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On the nilpotency of nil-algebras.

By Masayoshi NAGATA

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Introduction. We say in the present paper that a ring R satisfies the condition $c(n)$, if $u^n=0$ holds for all $u\in R$, n being a (given) natural number. The purpose of the present paper is to answer a problem on the nilpotency of algebras satisfying $c(n)$, raised by Y. Kawada and N. Iwahori, in proving the following

THEOREM. *Let $\mathfrak{A}$ be a ring with a coefficient field K. If $\mathfrak{A}$ satisfies the condition $c(n)$ and if K is of characteristic* 0, *then there exists a natural number $f(n)$ depending solely on n such that $\mathfrak{A}^{f(n)}=0$.*

In the last paragraph, we shall add some remarks concerning the case when K is a general coefficient ring.

1. Preliminaries on group rings of symmetric groups.

We denote by S_t the symmetric group on letters $1, 2, \cdots, t$; t being a natural number. Let K be the field in our theorem and $\mathfrak{o}_t$ the group ring of S_t over K.

We denote by (α) or (β) a Young diagram (of letters $1, \cdots, t$). Furthermore, for an arbitrary Young diagram (α), we denote by A_α the totality of $(-1)^{\delta(q)}q$ $(q\in S_t)$ such that i and $q(i)$ are in the same column of (α) for each i $(1\leq i\leq t)$ and that $\delta(q)=1$ or 0 according as the permutation q is odd or even; and by S_α the totality of p $(p\in S_t)$ such that i and $p(i)$ are in the same row of (α) for each i $(1\leq i\leq t)$. Further we set $A_\alpha^*=\sum_{a\in A_\alpha} a$ $(\in\mathfrak{o}_t)$, $S_\alpha^*=\sum_{s\in S_\alpha} s$ $(\in\mathfrak{o}_t)$.

Remark. A_α is a subgroup of $S_t\times\{1,-1\}$ and S_α is a subgroup of S_t.

Now the following facts are well known:

(1) $\mathfrak{o}_t S_\alpha^* A_\alpha^*$ is a simple left ideal of $\mathfrak{o}_t$.

(2) Every simple left ideal of $\mathfrak{o}_t$ is operator isomorphic to $\mathfrak{o}_t S_\alpha^* A_\alpha^*$ with a suitable (α).

From these facts follows easily

LEMMA 1. $\sum_{(\alpha)} \mathfrak{o}_t S_\alpha^* A_\alpha^* = \mathfrak{o}_t$ (where (α) runs over all Young diagrams of letters 1, 2, $\cdots$, t).

Now let g be a given natural number.

a) For a set of arbitrary g letters $i_1, i_2, \cdots, i_g$ $(i_1 < i_2 < \cdots i_g)$ among $1, 2, \ldots, t$ $(t \geq g)$, let $S(i_1, i_2, \cdots, i_g)$ be the symmetric group on letters $i_1, \cdots, i_g$ and set $S^*(i_1, \cdots, i_g) = \sum_{s \in S(i_1, \cdots, i_g)} s$. Further we set $\mathfrak{l}_1 = \sum_{(i_1, \cdots, i_g)} \mathfrak{o}_t S^*(i_1, \cdots, i_g)$.

b) For a set of arbitrary g letters $j_1, j_2, \cdots, j_g$ such that $j_1 < j_1 + 1 < j_2 < j_2 + 1 < \cdots < j_g < j_g + 1 \leq t$ $(t \geq 2g)$, let $A(j_1, \cdots, j_g)$ be the totality of elements σ of S_t such that (1) σ permutes only $j, j_1 + 1, j_2, \cdots, j_g, j_g + 1$ and (2) σ transforms $\{j_1, \cdots, j_g\}$ onto itself and $\sigma(j_k + 1) = \sigma(j_k) + 1$ for every $k = 1, \cdots, g$. We set $A^*(j_1, \cdots, j_g) = \sum_{a \in A(j_1, \cdots, j_g)} a$. Further we set $\mathfrak{l}_2 = \sum_{(j_1, \cdots, j_g)} \mathfrak{o}_t A^*(j_1, \cdots, j_g)$. Then we have

LEMMA 2. If $t \geq g(g^2 - 2g + 2)$, then $\mathfrak{l}_1 + \mathfrak{l}_2 = \mathfrak{o}_t$.

PROOF. Since $\sigma S^*(i_1, \cdots, i_g)\sigma^{-1} = S^*(\sigma(i_1), \cdots, \sigma(i_g))$ $(\sigma \in S)$, $\mathfrak{l}_1$ is a two-sided ideal. If (α) is a Young diagram with columns not less than g, $S_\alpha^* \in \mathfrak{l}_1$, whence $S_\alpha^* A_\alpha^* \in \mathfrak{l}_1$. Now it is sufficient to show, by virtue of Lemma 1, that for any Young diagram (β) with columns less than g, $A_\beta^* \in \mathfrak{l}_2$.

Let B be the set of letters which are in the first column of (β) and set $B' = \{s;\ s \in B,\ s+1 \in B\}$, $B'' = \{s;\ s \in B,\ s \neq t,\ s+1 \notin B\}$.

(1) When the number of letters of B' is not less than $2g-1$, we can select $j_1, j_2, \cdots, j_g \in B'$ as in $b)$ above. Then since $A(j_1, \cdots, j_g) \subseteq A_\beta$, we see that $A_\beta^* \in \mathfrak{l}_2$.

(2) When the number of letters of B' is less than $2g-1$, that of B'' is not less than $g^2 - 3g + 3 = (g-1)(g-2) + 1$. For, since $t \geq g(g^2 - 2g + 2)$ and since (β) has at most $g-1$ columns, the number of letters of B is not less than $g^2 - g + 2$ (observe that $(g^2 - g + 1)(g-1) = g^3 - 2g^2 + 2g - 1$). For all $s \in B''$, we consider $s+1$; they are in columns other than the first. Therefore at least one contains at least g of such $s+1$, i.e., we can select $j_1 < \cdots < j_g$ from B'' such that $j_1 + 1, \cdots, j_g + 1$ are in the same column. Then $A(j_1, \cdots, j_g) \subseteq A_\beta$, whence $A_\beta^* \in \mathfrak{l}_2$. Thus the proof is completed.

2. Preliminaries on rings satisfying the condition $c(n)$.

We denote by R a ring which has K as a coefficient field. When $y_1, \cdots, y_t \in R$ and $X = \sum_i a_i \sigma_i$ $(a_i \in K,\ \sigma_i \in S_t)$, we denote by $X(y_1 \cdots y_t)$ the sum $\sum_i a_i y_{\sigma_i(1)} \cdots y_{\sigma_i(t)}$. Let (K, R) be the ring obtained from R by adjoining an identity having K as the coefficient field.

LEMMA 3. Let d and t be given natural numbers and let $V=\{(i_1, \cdots, i_t)\}$ be the totality of vectors of dimension t such that (1) each component i_k is a non-negative integer and (2) the sum $\sum_{k=1}^{t} i_k$ of the components i_k is equal to d (for every vector $(i_1, \cdots, i_t) \in V$). Now suppose that to every vector $(i_1, \cdots, i_t) \in V$ there corresponds an element $u(i_1, \cdots, i_t)$ of R. If for arbitrary elements $c_2, \cdots, c_t$ of K it holds $\sum c_2^{i_2} \cdots c_t^{i_t} u(i_1, \cdots, i_t)=0$, then each $u(i_1, \cdots, i_t)$ is 0.

PROOF. When $t=1$ our assertion is evident. When $t=2$, since the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & a_1^2 & \cdots & a_1^d \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_d & a_d^2 & \cdots & a_d^d \end{vmatrix}$$

is the fundamental alternative function of $1, a_1, \cdots, a_d$, our assertion follows easily. Now assuming that our assertion holds when $t=s$, we consider the case $t=s+1$. Since

$$\sum_{i_{s+1}=0}^{d} c_{s+1}^{i_{s+1}} \Big(\sum_{i_{s+1} \text{ fixed}} c_2^{i_2} \cdots c_s^{i_s} u(i_1, \cdots, i_{s+1}) \Big)=0$$

we see, by the case $t=2$, that $\sum_{i_{s+1} \text{ fixed}} c_2^{i_2} \cdots c_s^{i_s} u(i_1, \cdots, i_{s+1}))=0$, which is the case $t=s$. Therefore $u(i_1, \cdots, i_{s+1})=0$.

LEMMA 4. Assume that R satisfies the condition $c(n)$. Then for arbitrary elements $y_1, \cdots, y_n$ of R, we have

$$S_n^*(y_1 \cdots y_n)=0 \quad (S_n^*=\sum_{s \in S_n} s).$$

PROOF. Since $(y_1+c_2y_2+\cdots+c_ny_n)^n=0$ for arbitrary elements $c_2, \cdots, c_n$ of K, we have our assertion by Lemma 3.

COROLLARY. With the same R, we have $y^{n-1}Ry^{n-1}=0$ for every $y \in R$.

LEMMA 5. Assume again that R satisfies the condition $c(n)$. Let m be the least integer greater than $n/2$. Then for an arbitrary element u of R, $\bar{\mathfrak{o}}=u(K, R)+(u^2)/(u^2)$ satisfies the condition $c(m)$, where $(u^2)=(K, R)u^2(K, R)$.

PROOF. It is sufficient to prove that for an arbitrary element z of R, $(uz)^{m-1}u\equiv 0$ (mod. u^2). In the equalities $S_n^*(y_1\cdots y_n)=0$ and $S_n^*(y_1\cdots y_n)u=0$, we put $y_1=\cdots=y_{n+1-m}=u$, $y_{n+2-m}=\cdots=y_n=z$. Then we see $(uz)^{m-1}u\equiv 0$ (mod. u^2) from the former or the latter according as n is odd or even.

LEMMA 6. Let R, n and m be the same as in Lemma 5. Let r be the least integer such that $n-1\leq 2^r$. If there exists $f(m)$ (in the theorem with n replaced by m), and if $u\in R$, then $(u)^{g(n)}=0$, where $g(n)$ is $2f(m)^r$ or $f(m)^r$ according as $n-1=2^r$ or $n-1<2^r$.

PROOF. By Lemma 5, we see that $(u(K, R))^{f(m)}\subseteq(u^2)$. Therefore $(u)^{f(m)}\subseteq(u^2)$. Thus we see $(u)^{f(m)^r}\subseteq(u^2)^{f(m)^{r-1}}\subseteq\cdots\subseteq(u^{2^r})$. Now our assertion follows from the corollary to Lemma 4.

LEMMA 7. With the same R and $g(n)$ (and assuming the existence of $f(m)$), let $y_1, \cdots, y_t$ $(t\geq g(n))$ be arbitrary elements of R and let $i_1<i_2<\cdots<i_{g(n)}$ be arbitrary integers among $1, 2, \cdots, t$. Then

$$S^*(i_1, \cdots, i_{g(n)})(y_1\cdots y_t)=0\,.$$

PROOF. We may assume without loss of generality that $i_1=1$ and $i_{g(n)}=t$. Take $r_1=1, s_1, r_2, s_2, \cdots, r_k, s_k=t$ such that $\{i_1, \cdots, i_{g(n)}\}=\{r_1=1, 2, \cdots, s_1, r_2, r_2+1, \cdots, s_2, \cdots, r_k, r_k+1, \cdots, s_k\}$ and that $r_{j+1}>s_j+1$ $(j=1, \cdots, k-1)$. We set $u=(y_{i_1}+c_2y_{i_2}+\cdots+c_{g(n)}y_{i_{g(n)}})$ witharbitrary elements $c_2, \cdots, c_{g(n)}$ of K. Then by Lemma 6 we have

$$u^{s_1}y_{s_1+1}\cdots y_{r_2-1}u^{s_2-r_2+1}y_{s_2+1}\cdots y_{r_k-1}u^{s_k-r_k+1}=0.$$

Since $c_2, \cdots, c_{g(n)}$ are arbitrary, we have our assertion by Lemma 3.

COROLLARY. When $t\geq 2g(n)$, take $j_1, \cdots, j_{g(n)}$ such that $A(j_1, \cdots, j_{g(n)})$ can be defined. Then

$$A^*(j_1, \cdots, j_{g(n)})(y_1\cdots y_t)=0.$$

3. Proof of Theorem.

Since we may set $f(2)=3$, as is easily seen, we prove the theorem by induction on n: We assume the existence of $f(m)$, m being the least integer greater than $n/2$ (observe that when $n \geq 3$, $m < n$).

We take $g=g(n)$ given by Lemma 6 and let $t=f(n)=g(g^2-2g+2)$: We prove $\mathfrak{A}^t=0$. For this purpose, we may assume without loss of generality that $\mathfrak{A}=F/\mathfrak{N}$, F being the ring freely generated by sufficiently many indeterminates x_λ over K and $\mathfrak{N}$ being the (two-sided) ideal of F generated by all of the n-th powers of elements of F.

Let $x_1, \cdots, x_t$ be arbitrary, mutually distinct elements among x_λ. Then, as $\mathfrak{N}$ is left-invariant under $\mathfrak{o}_t$, $\mathfrak{l}=\{X;\ X\in\mathfrak{o}_t,\ X(x_1\cdots x_t)\in\mathfrak{N}\}$ forms a left ideal of $\mathfrak{o}_t$. By Lemma 7 and its corollary, every $S^*(i_1, \cdots, i_g)$ and every $A^*(j_1, \cdots, j_g)$ are in $\mathfrak{l}$. Therefore by Lemma 2 $\mathfrak{l}\ni 1$, whence $x_1\cdots x_t\in\mathfrak{N}$, and this shows $\mathfrak{A}^t=0$.

4. A more restricted case.

We say in this paragraph that a ring R^* satisfies the condition $c^*(n)$, if R^* satisfies the condition $c(n)$ and R^* possesses a system M of generators such that $x^2=0$, $xR^*x=0$ for every $x\in M$.

By our theorem we see that there exists a natural number $h(n)$ such that $\mathfrak{B}^{h(n)}=0$, if $\mathfrak{B}$ satisfies the condition $c^*(n)$ and if $\mathfrak{B}$ has a coefficient field, say K, of characteristic 0.

It is remarkable that *for this* $h(n)$ *we have* $\mathfrak{A}^{h(n)}=0$ for any ring $\mathfrak{A}$ which satisfies the condition $c(n)$ and which has also a coefficient field of characteristic 0, that is *if* $f(n)$ *and* $h(n)$ *are chosen to be the least possible value, then* $f(n)=h(n)$.

Indeed, let F, $\mathfrak{N}$ and $\mathfrak{A}$ be the same as in § 3, and let $\mathfrak{N}_1$ be the ideal generated by $\mathfrak{N}$ and all of both $x_\lambda F x_\lambda$ and x_λ^2. We may assume that $\mathfrak{B}=F/\mathfrak{N}_1$. $\mathfrak{B}^{h(n)}=0$ shows $x_1\cdots x_{h(n)}\in\mathfrak{N}_1$ for any (mutually distinct) $x_1, \cdots, x_{h(n)}\in\{x\}$. This shows that

$$x_1\cdots x_{h(n)}=\sum_i a_i u_i s_i v_i+Q,$$

where (1) $s_i=S_n^*(y_1^{(i)}\cdots y_n^{(i)})$ with $y_j^{(i)}\in F$, (2) $a_i\in K$, (3) each of u_i and v_i are in F unless it is the identity and (4) Q is a sum of terms which are of weight greater than 1 on some x_λ. It is evident that when $y_k=z_k+w_k$ for one $k\ (1\leq k\leq n)$, then $S_n^*(y_1\cdots y_n)=S_n^*(z_1\cdots z_n)+S_n^*(w_1\cdots w_n)$ with $y_i=z_i=w_i$ for every $i\neq k$. Therefore we may assume without loss of generality that (1) u_i, v_i and $y_j^{(i)}$ are monomials on $x_1, \cdots, x_{h(n)}$ (u_i and v_i may be 1) and (2) every $u_i s_i v_i$ contains no term of weight greater than 1 for any x_λ. Then since $x_1\cdots x_{h(n)}$ is of weight

1 for every $x_1, \cdots, x_{h(n)}$, we have $Q=0$, i. e., $x_1 \cdots x_{h(n)} \in \mathfrak{N}$, which shows $\mathfrak{A}^{h(n)}=0$.

5. Remarks.

(I) *When K is a field of characteristic $p \neq 0$: If $p \leq n$, the conclusion of our theorem does not hold.* A counter-example can be easily constructed even when (1) $n=p$ and (2) $\mathfrak{A}$ is commutative; namely as follows: *Let A_k be the algebra over K of the rank n^k-1 with the generating elements $x_1, \cdots, x_k$ with the fundamental relations $x_i^n=0$, $x_i x_j = x_j x_i$ for $i, j=1, 2, \cdots, k$; and put $A=\sum_{k=1}^{\infty} A_k$ (direct sum!). Then A satisfies $c(p)$, but A is not nilpotent.*

If p is greater than $f(n)$ given by the theorem, then the same conclusion holds. This fact is established by an easy modification of our above proof.

(II) *When K is a commutative ring with identity: The conclusion of our theorem holds for K, if and only if it holds for every residue fields of K.* The proof is easily obtained if we consider the group rings of symmetric groups over K.

(III) *Conjecture: The conclusion of our theorem will hold, also when K is a field of characteristic $p > n$.*

Nagoya University.