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On Killing vector fields in a Kaehlerian space.

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§ 0. Introduction.

S. Bochner [1, 2][1] has shown a remarkable contrast between harmonic vectors and Killing vectors in a real compact Riemannian space by proving the following theorems:

THEOREM I. *In a compact Riemannian space, there exists no harmonic (Killing) vector field, other than zero vector, which satisfies the relation*

$$R_{jk}\xi^{j}\xi^{k}\geqq 0, \qquad (R_{jk}\xi^{j}\xi^{k}\leqq 0)$$

unless we have $\xi_{j;k}=0$. *If the space has positive (negative) Ricci curvature throughout, then the exceptional case cannot arise.*

THEOREM II. *If, in a compact Riemannian space, there exist a harmonic vector field* ξ_{i} *and a Killing vector field* η^{i}, *then we have*

$$\xi_{i}\eta^{i}=\text{constant}.$$

S. Bochner has shown also a remarkable contrast between covariant analytic vectors and contravariant analytic vectors in a compact Kaehlerian space by proving the following theorems:

THEOREM III. *In a compact Kaehlerian space, there exists no self-adjoint covariant (contravariant) vector field, other than zero vector, the components of which are analytic functions of coordinates and which satisfies the relation*

$$R_{\alpha\bar{\beta}}\xi^{\alpha}\xi^{\bar{\beta}}\geqq 0, \qquad (R_{\alpha\bar{\beta}}\xi^{\alpha}\xi^{\bar{\beta}}\leqq 0)$$

unless the vector field has vanishing covariant derivative. If $R_{\alpha\bar{\beta}}\xi^{\alpha}\xi^{\bar{\beta}}$ *is positive (negative) definite throughout, then the exceptional case cannot arise.*

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1) See the Bibliography at the end of the paper.

THEOREM IV. *If, in a compact Kaehlerian space, there exist a covariant analytic vector field* ξ_α *and a contravariant analytic vector field* η^α, *then we have*

$$\xi_\alpha \eta^\alpha = \text{constant}.$$

On the other hand, following theorem is well known.

THEOREM V. *In a compact Kaehlerian space, a harmonic vector field has covariant components* ξ_α *which are analytic functions of coordinates* z^λ *and* $\xi_{\bar\alpha}$ *which are analytic functions of coordinates* $\bar z^\lambda$. *The converse is also true.*

The purpose of the present paper is to study properties of a Killing vector field in a compact Kaehlerian space and to obtain a theorem corresponding to Theorem V.

To show clearly the contrast between harmonic and Killing vectors, we shall give, in § 1, a sketch of the proof of Theorem V by a method which can be used, in § 2, for the proof of the corresponding theorem in the case of Killing vectors.

§ 1. Harmonic vectors in a compact Kaehlerian space.

We consider a compact Kaehlerian space with positive definite metric

$$ds^2 = g_{ij} dz^i dz^j$$

or

$$ds^2 = 2 g_{\alpha\bar\beta}\, dz^\alpha\, d\bar z^\beta, \tag{1.1}$$

the coefficients g_{ij} satisfying

$$g_{ij} = g_{ji}, \qquad g_{\alpha\beta} = g_{\bar\alpha\bar\beta} = 0, \qquad g_{\alpha\bar\beta} = \overline{g_{\bar\alpha\beta}}$$

and

$$\frac{\partial g_{\alpha\bar\beta}}{\partial \bar z^\gamma} = \frac{\partial g_{\alpha\bar\gamma}}{\partial \bar z^\beta}, \tag{1.2}$$

where the indices $i, j, k, \cdots$ take the values $1, 2, \cdots\cdots, n, \bar 1, \bar 2, \cdots\cdots, \bar n$, the indices $\alpha, \beta, \gamma, \cdots\cdots$ the values $1, 2, \cdots\cdots, n$ and the indices $\bar\alpha, \bar\beta, \bar\gamma, \cdots\cdots$ the values $\bar 1, \bar 2, \cdots\cdots, \bar n$ and

$$z^{\bar\alpha} = \bar z^\alpha,$$

the bar on a central letter denoting its complex conjugate.

On account of (1.2), the only non-zero Christoffel symbols formed with g_{ij} are

$$(1.3)\qquad \Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\bar{\varepsilon}}\frac{\partial g_{\bar{\varepsilon}\beta}}{\partial z^{\gamma}}\quad\text{and}\quad \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}=\frac{1}{2}g^{\bar{\alpha}\varepsilon}\frac{\partial g_{\varepsilon\bar{\beta}}}{\partial \bar{z}^{\gamma}},$$

where g^{ij} are defined by $g^{ij}g_{jk}=\delta^i_k$, and consequently, the only non-zero components of the curvature tensor $R^i{}_{jkl}$ are

$$(1.4)\qquad R^{\alpha}{}_{\beta\gamma\bar{\delta}}=-R^{\alpha}{}_{\beta\bar{\delta}\gamma}=\frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial\bar{z}^{\delta}}\quad\text{and}\quad R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}=-R^{\bar{\alpha}}{}_{\bar{\beta}\delta\bar{\gamma}}=\frac{\partial\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}}{\partial z^{\delta}}.$$

From (1.4), we can see that the only non-zero components of the Ricci tensor R_{ij} are

$$(1.5)\qquad R_{\beta\bar{\gamma}}=R_{\bar{\gamma}\beta}=R^{\alpha}{}_{\beta\bar{\gamma}\alpha}.$$

The quantities Γ^i_{jk}, $R^i{}_{jkl}$ and R_{jk} are all self-adjoint, that is to say, they satisfy

$$\overline{\Gamma^{\alpha}_{\beta\gamma}}=\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}},\quad \overline{R^{\alpha}{}_{\beta\gamma\bar{\delta}}}=R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta},\quad \overline{R_{\beta\bar{\gamma}}}=R_{\bar{\beta}\gamma}.$$

Now, we know [3] that, in a real compact orientable Riemannian space, the necessary and sufficient condition that a vector ξ_i be harmonic is that it satisfy

$$(1.6)\qquad g^{jk}\xi_{i;j;k}-\xi_a R^a{}_i=0,$$

where the semi-colon denotes covariant differentiation with respect to Christoffel symbols formed with g_{jk}.

This theorem is true also in a compact (orientable) Kaehlerian space, because if we consider a real representation of the Kaehlerian space, then equation (1.6) takes the form

$$(1.7)\qquad g'^{jk}\xi'_{i;j;k}-\xi'_a R'^a{}_i=0,$$

where g'_{ij}, g'^{jk}, Γ'^i_{jk} and $R'^a{}_i$ are all real and consequently equation (1.7) shows that the real and imaginary parts of ξ'_i are both harmonic vectors.

Now, we suppose that there exists a harmonic vector $\xi_i=(\xi_\alpha, \xi_{\bar{\alpha}})$

in a compact Kaehlerian space, then ξ_i satisfies (1.6) or

$$g^{jk}\,\xi_{\alpha;\,j;\,k}-\xi_{\varepsilon}R^{\varepsilon}{}_{\alpha}=0\,,$$

$$g^{jk}\,\xi_{\bar{\alpha};\,j;\,k}-\xi_{\bar{\varepsilon}}\,R^{\bar{\varepsilon}}{}_{\bar{\alpha}}=0\,.$$

Thus, if $\xi_i=(\xi_\alpha, \xi_{\bar{\alpha}})$ is a harmonic vector, then the vectors

$$\xi^*{}_i=(\bar{\xi}_{\bar{\alpha}}\,,\bar{\xi}_{\alpha})\,,\quad \eta_i=(\xi_\alpha\,,0)\,,\quad \zeta_i=(0,\xi_{\bar{\alpha}})$$

are also all harmonic vectors and consequently we have

$$\eta_{i;\,j}=\eta_{j;\,i}\,,$$

from which, putting $i=\alpha$, $j=\bar{\beta}$, we find

$$\eta_{\alpha;\,\bar{\beta}}\equiv\frac{\partial\xi_\alpha}{\partial\bar{z}^\beta}=0\,.$$

Also we have

$$\zeta_{i;\,j}=\zeta_{j;\,i}\,,$$

from which, putting $i=\bar{\alpha}$, $j=\beta$, we find

$$\zeta_{\bar{\alpha};\,\beta}\equiv\frac{\partial\xi_{\bar{\alpha}}}{\partial z^\beta}=0\,.$$

Thus, ξ_α are analytic functions of z^λ and $\xi_{\bar{\alpha}}$ analytic functions of $\bar{z}^\lambda$.

Conversely, if the covariant components ξ_α are analytic functions of the coordinates z^λ and $\xi_{\bar{\alpha}}$ those of $\bar{z}^\lambda$, then we have

$$\xi_{\alpha;\,\bar{\gamma}}=0\,,\quad \xi_{\bar{\alpha};\,\gamma}=0\,.$$

On the other hand, we have the Ricci identities

$$\xi_{\alpha;\,\gamma;\,\bar{\delta}}-\xi_{\alpha;\,\bar{\delta};\,\gamma}=-\xi_\varepsilon\,R^{\varepsilon}{}_{\alpha\gamma\bar{\delta}}\,,$$

$$\xi_{\bar{\alpha};\,\gamma;\,\bar{\delta}}-\xi_{\bar{\alpha};\,\bar{\delta};\,\gamma}=-\xi_{\bar{\varepsilon}}R^{\bar{\varepsilon}}{}_{\bar{\alpha}\gamma\bar{\delta}}\,,$$

from which

$$g^{jk}\xi_{\alpha;\,j;\,k}-\xi_a R^a{}_\alpha=0\,,$$

$$g^{jk}\xi_{\bar{\alpha};\,j;\,k}-\xi_a R^a{}_{\bar{\alpha}}=0\,,$$

or

$$g^{jk}\xi_{i;\,j;\,k}-\xi_a R^a{}_i=0\,,$$

which shows that ξ_i is a harmonic vector. Thus Theorem V is proved.

§ 2. Killing vectors in a compact Kaehlerian space.

We know [3] that, in a real compact orientable Riemannian space, the necessary and sufficient condition that a vector ξ^i be a Killing vector is that it satistfy

$$g^{jk}\xi^i{}_{;\,j;\,k}+R^i{}_j\xi^j=0 \quad \text{and} \quad \xi^i{}_{;\,i}=0\,. \tag{2.1}$$

This theorem is true also in a compact Kaehlerian space.

Now, we suppose that there exists a Killing vector $\xi^i=(\xi^\alpha, \xi^\alpha)$ such that $\xi^\alpha{}_{;\,\alpha}=0$, and consequently

$$\xi^\alpha{}_{;\,\alpha}=\xi^{\bar\alpha}{}_{;\,\bar\alpha}=0 \tag{2.2}$$

in a Kaehlerian space, then ξ^i satisfies (2.1) or

$$g^{jk}\xi^\alpha{}_{;\,j;\,k}+R^\alpha{}_\beta\xi^\beta=0\,, \qquad \xi^\alpha{}_{;\,\alpha}=0$$

and

$$g^{jk}\xi^{\bar\alpha}{}_{;\,j;\,k}+R^{\bar\alpha}{}_{\bar\beta}\xi^{\bar\beta}=0\,, \qquad \xi^{\bar\alpha}{}_{;\,\bar\alpha}=0\,.$$

Thus we have

THEOREM 2.1. *If $\xi^i=(\xi^a, \xi^{\bar\alpha})$ is a Killing vector satisfying* (2.2) *in a compact Kaehlerian space, then the vectors*

$$\xi^{*i}=(\overline{\xi^{\bar\alpha}}, \overline{\xi^\alpha})\,, \qquad \eta^i=(\xi^\alpha, 0)\,, \qquad \zeta^i=(0, \xi^{\bar\alpha})$$

are also all Killing vectors.

Consequently, if $\xi^i=(\xi^\alpha, \xi^{\bar\alpha})$ is a Killing vector satisfying (2.2), then

$$\eta_i=(0, \xi_{\bar\alpha})\,, \qquad \zeta_i=(\xi_\alpha\,, 0)$$

satisfy

$$\eta_{i;\,j}+\eta_{j;\,i}=0\,, \qquad \zeta_{i;\,j}+\zeta_{j;\,i}=0\,, \tag{2.3}$$

where

(2.4) $$\xi_{\bar{\alpha}}=g_{\bar{\alpha}\beta}\,\xi^{\beta}\,,\qquad \xi_{\alpha}=g_{\alpha\bar{\beta}}\,\xi^{\bar{\beta}}\,.$$

Putting $i=\alpha$, $j=\bar{\beta}$ in (2.3), we find

$$\eta_{\bar{\beta};\,\alpha}\equiv\frac{\partial\xi_{\bar{\beta}}}{\partial z^{\alpha}}=0\,,\qquad \zeta_{\alpha;\,\bar{\beta}}\equiv\frac{\partial\xi_{\alpha}}{\partial\bar{z}^{\beta}}=0\,.$$

These equations show that ξ_{α} are analytic functions of coordinates z^{λ} and $\xi_{\bar{\alpha}}$ those of $\bar{z}^{\lambda}$. Thus, from Theorem V, we have

THEOREM 2.2. *If $\xi^{i}=(\xi^{\alpha},\xi^{\bar{\alpha}})$ is a Killing vector satisfying* (2.2), *then it is necessarily a harmonic vector.*

Thus, if $\xi^{i}=(\xi^{\alpha},\xi^{\bar{\alpha}})$ is a Killing vector satisfying (2.2), then it being necessarily a harmonic vector, we have

$$\xi_{i;\,j}+\xi_{j;\,i}=0\,,\qquad \xi_{i;\,j}-\xi_{j;\,i}=0\,,$$

from which

$$\xi_{i;\,j}=0\,.$$

Thus we have

THEOREM 2.3. *If $\xi^{i}=(\xi^{\alpha},\xi^{\bar{\alpha}})$ is a Killing vector satisfying* (2.2), *then it is a parallel vector field.*

Thus, if $\xi^{i}=(\xi^{\alpha},\xi^{\bar{\alpha}})$ is a Killing vector satisfying (2.2), it being a parallel vector field, we should have

$$\xi^{i}{}_{;\,j}=0\,,$$

from which

$$\xi^{\alpha}{}_{;\,\bar{\beta}}\equiv\frac{\partial\xi^{\alpha}}{\partial\bar{z}^{\beta}}=0\,,\qquad \xi^{\bar{\alpha}}{}_{;\,\beta}\equiv\frac{\partial\xi^{\bar{\alpha}}}{\partial z^{\beta}}=0\,,$$

and consequently, we have

THEOREM 2.4. *If $\xi^{i}=(\xi^{\alpha},\xi^{\bar{\alpha}})$ is a Killing vector satisfying* (2.2), *then its contravariant components ξ^{α} are analytic functions of coordinates z^{λ} and $\xi^{\bar{\alpha}}$ those of $\bar{z}^{\lambda}$.*

Conversely, we suppose that a vector $\xi^{i}=(\xi^{\alpha},\xi^{\bar{\alpha}})$ has contravariant components which satisfy (2.2) and ξ^{α} are analytic functions of coordinates z^{λ} and $\xi^{\bar{\alpha}}$ those of $\bar{z}^{\lambda}$.

Then, from the Ricci identities

$$\xi^{\alpha}{}_{;\,\gamma;\,\bar{\delta}}-\xi^{\alpha}{}_{;\,\bar{\delta};\,\gamma}=\xi^{\beta}\,R^{\alpha}{}_{\beta\gamma\bar{\delta}}\,,$$

$$\xi^{\bar\alpha}{}_{;\gamma;\bar\delta}-\xi^{\bar\alpha}{}_{;\bar\delta;\gamma}=\xi^{\bar\beta}R^{\bar\alpha}{}_{\bar\beta\gamma\bar\delta},$$

we have

$$g^{jk}\xi^{\alpha}{}_{;j;k}+R^{\alpha}{}_{\beta}\xi^{\beta}=0, \qquad \xi^{\alpha}{}_{;\alpha}=0,$$

$$g^{jk}\xi^{\bar\alpha}{}_{;j;k}+R^{\bar\alpha}{}_{\bar\beta}\xi^{\bar\beta}=0, \qquad \xi^{\bar\alpha}{}_{;\bar\alpha}=0,$$

or

$$g^{jk}\xi^{i}{}_{;j;k}+R^{i}{}_{j}\xi^{j}=0; \qquad \xi^{i}{}_{;i}=0,$$

which shows that the vector ξ^i is a Killing vector. Thus we have

THEOREM 2.5. *If a vector $\xi^i=(\xi^\alpha,\xi^{\bar\alpha})$ in a compact Kaehlerian space has contravariant components which satisfy* (2.2) *and ξ^α are analytic functions of coordinates z^λ and $\xi^{\bar\alpha}$ those of $\bar z^\lambda$, then it is a Killing vector.*

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