, then

$$F(z)=\sum_{n=0}^{\infty}a_n e^{-nz}\quad(z=\sigma+it). \tag{1}$$

$F(z)$ is regular for $\sigma>0$ and $F(z)\to 0$, when z tends to $z=0$ along an analytic Jordan arc C ending at $z=0$, where we denote the image of C on the z-plane by the same letter C.

$$|F'(z)|\leqq\sum_{n=1}^{\infty}n|a_n|e^{-n\sigma}\leqq\sum_{n=1}^{\infty}e^{-n\sigma}=\frac{1}{e^{\sigma}-1}<\frac{1}{\sigma}\quad(\sigma>0). \tag{2}$$

Let Δ: $|\arg z|<\alpha<\frac{\pi}{2}$ be an angular domain and $z'=\sigma+it'$, $z''=\sigma+it''$ be two points of Δ, which lie on the same vertical line, then by (2),

$$|F(z')-F(z'')|<\frac{|t'-t''|}{\sigma}<2\tan\alpha. \tag{3}$$

We consider two cases, according as (i) C is a Stolz path, or (ii) C is not a Stolz path.

(i) C is a Stolz path.

Then C is contained in an angular domain Δ: $|\arg z|<\alpha<\frac{\pi}{2}$.

Let z' be a point of C, then we may assume that $|F(z')| \leqq 1$. Then by (3), for any z in $\varDelta$, which lies on the same vertical line as z',

$$|F(z)| \leqq |F(z')| + 2 \tan \alpha \leqq 1 + 2 \tan \alpha .$$

Hence $F(z)$ is bounded in $\varDelta$ and $F(z) \to 0$, when $z \to 0$ along C, hence $F(z) \to 0$, when $z \to 0$ on the real axis, so that by Theorem 1, $\sum_{n=0}^{\infty} a_n = 0$.

(ii) C is not a Stolz path.

We assume that C meets any half-line L: $\arg z = \frac{\pi}{2} - \alpha$ for any small $\alpha > 0$. Let D be any simply connected domain, which lies in an angular domain $\varDelta : \frac{\pi}{2} - \alpha < \arg z < \frac{\pi}{2}$ and is bounded by a part of C and a segment AB on L, where C meets L at A, B. We may assume that $|F(z)| \leqq 1$ on C. Then we shall prove that

$$|F(z)| \leqq K = 1 + 8 \cot \frac{\alpha}{2} \quad \text{in } D . \tag{4}$$

Let O be the origin and $\overline{OA} < \overline{OB}$. First suppose that A is different from O, then $F(z)$ is regular in the closed domain $\bar{D}$. Let $M =$ Max. $|F(z)|$ on AB, then $M = |F(z_0)|$ at a point z_0 on AB. Since (4) holds evidently, if $M \leqq 1$, we assume that $M > 1$, then $z_0 \neq A, \neq B$. Let L': $\arg z = \frac{\pi}{2} - \frac{\alpha}{2}$ be a half-line and z' be the point of L', which lies on the same vertical line as z_0. If z' lies in D, then applying the lemma for $D \subset \varDelta$, we have

$$|F(z')| \leqq \sqrt{M} . \tag{5}$$

On the other hand, we have by (3),

$$|F(z')| \geqq |F(z_0)| - 2 \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = M - 2 \cot \frac{\alpha}{2},$$

so that

$$M - 2 \cot \frac{\alpha}{2} \leqq \sqrt{M} ,$$

hence

$$M \leq \left(\frac{1+\sqrt{1+8 \cot \frac{\alpha}{2}}}{2} \right)^2 < 1+8 \cot \frac{\alpha}{2} = K. \tag{6}$$

If z' lies outside D, then the segment $z_0 z'$ meets C at z'', so that by (3),

$$M = |F(z_0)| \leq |F(z'')| + 2 \cot \frac{\alpha}{2} < 1 + 2 \cot \frac{\alpha}{2} < K. \tag{7}$$

Hence in any case, $M \leq K$, so that by the maximum principle,

$$|F(z)| \leq K \text{ in } D. \tag{4}$$

If A coincides with O, then we apply (4) for the part of D, which lies between the positive imaginary axis and a line L', which passes through it $(t > 0)$ and is parallel to L and then we make $t \to 0$, then we have (4). Hence (4) holds in general.

Suppose that C meets a half-line L: $\arg z = \frac{\pi}{2} - \alpha \left(0 < \alpha < \frac{\pi}{2}\right)$ infinitely often, then the part of C, which lies between L and the positive imaginary axis consists of a countable number of Jordan arcs $\{C_\nu\}$, whose end points A_ν, B_ν lie on L. C_ν and the segment $A_\nu B_\nu$ bounds a simply connected domain D_ν. Let $|F(z)| \leq \delta_\nu$ on C_ν, then $\delta_\nu \to 0$ with $\nu \to \infty$. By (4), $|F(z)| \leq K$ on $A_\nu B_\nu$.

Since C is not a Stolz path, a half-line L': $\arg z = \frac{\pi}{2} - \frac{\alpha}{2}$ penetrates into infinitely many D_ν, $D_{\nu_k} (k=1, 2, \cdots)$, say.

Then by the lemma,

$$|F(z)| \leq \sqrt{K \delta_{\nu_k}} \tag{8}$$

on the part of L', which lies in D_{ν_k}.

Now the part of D_{ν_k}, which lies between L and L' is decomosed into a finite number of simply connected domains. Let $D^0_{\nu_k}$ be such one, which abutts on L. We modify C to a curve C', by replacing C_{ν_k} by the boundary of $D^0_{\nu_k}$, except the segment $A_{\nu_k} B_{\nu_k} (k=1, 2, \cdots)$. Then C' lies between L' and the negative imaginary axis. From (8), $F(z) \to 0$, when $z \to 0$ along C'. If C' is a Stolz path, then by (i),

$\sum_{n=0}^{\infty} a_n = 0$.

If C' is not a Stolz path, then C' meets the real axis infinitely often. Then we modify C' to C'' as before, such that C'' lies between L' and the real axis and $F(z) \to 0$, when $z \to 0$ along C''. Hence by (i), $\sum_{n=0}^{\infty} a_n = 0$.

If C meets any half-line L: $\arg z = \frac{\pi}{2} - \alpha\left(0 < \alpha < \frac{\pi}{2}\right)$ only finite times, then C touches the positive imaginary axis. If we define $\{D_\nu\}$ for L as before, then there exists one D_{ν_0}, which has $z=0$ on its boundary. Since by (4), $F(z)$ is bounded in D_{ν_0} and $F(z) \to 0$, when $z \to 0$ along C, which touches the positive imaginary axis, we see that $F(z) \to 0$, when $z \to 0$ on a half-line L': $\arg = \frac{\pi}{2} - \frac{\alpha}{2}$. Since L' is a Stolz path, we have $\sum_{n=0}^{\infty} a_n = 0$. Hence our theorem is proved.

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