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An application of Ahlfors's theory of covering surfaces.

By Zuiman YÛJÔBÔ

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We shall give here an alternative proof of the following theorem of Ahlfors[1)] using his theory of covering surfaces.[2)]

THEOREM. *Let* $w=f(z)$ *be a meromorphic function in* $|z|<R$, *and* $D_1, D_2, \cdots, D_q$ $(q \geqq 3)$ *be simply connected closed domains on the Riemann sphere lying outside each others. If*

$$R \geqq k \frac{1+|f(0)|^2}{|f'(0)|},$$

k being a constant depending only on D_i $(i=1, 2, \cdots, q)$, *then we have*

$$\sum_{i=1}^{q}\left(1-\frac{1}{\mu_i}\right) \leqq 2,$$

$f(z)$ *ramifying at least* μ_i*-ply on* D_i $(i=1, 2, \cdots, q)$.[3)]

PROOF. Suppose that the latter inequality does not hold. Then, since μ_i are positive integers or $=\infty$, it is easily verified that there holds for any $r\ (\leqq R)$

$$\sum_{i=1}^{q}\left(1-\frac{1}{\mu_i(r)}\right) \geqq \sum_{i=1}^{q}\left(1-\frac{1}{\mu_i}\right) \geqq 2+\frac{1}{42}, \tag{1}$$

where $f(z)$ ramifies at least $\mu_i(r)$-ply on D_i $(i=1, 2, \cdots, q)$ in $|z| \leqq r \leqq R$ $\left(\mu_i(r) \geqq \mu_i(R)=\mu_i\right)$.

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1) L. Ahlfors, Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant à une région donnée. (Acta Soc. Sci. Fenn. N. s. 2 Nr. 2 (1933)).

2) L. Ahlfors, Zur Theorie der Überlagerungsflächen (Acta Math. 65 (1935)); or R. Nevanlinna, Eindeutige analytische Funktionen.

3) By this expression we mean that the Riemann image of $|z|<R$ by $f(z)$ contains no connected island above D_i whose number of sheets is $<\mu_i$.

On the other hand we have the following inequality[4] which Ahlfors obtained from his theory of covering surfaces:

$$\sum_{i=1}^{q}\left(1-\frac{1}{\mu_i(r)}\right)\leqq 2+h\frac{L(r)}{A(r)}, \tag{2}$$

where $h\,(>0)$ depends only on D_i, and $A(r)$, $L(r)$ are respectively the area and the length of the Riemann images of $|z|<r$ and $|z|=r$ by $f(z)$. Then we have from (1) and (2),

$$\frac{L(r)}{A(r)}\geqq\frac{1}{42h}.$$

Next from this and the inequality (obtained easily using Schwarz's inequality)

$$\log\frac{R}{r_0}\leqq 2\pi\int_{r_0}^{R}\frac{dA(r)}{L(r)^2}\ {}^{5)},$$

we have

$$\log\frac{R}{r_0}\leqq 2\pi\,(42h)^2\int_{r_0}^{R}\frac{dA(r)}{A(r)^2}<\frac{3528\pi h^2}{A(r_0)}$$

or

$$A(r_0)<\frac{3528\pi h^2}{\log\,(R/r_0)}.$$

So we have for $r_0=R\exp\,(-7056h^2)$

$$A(r_0)<\pi/2.$$

Next we have for any $0<r_1<r_0$,

$$\log\frac{r_0}{r_1}\leqq 2\pi\int_{r_1}^{r_0}\frac{dA(r)}{L(r)^2}<\frac{2\pi}{L(r_f)^2}A(r_0)<\frac{\pi^2}{L(r_f)^2},$$

where r_f is the radius which minimizes $L(r)$ in $r_1\leqq r\leqq r_0$. Therefore we have

$$L(r_f)<\pi/2 \qquad (r_1\leqq r_f\leqq r_0), \tag{3}$$

when we take $e^{-4}r_0$ for r_1. On the other hand we have of course

$$A(r_f)<\pi/2. \tag{4}$$

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4), 5) L. Ahlfors or R. Nevanlinna, loc. cit.

Now we rotate the Riemann sphere so as to bring $f(0)$ to $w=0$. Then we obtain a new function $f^*(z)$ such that $f^*(0)=0$ and $|f^{*\prime}(0)| = |f'(0)|/(1+|f(0)|^2)$, and such that $L(r)$ and $A(r)$ for f^* is the same as those for f. Further we have $|f^*(z)|<1$ in $|z|<r_f$, as is easily observed from (3), (4) and $f^*(0)=0$[6]. Then we have

$$\frac{\pi}{2} > L(r_f) = \int_{|z|=r_f} \frac{|f^{*\prime}(z)|}{1+|f^*(z)|^2} |dz|$$

$$> \frac{r_f}{2} \int_0^{2\pi} |f^{*\prime}(r_f e^{i\theta})| d\theta > \pi r_f |f^{*\prime}(0)| .$$

Therefore we obtain

$$R < \frac{1}{2} \exp(4+7056h^2) \frac{1+|f(0)|^2}{|f'(0)|}$$

and we conclude the proof by putting

$$k = \frac{1}{2} \exp(4+7056h^2).$$

Mathematical Institute,
Tokyo University

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6) This can be proved as follows: Let λ be the image of $|z|=r_f$ by f^* on the Riemann sphere. Then the complement of λ consists of a finite number of connected domains. If the southern pole lie on λ, then λ is obviously contained in the lower hemisphere on account of (3); then no point of the upper hemisphere is assumed by f^* in $|z|<r_f$, for otherwise the whole upper hemisphere would be covered by the image of $|z|<r_f$ by f^*, which contradicts (4). So we may consider only the case where the southern pole does not lie on λ. Now let G be the one containing the southern pole among the above-mentioned domains. Then G is obviously simply connected and all the points of G are assumed by f^* on account of $f^*(0)=0$.

Now let us suppose that λ is not contained in the lower hemisphere. Then there must exist intersection points of λ with the equator ($|w|=1$), for otherwise G must cover the whole lower hemisphere which is impossible on account of (4). Let us denote one of them by P and by C_P the locus of the points with spherical distance $\pi/4$ from P. Then C_P meets with the boundary of G, since C_P is a great circle which passes through the southern pole and G cannot cover a whole hemisphere. But this is clearly impossible on account of (3).

Now since λ is contained in the lower hemisphere, either all the points in the upper hemisphere or none of them are assumed by f^*. But the former case is impossible on account of (4).