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Note on Betti numbers of Riemannian manifolds II.

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§ 1. An extension of a theorem of Bochner-Lichnerowicz.

Consider a harmonic vector X_i and a symmetric tensor A_{ij}. By Green's theorem we have

$$(1.1)\quad 0=\int (A_{ij}X^{i;k}X^j)_{;k}\,dv=\int A_{ij;k}X^{i;k}X^j\,dv+\int A_{ij}\Delta X^i X^j\,dv$$
$$+\int A_{ij}X^{i;k}X^j_{;k}\,dv=\int A_{ij;k}X^{i;k}X^j dv+\int A_{ij}R^i_k X^k X^j\,dv$$
$$+\int A_{ij}X^{i;k}X^j_{;k}dv\,,$$

$$(1.2)\quad 0=\int (A_{ij;k}X^iX^j)^{;k}dv=\int (\Delta A_{ij})X^iX^j\,dv+2\int A_{ij;k}X^{i;k}X^j\,dv\,.$$

From (1.1) and (1.2) we have

$$(1.3)\quad 0=\int (A_{ij}R^i_k-\frac{1}{2}\Delta A_{jk})X^jX^k\,dv+\int A_{ij}X^{i;k}X^j_{;k}dv\,.$$

Hence we have the

THEOREM 1.1 *Let A_{ij} be any symmetric tensor such that the quadratic form*

$$(1.4)\qquad A_{ij}X^iX^j$$

is positive definite. Then if the quadratic form

$$(1.5)\qquad Q=\left(A_{ij}R^i_k-\frac{1}{2}\Delta A_{jk}\right)X^jX^k$$

is everywhere positive definite, we have

$$B_1=0.$$

If Q is everywhere positive semi-definite, we have

$$X_{i;j}=0.$$

for every harmonic vector X_i, and consequently $B_1\leqq n$.

Especially when

$$A_{ij}=\rho^2 g_{ij}\quad(\rho\neq 0), \tag{1.6}$$

we have the

THEOREM 1.2 *If the quadratic form*

$$Q'=\left(\rho^2 R_{jk}-\frac{1}{2}(\Delta\rho^2)\,g_{jk}\right)X^j X^k, \tag{1.7}$$

where ρ is a certain non zero scalar, is everywhere positive definite, we have

$$B_1=0.$$

If Q' is everywhere positive semi-definite, we have

$$X_{i;j}=0,\ \text{and}\ B_1\leqq n.$$

From (1.7) we have

$$\begin{aligned} Q'&=\left[\rho^2\left\{\frac{R}{n}g_{jk}+\left(R_{jk}-\frac{R}{n}g_{jk}\right)\right\}-\frac{1}{2}(\Delta\rho^2)g_{jk}\right]X^jX^k \\ &=\left(\frac{R}{n}\rho^2-\frac{1}{2}\Delta\rho^2\right)X_iX^i+\rho^2\left(R_{jk}-\frac{R}{n}g_{jk}\right)X^jX^k \\ &\geqq\left(\frac{R}{n}\rho^2-\frac{1}{2}\Delta\rho^2\right)X_iX^i-\rho^2\sqrt{\left(R_{jk}-\frac{R}{n}g_{jk}\right)\left(R^{jk}-\frac{R}{n}g^{jk}\right)}X_iX^i \\ &=\left(\frac{R}{n}\rho^2-\frac{1}{2}\Delta\rho^2-\rho^2\sqrt{R_{jk}R^{jk}-\frac{R^2}{n}}\right)X_iX^i. \end{aligned} \tag{1.8}$$

Hence we have the

THEOREM 1.3 *If there exists a scalar such that*

$$\frac{1}{2}\left(\frac{\Delta\rho^2}{\rho^2}\right) \leqq \frac{R}{n} - \sqrt{R_{jk}R^{jk} - \frac{R^2}{n}} \tag{1.9}$$

we have $X_{i;j}=0$ for every harmonic vector and consequently

$$B_1 \leqq n .$$

If in (1.9) *the equality does not hold, we have*

$$B_1 = 0 .$$

§ **2.**—Consider a harmonic tensor $X_{i(1)\cdots i(p)}$ and a symmetric tensor A_{ij}. We have from Green's theorem

$$\begin{aligned} 0 &= \int (A_{r\cdot}^{\cdot s} X^{i(1)\cdots i(p-1)r\,;\,k} X_{i(1)\cdots i(p-1)s})_{;\,k}\, dv \\ &= \int A_{r\cdot\,;\,k}^{\cdot s} X^{i(1)\cdots i(p-1)r\,;\,k} X_{i(1)\cdots i(p-1)s}\, dv \\ &\quad + \int A_{r\cdot}^{\cdot s} (\Delta X^{i(1)\cdots i(p-1)r}) X_{i(1)\cdots i(p-1)s}\, dv \\ &\quad + \int A_{r\cdot}^{\cdot s} X^{i(1)\cdots i(p-1)r\,;\,k} X_{i(1)\cdots i(p-1)s\,;\,k}\, dv , \end{aligned} \tag{2.1}$$

$$\begin{aligned} 0 &= \int (A_{rs\,;\,k} X^{i(1)\cdots i(p-1)r} X_{i(1)\cdots i(p-1)}{}^{s})^{;\,k}\, dv \\ &= \int (\Delta A_{rs}) X^{i(1)\cdots i(p-1)r} X_{i(1)\cdots i(p-1)\cdot}^{\quad\quad\quad\quad s}\, dv \\ &\quad + 2\int A_{rs\,;\,k} X^{i(1)\cdots i(p-1)r\,;\,k} X_{i(1)\cdots i(p-1)\cdot}^{\quad\quad\quad\quad s}\, dv . \end{aligned} \tag{2.2}$$

From (2.1) and (2.2) we have

$$\begin{aligned} 0 &= \int A_{rs} (\Delta X^{i(1)\cdots i(p-1)r}) X_{i(1)\cdots i(p-1)\cdot}^{\quad\quad\quad\quad s}\, dv \\ &\quad - \frac{1}{2}\int (\Delta A_{rs}) X^{i(1)\cdots i(p-1)r} X_{i(1)\cdots i(p-1)\cdot}^{\quad\quad\quad\quad s}\, dv \\ &\quad + \int A_{r\cdot}^{\cdot s} X^{i(1)\cdots i(p-1)r\,;\,k} X_{i(1)\cdots i(p-1)s\,;\,k}\, dv \\ &= \int \Big\{ (p-1)(p-2) A_{ad} R_{becf} + 2(p-1) g_{ad} R_{embf} A^{m}_{\cdot c} \end{aligned} \tag{2.3}$$

$$+(p-1)g_{ad}R_{be}A_{cf}+g_{ad}g_{be}R_{cm}A^{m}_{f}-\frac{1}{2}g_{ad}g_{be}\Delta A_{cf}\Big\}$$

$$X^{i(1)\cdots i(p-3)abc}X_{i(1)\cdots i(p-3)}{}^{def}{}_{\cdots\cdots\cdots}\,dv$$

$$+\int A_{r}{}^{s}_{\cdot}X^{i(1)\cdots i(p-1)r\,;\,k}X_{i(1)\cdots i(p-1)s\,;\,k}\,dv\,.$$

Hence we have the

THEOREM 2.1 *Let A_{ij} be any symmetric tensor such that the quadratic form*

$$A_{ij}f^{i}f^{j}$$

is positive definite. If the quadratic form

$$(2.4)\quad Q''=\Big\{(p-1)(p-2)A_{ad}R_{becf}+2(p-1)g_{ad}R_{embf}A^{m}_{c}$$
$$+(p-1)g_{ad}R_{be}A_{cf}+g_{ad}g_{be}R_{cm}A^{m}_{f}-\frac{1}{2}g_{ad}g_{be}\Delta A_{cf}\Big\}X^{abc}X^{def}\,,$$

where X^{abc} is any skew-symmetric tensor, is everywhene positive semi-definite, we have

$$X_{i(1)\cdots i(p)\,;\,r}=0\,,$$

for every harmonic tensor $X_{i(1)\cdots i(p)}$, and consequently $B_p\leqq\binom{n}{p}$.

If Q'' is everywhere positive definite, we have

$$B_p=0\,.$$

When $A_{ij}=\rho^2 g_{ij}\ (\rho\neq 0)$ we have the

THEOREM 2.2 *If there exists a scalar ρ such that the quadratic form*

$$(2{,}5)\quad Q'''=\Big[\rho^2\Big\{\frac{p(p-1)}{2}R_{abcd}+pg_{ac}R_{bd}\Big\}-\frac{1}{2}(\Delta\rho^2)g_{ac}g_{bd}\Big]f^{ab}f^{cd}\quad(f^{ab}=-f^{ba})$$

is everywhere positive semi-definite, we have

$$X_{i(1)\cdots i(p)\,;\,r}=0$$

for every harmonic tensor $X_{i(1)\cdots i(p)}$, *and consequently* $B_p \leqq \binom{n}{p}$.
If Q''' *is everywhere positive definite, we have*

$$B_p = 0 .$$

From (2.5) we have

$$(2.6) \quad Q''' \geqq \Big[p\rho^2 \Big\{ \frac{n-p}{n(n-1)} R - \sqrt{\left(\frac{p-1}{2}\right)^2 R_{ijkl}R^{ijkl} + \frac{n-4p+2}{4} R_{ij}R^{ij} + \left\{\frac{1}{4} - \frac{(n-p)^2}{2n(n-1)}\right\} R^2 } \Big\} - \frac{1}{2} \Delta\rho^2 \Big] f_{ab} f^{ab} .$$

Hence we have the

THEOREM 2.3 *If there exists a scalar such that*

$$(2.7) \quad \frac{1}{2}\left(\frac{\Delta\rho^2}{\rho^2}\right) \leqq p \Big\{ \frac{n-p}{n(n-1)} R - \sqrt{\left(\frac{p-1}{2}\right)^2 R_{ijkl}R^{ijkl} + \frac{n-4p+2}{4} R_{ij}R^{ij} + \left\{\frac{1}{4} - \frac{(n-p)^2}{2n(n-1)}\right\} R^2 } \Big\}$$

we have

$$X_{i(1)\cdots i(p);\, r} = 0$$

for every harmonic vector $X_{i(1)\cdots i(p)}$, *and consequently*

$$B_p \leqq \binom{n}{p} ,$$

If in (2.7) *the equality does not hold, we have*

$$B_p = 0 .$$

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