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Symbolic methods in the problem of three-line Latin rectangles.

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1. Introduction. In previous papers [1], [2] the author gave an asymptotic series for the number $f(3, n)$ of $3\times n$ Latin rectangles:

$$f(3, n)\sim e^{-3}(n\,!)^{3}\sum_{i=0}^{n}H_{i}\left(-\frac{1}{2}\right)/i\,!\,(n)_{i}$$

where $H_{i}(x)$ is Hermite polynomial and $(n)_{i}=n\,!/(n-i)\,!$ is Jordan factorial. The method of [2] was rather complicated, and the author has attempted in [1] (written actually after [2]) to clarify derivation of the asymptotic series. The present paper will make a slight improvement of [1].

One of the most powerful instruments in such a problem would be the *symbolic method*, recently recognized in its full power by Touchard, Fréchet, Kaplansky and others. This method also clarifies results on the number $f(3, n)$, and our Theorem 1 states $f(3, n)$ as a polynomial in the shift operator (to the left) applying on partial sums for e^{-3}. This explicit form, though complicated in itself, is undoubtedly the most reasonable for the asymptotic expansion (Theorem 2).

Mr. John Riordan of Bell Telephone Laboratories kindly communicated to the author an application of [1], obtaining a very neat recursive formula for the number $f(3, n)$ (*Memorandum*, Sept. 8, 1950). This would be the simplest recursion one can expect, and it is a great pleasure for the author to include it in this paper (Theorem 3).

Our method strongly suggests an extension to the general Latin rectangles. The first and the chief obstacle, however, lies in establishing analogue of Theorem 1, and this would have to be overcome only through elaborate inductions.

Several symbolisms are used in this paper to simplify description. One is the symbolic representation of sieve process (or Poincaré's formula in the theory of probability), most important in combinatorial analysis, another is that of the so-called Blissard umbra for sequences (in our case for Jordan powers), and shift operator may be regarded as one such.

2. Symbolic representation of sieve process. Let S be a field of subsets of Ω and m be a completely additive abstract measure defined on S, assuming measure values in a commutative topological group (written in additive form). Denoting intersection of subsets by product, and complementation by accent, we have for A's in S

$$mA'=m-mA=m(\Omega-A),\quad m=m\,\Omega,$$

$$mA_1'A_2'=m-mA_1-mA_2+mA_1A_2=m(\Omega-A_1)(\Omega-A_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$mA_1'A_2'\cdots A_r'=m-\sum mA_i+\sum mA_iA_j-\sum mA_iA_jA_k+\cdots$$
$$+(-)^r mA_1A_2\cdots A_r=m(\Omega-A_1)(\Omega-A_2)\cdots(\Omega-A_r).$$

The symbolism should be understood in an obvious way: in order to evaluate $m(\Omega-A_1)\cdots(\Omega-A_r)$, we first develop the *set part* as if it were an element of an ordinary (*not* Boolean) algebra with Ω as neutral element and with A's as idempotent bases, and then operate m term by term. The function m is additive on this algebra.

This is a symbolic representation of the sieve process and of Poincaré's formula in the thery of probability. We shall not enumerate its various applications in algebra and number-theory, but content ourselves with one example: the Möbius inversion formula of number-theoretical summatorial functions.

3. Latin rectangles. Consider the totality Ω of $k\times n$ "rectangles", i.e. $k\times n$ matrices whose k rows are permutations of integers $1,2,\cdots,n$. Let $S=2^\Omega$ and let mA denote the number of elements in A. Denote now by $A_p(u,v)$ the set of "rectangles", whose (p,u)-element is the integer v, and define $A_{pq\ldots}(u,v)=A_p(u,v)\,A_q(u,v)\cdots$. Then Latin rectangles constitute the subset

$$L=\prod_{u,v}\prod_{(p,q)}(\Omega-A_{pq}(u,v)) \tag{1}$$

and $f(k,n)=mL$ is the number of Latin rectangles. Note that if $u\neq u',\ v\neq v'$

(2) $$A_{pq}(u, v)A_{pq'}(u', v) = A_{pq}(u,v)A_{pq'}(u, v') = \phi = \text{empty subset}.$$

4. The case $k=3$. We consider the special case of $k=3$ in the rest of the present paper. Developing inner product in (1) first, we find

$$\prod_{(p,q)}(\Omega - A_{pq}) = \Omega - (A_{12} + A_{13} + A_{23}) + 2A_{123} = \Omega - B + 2C,$$

if we define *symbolic subsets*

(3) $$B = A_{12} + A_{13} + A_{23}, \quad C = A_{123}.$$

Making use of these, (1) is written as

(4) $$\begin{aligned} L &= \prod_{u,v}(\Omega - B(u, v) + 2C(u, v)) \\ &= \prod_{u,v}(\Omega - B(u, v)) + 2\sum C(w_1, z_1)\prod_{u,v}(\Omega - B(u, v)) + \cdots \\ &\quad + 2^i \sum C(w_1, z_1)\cdots C(w_i, z_i)\prod_{u,v}(\Omega - B(u, v)) + \cdots. \end{aligned}$$

In the i-th summand the product may be taken over all u, v such that $u \neq w_1, \cdots, \neq w_i$ and that $v \neq z_1, \cdots, \neq z_i$, since if $v = z_j$ then $C(w_j, z_j) B(w_j, v) = \phi$, i.e., $C(w_j, z_j)(\Omega - B(w_j, v)) = C(w_j, z_j)$ according to (2) and (3), and we may spare the factor $\Omega - B(w_j, v)$. The same holds for factors with $u \neq w_j, v = z_j$. Then the summation in this i-th summand is over $(n)_i^2/i!$ *combinations* $(w_1, z_1), \cdots, (w_i, z_i)$ such that all w's are $\neq$ and that all z's are $\neq$, since other terms are *suppressed* at the restitution m, by the same reason.

Let us define the subset

$$\tilde{L} = \prod_{u,v}(\Omega - B(u, v))$$

and put $m\tilde{L} = \lambda = \lambda_n$. We assert that

(5) $$mC(w_1, z_1)\cdots C(w_i, z_i)\prod_{u,v}(\Omega - B(u, v)) = \lambda_{n-i}$$

for any one of the $(n)_i^2/i!$ *permissible combinations* $(w_1, z_1), \cdots, (w_i, z_i)$. Indeed $\prod C(w_j, z_j)$ consists of "rectangles" whose w_j-th columns are filled up with the integer z_j for $j = 1, \cdots, i$. If we delete these i columns, we would have $3 \times (n-i)$ "rectangles" in the integers different from $z_1, \cdots, z_i$. Since in the ensuing product neither $u = w_j$ nor $v = z_j$, this product bears the same meaning as $\tilde{L}$ for the case of $3 \times (n-i)$

"rectangles". We thus have proved

$$mL=\sum_{i=0}^{n}\frac{2^i(n)_i^2}{i!}\lambda_{n-i},$$

or if we write

$$\phi_n=f(3, n)/(n!)^3, \quad \psi_n=\lambda_n/(n!)^3$$

the relation

$$\phi_n=\sum_{i=0}^{n}\frac{2^i\psi_{n-i}}{i!(n)_i}=\sum_{i=0}^{n}\frac{(2F)^i}{i!(n)_i}\psi_n. \tag{6}$$

5. In the present paper throughout the shift operator F will be applied only to sequences denoted by small Greeks, ex. gr., α_n $(n=0, 1, \cdots)$, and *diminish* argument by 1; ex. gr., $F\alpha_n=\alpha_{n-1}$ $(n\geqq 1)$, $F\alpha_0=0$. This operator may be considered as an inverse of the usual shift operator E such that $E\alpha_n=\alpha_{n+1}$.

6. Now let us proceed to evaluate $\lambda=m\tilde{L}$. Developing the set portion we have

$$\tilde{L}=\sum_{k=0}^{n}(-)^k\sum B(u_1, v_1)\cdots B(u_k, v_k).$$

Here the inner summation may be restricted over $(n)_k^2/k!$ *combinations* of k pairs $(u_1, v_1), \cdots, (u_k, v_k)$ such that u's are all $\neq$ and that v's are all $\neq$. Other terms are again to be suppressed as before. We assert again that any of these $(n)_k^2/k!$ *permissible symbolic subsets* $B(u_1, v_1)\cdots B(u_k, v_k)$ has the same measure. Indeed such a symbolic subset is the sum of 3^k subsets substituting each factor B by A_{12}, A_{13} or A_{23}. Suppose a, b and c of the B's are replaced by A_{12}, A_{13} and A_{23} respectively $(a+b+c=k)$. Then, since the measure of this replaced term is

$$(n-a-b)!(n-a-c)!(n-b-c)!=(n!)^3/(n)_{a+b}(n)_{a+c}(n)_{b+c},$$

the measure of symbolic subset $B(u_1, v_1)\cdots B(u_k, v_k)$ is

$$\sum_{a+b+c=k}\frac{k!}{a!b!c!}\frac{(n!)^3}{(n)_{a+b}(n)_{a+c}(n)_{b+c}}.$$

We have found that

$$\lambda_n=\sum_{k=0}^{n}(-)^k\frac{(n)_k^2}{k!}\sum_{a+b+c=k}\frac{k!}{a!\,b!\,c!}\,\frac{(n!)^3}{(n)_{a+b}(n)_{a+c}(n)_{b+c}},$$

or

$$\psi_n=\sum_{k=0}^{n}\frac{1}{(n)_k}\sum_{a+b+c=k}\binom{-(n-k+1)}{a}\binom{-(n-k+1)}{b}\binom{-(n-k+1)}{c} \tag{7}$$

i.e.,

$$\psi_n=\sum_{k=0}^{n}\frac{1}{(n)_k}\binom{-3(n-k+1)}{k}. \tag{8}$$

The formulas (6) and (8) express the number ϕ_n in terms of known functions. In [1] the author has developed (8) in negative Jordan powers, but since this short cut does not seem to clarify the matters enough, we prefer in this paper to stop at (7) and develop it into negative Jordan powers. This method seems to explain best the intrusion of Hermite polynomials in our problem.

7. The next step is to develop (7) in *negative Jordan powers* $1/i!\,(n)_i$ of n. And this is carried out by expanding

$$(-(n-k+1))_a\,(-(n-k+1))_b\,(-(n-k+1))_c$$

in terms of $(-(n-k+1))_j$. Thus we are led to define symmetric functions P_i of a, b, c by

$$(x)_a(x)_b(x)_c=\sum_{i=0}^{k}\frac{P_i(a,b,c)}{i!}(x)_{k-i} \qquad (k=a+b+c). \tag{9}$$

Then (7) is transformed as follows:

$$\begin{aligned}\psi_n&=\sum_{k=0}^{n}\sum_{i=0}^{k}\frac{(-(n-k+1))_{k-i}}{(n)_k\,i!}\sum_{a+b+c=k}\frac{P_i(a,b,c)}{a!\,b!\,c!}\\&=\sum_{i=0}^{n}(-)^i\frac{1}{i!\,(n)_i}\sum_{k=i}^{n}(-)^k\sum_{a+b+c=k}\frac{P_i(a,b,c)}{a!\,b!\,c!}.\end{aligned} \tag{10}$$

In order to find P_i in (9), we make use of the *Vandermonde binomial theorem* for *Jordan powers*

$$(x+y)_a=\sum_{j=0}^{a}\binom{a}{j}(x)_j\,(y)_{a-j}$$

or

$$(x+a)_b=\sum_{j=0}^{a,b}\frac{(a)_j\,(b)_j}{j\,!}\,(x)_{b-j}$$

the upper end of summation being min. (a, b) for integral values of a and b. Now we find

$$(x)_a(x)_b=(x)_a\,((x-a)+a)_b=\sum_{j=0}^{a,b}\frac{(a)_j\,(b)_j}{j\,!}\,(x)_{a+b-j}\,,$$

$$(x)_a(x)_b(x)_c=\sum_{j=0}^{a,b}\frac{(a)_j\,(b)_j}{j\,!}\sum_{h=0}^{a+b-j,c}\frac{(a+b-j)_h\,(c)_h}{h\,!}\,(x)_{k-j-h}\,.$$

Hence P_i in (9) is given by

$$P_i=\sum_{j=0}^{i}\binom{i}{j}(a)_j\,(b)_j\,(a+b-j)_{i-j}\,(c)_{i-j}\,.$$

Expanding again $(a+b-j)_{i-j}=((a-j)+(b-j)+j)_{i-j}$ according to the *Vandermonde multinomial theorem* we find

$$P_i=\sum_{j=0}^{i}\frac{i\,!}{j\,!\,(i-j)\,!}\sum_{r+s+t=i-j}\frac{(i-j)\,!}{r\,!\,s\,!\,t\,!}\,(a)_{j+r}\,(b)_{j+s}\,(c)_{i-j}\,(j)_t\,.$$

Noting that P_i is symmetric *polynomial* in a, b, and c we write for the moment usual power products $a^p b^q c^r$ instead of Jordan power products $(a)_p(b)_q(c)_r$ (method of *Blissard umbra*). Then

$$P_i=\sum_{j=0}^{i}\sum_{t+u=i-j}\frac{i\,!}{(j-t)!\,t!\,u!}\,c^{t+u}\sum_{u=r+s}\frac{u!}{r!\,s!}\,a^r\,b^s\,(ab)^j$$

$$=\sum_{w+2t=i}\frac{i\,!}{t\,!\,w\,!}\sum_{u+j-t=w}\frac{w\,!}{u\,!\,(j-t)\,!}(a+b)^u c^u\cdot(ab)^{j-t}\,(abc)^t$$

$$=\sum_{j=0}^{[i/2]}\frac{(i)_{2j}}{j\,!}\,(ab+ac+bc)^{i-2j}\,(abc)^j$$

$$=\overline{H}_i\,(-(ab+ac+bc),-abc)\,,$$

if we define *modified Hermite polynomials*

$$\overline{H}_i\,(\xi\,,\eta)=(-)^i\sum_{j=0}^{[i/2]}(-)^j\,\frac{(i)_{2j}}{j\,!}\,\xi^{i-2j}\eta^j\,.$$

8. These modified Hermite polynomials are connected with the usual Hermite polynomials $H_i(x)$ by the relations

(11) $$\overline{H}_i(\xi,\eta)=\sqrt{\eta}^{\,i}\,H_i(-\xi/2\sqrt{\eta}),$$

(12) $$H_i(x)=\overline{H}_i(-2x,\ 1).$$

These polynomials have the exponential generating function

(13) $$\exp.\ \overline{H}t=\sum_{i=0}^{\infty}\frac{\overline{H}_i(\xi,\eta)}{i\,!}\,t^i=e^{-\xi t-\eta t^2},$$

and satisfy the recurrence relation

(14) $$\overline{H}_i(\xi,\eta)=-\xi\overline{H}_{i-1}(\xi,\eta)-2(i-1)\,\eta\,\overline{H}_{i-2}(\xi,\eta)\qquad(i\geqq 2),$$
$$\overline{H}_0(\xi,\eta)=1,\quad \overline{H}_1(\xi,\eta)=-\xi.$$

They have also the reproducible property

(15) $$\sum_{j=0}^{i}(-)^{i-j}\binom{i}{j}\overline{H}_j(\xi,\eta)\,\zeta^{i-j}=\overline{H}_i(\xi+\zeta,\eta),$$

and the homogeneity property

(16) $$\overline{H}_i(t\xi,\ t^2\eta)=t^i\,\overline{H}_i(\xi,\eta).$$

9. Now let us return to ψ_n, (10). Consider the generic term $(a)_p\,(b)_q\,(c)_r$ of P_i instead of P_i itself. Then

$$\sum_{k=i}^{n}(-)^k\sum_{a+b+c=k}\frac{(a)_p\,(b)_q\,(c)_r}{a\,!\,b\,!\,c\,!}$$
$$=\sum_{k=p+q+r}^{n}(-)^k\sum_{a+b+c=k}\frac{1}{(a-p)\,!\,(b-q)\,!\,(c-r)\,!}$$
$$=\sum_{k=p+q+r}^{n}(-)^k\frac{3^{k-(p+q+r)}}{(k-(p+q+r))\,!}$$
$$=(-)^{p+q+r}\sigma_{n-(p+q+r)}=(-F)^{p+q+r}\sigma_n,$$

with $\sigma_n=\sum_{u=0}^{n}(-3)^u/u\,!$, since in the inner sum of the first member we have only to sum those terms such as $a\geqq p$, $b\geqq q$ and $c\geqq r$; and hence in particular we may restrict the summation range of k to

$p+q+r \geqq k \geqq n.$

Thus the generic power product $a^p b^q c^r$ in the *symbolic* representation of P_i contributes $(-F)^{p+q+r}\sigma_n$ to ψ_n in (10). We know on the other hand that the symbolic representation of P_i is the modified Hermite polynomial $\overline{H}_i(-(ab+ac+bc), -abc)$. Hence we may write

$$\psi_n=\sum_{i=0}^{n}(-)^i \frac{\overline{H}_i(-3F^2, F^3)}{i!(n)_i}\sigma_n=\sum_{i=0}^{n}\frac{\overline{H}_i(3F^2, F^3)}{i!(n)_i}\sigma_n .$$

by (16), and we may write ϕ_n by (6), (15), (16) and (11),

$$\phi_n=\sum_{i=0}^{n}\frac{2^i}{i!(n)_i}\sum_{j=0}^{n-i}\frac{\overline{H}_j(3F^2, F^3)}{j!(n-i)_j}F^i\sigma_n$$

$$=\sum_{i=0}^{n}\frac{1}{i!(n)_i}\sum_{j=0}^{i}\binom{i}{j}\overline{H}_j(3F^2, F^3)(2F)^{i-j}\sigma_n$$

$$=\sum_{i=0}^{n}\frac{\overline{H}_i(-2F+3F^2, F^3)}{i!(n)_i}\sigma_n$$

$$=\sum_{i=0}^{n}\frac{\overline{H}_i(-2+3F, F)F^i}{i!(n)_i}\sigma_n .$$

We have proved

THEOREM 1. *The number $f(3, n)$ of three-line Latin rectangles is given by*

$$f(3, n)=(n!)^3\sum_{i=0}^{n}\frac{\sqrt{F}^{3i}H_i((2-3F)/2\sqrt{F})}{i!(n)_i}\sigma_n$$

where σ_n is the partial sum of order n for e^{-3}, F is a shift operator to the left, and $H_i(x)$ is Hermite polynomial.

Since $F^i\sigma_n \rightarrow e^{-3}$ as $n \rightarrow \infty$ we obtain

THEOREM 2. *The number $f(3, n)$ is asymptotically given by*

$$e^{-3}(n!)^3\sum_{i=0}^{n}\frac{H_i\left(-\frac{1}{2}\right)}{i!(n)_i} .$$

10. John Riordan has derived a new recursive formula for the number $f(3, n)$ from our results. We state it here in our terminology.

THEOREM 3 (RIORDAN). *The numbers ϕ_n satisfy the recursive relation*

$$\phi_n=\phi_{n-1}+\frac{1}{(n)_2}\phi_{n-2}+\frac{2}{(n)_3}\phi_{n-3}+\eta_n \qquad (n\geqq 3),$$

$$\phi_0=1,\quad \phi_1=\phi_2=0,$$

where

$$\zeta_n=n!\,\eta_n=(-)^n\,3\sum_{u=0}^{n}(-2)^u/u!\,-2^{n+1}/n!.$$

The auxiliary numbers ζ_n satisfy the recursive formula

$$\zeta_n+\zeta_{n-1}+\frac{2^n(n-1)}{n!}=0 \qquad (n\geqq 1),$$

$$\zeta_0=1.$$

11. PROOF. Let us start from the recurrence for the negative Jordan powers

$$\frac{1}{i!\,(n)_i}=\frac{1}{i!\,(n-1)_i}-\frac{1}{(n)_2}\,\frac{1}{(i-1)!\,(n-2)_{i-1}} \qquad (n-1\geqq i\geqq 1)$$

and the recurrence (14) for $\overline{H}_i(-2+3F, F)$, which may be written

$$\overline{H}_i=(2-3F)\,\overline{H}_{i-1}-2\,(i-1)\,F\overline{H}_{i-2} \qquad (i\geqq 2).$$

It is convenient to define $\alpha_n=(1-F)\,\sigma_n=\sigma_n-\sigma_{n-1}=(-3)^n/n!$, and to put

$$(17) \qquad \rho_n=\sum_{i=0}^{n}\frac{\overline{H}_iF^i}{i!\,(n)_i}\,\alpha_n=\sum_{i=0}^{n}\frac{\overline{H}_i\,F^i\,(1-F)}{i!\,(n)_i}\,\sigma_n\,,$$

$$\xi_n=\sum_{i=0}^{n}\frac{\overline{H}_{i+1}F^i}{i!\,(n)_i}\,\sigma_n\,.$$

Then we deduce from the two recurrence relations that

$$\phi_n=\phi_{n-1}+\rho_n-\frac{1}{(n)_2}\,\xi_n\,.$$

In the same manner we can rewrite ξ_n as

$$\xi_n = 2\,\phi_n - 3\,(\phi_n - \rho_n) - \frac{2}{n}(\phi_{n-1} - \rho_{n-1})$$

$$= -\phi_n - \frac{2}{n}\phi_{n-1} + 3\,\rho_n + \frac{2}{n}\rho_{n-1}\,.$$

Combining these two we obtain for $n \geqq 3$,

(18) $$\phi_n = \phi_{n-1} + \frac{1}{(n)_2}\phi_{n-2} + \frac{2}{(n)_3}\phi_{n-3} + \left(\rho_n - \frac{3}{(n)_2}\rho_{n-2} - \frac{2}{(n)_3}\rho_{n-3}\right).$$

It is natural to introduce

$$\eta_n = \rho_n - \frac{3}{(n)_2}\rho_{n-2} - \frac{2}{(n)_3}\rho_{n-3} \qquad (n \geqq 3),$$

$$\eta_0 = \rho_0,\quad \eta_1 = \rho_1,\quad \eta_2 = \rho_2 - \frac{3}{2}\rho_0\,,$$

and

$$\zeta_n = n\,!\,\eta_n\,,\qquad \omega_n = n\,!\,\rho_n\,.$$

Since

$$\zeta_n = \omega_n - 3\omega_{n-2} - 2\,\omega_{n-3}\,,$$

the generating functions $Z(t) = \sum_0^\infty \zeta_n t^n$, and $G(t) = \sum_0^\infty \omega_n t^n$ of ζ's and of ω's are linked by

(19) $$Z(t) = (1 - 3t^2 - 2t^3)\,G(t)\,,$$

and these generating functions converge for $|t| < 1/3$, as seen from $\omega_n \sim (-3)^n$.

12. Now $G(t)$ is readily evaluated. Consider its generic part

$$\sum_0^\infty n\,!\,t^n \frac{F^k F^i}{(n)_i}\alpha_n\,.$$

If we put $\beta = -3$ for convenience this is transformed as follows:

$$= \sum_{n-i+k}^{\infty} (n-i)\,!\,t^n \frac{\beta^{n-i-k}}{(n-i-k)\,!}$$

$$= \sum_{n-i}^{\infty} (n-i)_k\, t^n\, \beta^{n-i-k}$$

$$=t^i\left(\frac{\partial}{\partial\beta}\right)^k \sum_{n=i}^{\infty}\beta^{n-i}t^{n-i}=t^i\left(\frac{\partial}{\partial\beta}\right)^k(1-\beta t)^{-1}.$$

Multiplying by suitable factors and summing over k we find that the i-th term $\overline{H}_i(-2+3F, F)F^i\alpha_n/i!(n)_i$ of (17) contributes to $G(t)$ the fraction $(t^i/i!)\overline{H}_i\left(-2+3\frac{\partial}{\partial\beta}, \frac{\partial}{\partial\beta}\right)\cdot(1-\beta t)^{-1}$. Again summing over i and using the relation (13) we finally obtain

$$G(t)=\exp.\ t\overline{H}\left(-2+3\frac{\partial}{\partial\beta}, \frac{\partial}{\partial\beta}\right)\cdot(1-\beta t)^{-1}$$

$$=e^{2t}\exp.\left(-(3t+t^2)\frac{\partial}{\partial\beta}\right)\cdot(1-\beta t)^{-1}$$

$$=e^{2t}(1-(\beta-3t-t^2)t)^{-1}=e^{2t}(1+t)^{-3},$$

by Taylor expansion for $(1-\beta t)^{-1}$. Since generating function

$$Z(t)=G(t)(1-3t^2-2t^3)=e^{2t}\left(\frac{3}{1+t}-2\right)$$

for ζ_n was found, we may regard Theorem 3 as proved.

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References

[1] K. Yamamoto: Latin kukei no zenkinsu to symbolic method, Sugaku 2 (1949), 159-162.

[2] ——: An asymptotic series for the number of three-line Latin rectangles, J. Math. Soc. Japan, 1 (1950), 226-241.

Added during proof: It might be pointed out that Riordan has published his result entitled ***A recurrence relation for three-line Latin rectangles*** in Amer. Math. Monthly, Vol. 59 (1952), pp. 159-162.