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Note on Betti numbers of Riemannian manifolds I.

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In this paper, we give some applications of a theorem of Bochner—Lichnerowicz on the Betti numbers of a Riemannian manifold. We consider a Riemannian manifold R_n whose fundamental tensor g_{ij} is positive definite and assume that R_n is compact and orientable.

THEOREM I. (BOCHNER-LICHNEROWICZ)

In R_n, if the quadratic form

$$\left(\frac{p-1}{2}R_{ijkl}+R_{ik}g_{jl}\right)f^{ij}f_{kl} \quad (f^{ij}=-f^{ji}) \tag{1}$$

is everywhere positive semi-definite, then, for any harmonic tensor $X_{i(1)\cdots i(p)}$ of degree p, it holds that

$$X_{i(1)\cdots i(p);r}=0\,,$$

and hence we have

$$B_p \leqq \binom{n}{p}$$

where B_p denotes the p-th Betti number and $p \geqq 2$.

When $p=1$, the quadratic form (1) *can be replaced by*

$$R_{ij}f^i f^j\,, \tag{2}$$

and if this form is everywhere positive semi-definite, then the covariant derivative of any harmonic vector vanishes, and hence we have

$$B_1 \leqq n\,.$$

If the quadratic form (1) *or* (2) *is everywhere positive definite, then the harmonic vector or tensor should be identically zero, and hence we have*

$$B_p=0 \text{ or } B_1=0\,.$$

We put

$$R_{ij}=\frac{1}{n}Rg_{ij}+S_{ij}. \tag{3}$$

If S_{ij} is zero, our manifold is an Einstein space.
The quadratic form (2) becomes

$$Q=R_{ij}f^i f^j=\frac{1}{n}Rf_i f^j+S_{ij}f^i f^j. \tag{4}$$

Let t be any real number, then we have

$$(S_{ij}+tf_i f_j)(S^{ij}+tf^i f^j)\geqq 0, \tag{5}$$

that is

$$t^2(f_i f^i)^2+2tS_{ij}f^i f^j+S_{ij}S^{ij}\geqq 0. \tag{6}$$

Hence we have

$$(S_{ij}f^i f^j)^2\leqq(f_a f^a)^2 S_{ij}S^{ij}. \tag{7}$$

From (4) and (7) we have

$$\begin{aligned} Q&\geqq\frac{1}{n}Rf_i f^i-\sqrt{S_{ij}S^{ij}}f_a f^a \\ &=\left(\frac{R}{n}-\sqrt{S_{ab}S^{ab}}\right)f_i f^i=\left(\frac{R}{n}-\sqrt{R_{ab}R^{ab}-\frac{R^2}{n}}\right)f_i f^i. \end{aligned} \tag{8}$$

Hence we have the

THEOREM 2. *Let*

$$Q=\frac{R}{n}-\sqrt{R_{ij}R^{ij}-\frac{R^2}{n}}.$$

If Q is everywhere non-negative, the covariant derivative of any harmonic vector should vanish and hence, $B_1\leqq n$. If Q is everywhere positive, the first Betti number B_1 is zero.

Next we consider the case where $p>1$. We put

$$R_{ijkl}=\frac{R}{n(n-1)}(g_{jk}g_{il}-g_{jl}g_{ik})+S_{ijkl}. \tag{9}$$

If S_{ijkl} is zero, our manifold becomes the space of constant curvature.

It follows from (9) that

(10) $$R_{jk}=\frac{R}{n}g_{jk}+S_{jk},$$

where

(11) $$S_{jk}=S^{i}{}_{jki}.$$

The quadratic form (1) becomes

(12) $$Q'=\frac{p-1}{2}\left\{\frac{R}{n(n-1)}(g_{jk}g_{il}-g_{jl}g_{ik})+S_{ijkl}\right\}X^{ij}X^{kl}$$

$$+\left(\frac{R}{n}g_{ik}+S_{ik}\right)g_{jl}X^{ij}X^{kl}$$

$$=\frac{R(n-p)}{n(n-1)}X_{ij}X^{ij}+\left(\frac{p-1}{2}S_{ijkl}+S_{ik}g_{jl}\right)X^{ij}X^{kl}$$

$$=\frac{R(n-p)}{n(n-1)}X_{ij}X^{ij}$$

$$+\left[\frac{p-1}{2}S_{ijkl}+\frac{1}{4}(S_{ik}g_{jl}-S_{jk}g_{il}+S_{jl}g_{ik}-S_{il}g_{jk})\right]X^{ij}X^{kl}$$

$$=\frac{R(n-p)}{n(n-1)}X_{ij}X^{ij}+M_{ijkl}X^{ij}X^{kl},$$

where

(13) $$M_{ijkl}=\frac{p-1}{2}S_{ijkl}+\frac{1}{4}(S_{ik}g_{jl}-S_{jk}g_{il}+S_{jl}g_{ik}-S_{il}g_{jk})$$

$$=\frac{p-1}{2}R_{ijkl}+\frac{1}{4}(R_{ik}g_{jl}-R_{jk}g_{il}+R_{jl}g_{ik}-R_{il}g_{jk})$$

$$+\frac{R(n-p)}{2n(n-1)}(g_{jk}g_{il}-g_{ik}g_{jl}).$$

On the other hand, we have, for an arbitrary real number t,

(14) $$(M_{ijkl}+t\,X_{ij}X_{kl})(M^{ijkl}+t\,X^{ij}X^{kl})$$

$$=t^2(X_{ij}X^{ij})^2+2tM_{ijkl}X^{ij}X^{kl}+M_{ijkl}M^{ijkl}\geqq 0.$$

Hence we have

$$(M_{ijkl}X^{ij}X^{kl})^2\leqq M_{ijkl}M^{ijkl}(X_{ab}X^{ab})^2, \tag{15}$$

that is

$$|M_{ijkl}X^{ij}X^{kl}|\leqq\sqrt{M_{ijkl}M^{ijkl}}\,X_{ab}X^{ab}. \tag{16}$$

From (12) and (16) we have

$$Q'\geqq\frac{R(n-p)}{n(n-1)}X_{ij}X^{ij}-\sqrt{M_{ijkl}M^{ijkl}}\,X_{ab}X^{ab}. \tag{17}$$

Since

$$M_{ijkl}M^{ijkl}=\frac{(p-1)^2}{4}R_{ijkl}R^{ijkl}+\frac{n-4p+2}{4}R_{ij}R^{ij}+\left(\frac{1}{4}-\frac{(n-p)^2}{2n(n-1)}\right)R^2 \tag{18}$$

we have

THEOREM 3. *Let*

$$T=\frac{n-p}{n(n-1)}R$$

$$-\sqrt{\frac{(p-1)^2}{4}R_{ijkl}R^{ijkl}+\frac{n-4p+2}{4}R_{ij}R^{ij}+\left(\frac{1}{4}-\frac{(n-p)^2}{2n(n-1)}\right)R^2}\quad(p\geqq 2).$$

If T is everywhere non-negative, the covariant derivative of any harmonic tensor of degree p should vanish, and hence

$$B_p\leqq\binom{n}{p}.$$

If T is everywhere positive, then

$$B_p=0.$$

Next we have

$$Q' = R_{ik}\, g_{jl}\, X^{ij}\, X^{kl} + \frac{p-1}{2} R_{ijkl}\, X^{ij}\, X^{kl} \tag{19}$$

$$\geqq R_{ik}\, g_{jl}\, X^{ij}\, X^{kl} - \frac{p-1}{2} \sqrt{R_{ijkl}\, R^{ijkl}}\, X_{ab}\, X^{ab}$$

$$= \left(\frac{R_{ik}\, g_{jl}\, X^{ij}\, X^{kl}}{X_{ab}\, X^{ab}} - \frac{p-1}{2} \sqrt{R_{ijkl}\, R^{ijkl}} \right) X_{ab}\, X^{ab}.$$

Hence we have

THEOREM 4. *If*

$$\frac{R_{ik}\, g_{jl}\, X^{ij}\, X^{kl}}{X_{ab}\, X^{ab}} \geqq \frac{p-1}{2} \sqrt{R_{ijkl}\, R^{ijkl}} \quad (X_{ij} = -X_{ji})$$

holds everywhere, we have the same result as in Theorem 3. If the inequality sign holds everywhere, then B_p is zero.

Moreover we have

$$Q' = \frac{p-1}{2} R_{ijkl}\, X^{ij}\, X^{kl} + \frac{1}{4} (R_{ik} g_{jl} - R_{jk} g_{il} - R_{il} g_{jk} + R_{jl} g_{ik}) X^{ij} X^{kl} \tag{20}$$

$$\geqq \frac{p-1}{2} R_{ijkl}\, X^{ij}\, X^{kl} - \sqrt{Q_{ijkl}\, Q^{ijkl}}\, X_{ab}\, X^{ab},$$

where

$$Q_{ijkl} = \frac{1}{4} (R_{ik}\, g_{jl} - R_{jk}\, g_{il} - R_{il}\, g_{jk} + R_{jl}\, g_{ik}). \tag{21}$$

Since

$$Q_{ijkl}\, Q^{ijkl} = \frac{(n-2)\, R_{ij}\, R^{ij} + R^2}{4}, \tag{20}$$

we have

THEOREM 5. *If the curvature tensor satisfies the inequality*

$$\frac{R_{ijkl}\, X^{ij}\, X^{kl}}{X_{ab}\, X^{ab}} \geqq \frac{\sqrt{(n-2)\, R_{ij}\, R^{ij} + R^2}}{p-1},$$

then we have the same result as in Theorem 3. *If the inequality sign holds everywhere, then B_p is zero.*

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