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A proof of a transformation formula in the theory of partitions.

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Let us denote by $p(n)$ the number of unrestricted partitions of a positive integer n. Then we have

$$f(x)=1+\sum_{n=1}^{\infty} p(n)x^{n}=\prod_{n=1}^{\infty}\frac{1}{1-x^{n}},$$

where $f(x)$ is defined and regular for $|x|<1$. Now this function $f(x)$ is known to satisfy a remarkable transformation formula (see Rademacher [3]):

$$(1)\qquad f\left(e^{\frac{2\pi ih}{k}-\frac{2\pi z}{k}}\right)=\omega_{h,k}\sqrt{z}\,e^{\frac{\pi}{12kz}-\frac{\pi z}{12k}}f\left(e^{\frac{2\pi iH}{k}-\frac{2\pi}{kz}}\right),$$

where h, k and H are positive integers such that

$$(h,k)=1,\qquad hH\equiv -1 \pmod{k},$$

and

$$\omega_{h,k}=\exp\left(\pi i\sum_{m=1}^{k-1}\frac{m}{k}\left(\frac{hm}{k}-\left[\frac{hm}{k}\right]-\frac{1}{2}\right)\right),$$

an empty sum meaning zero; further, z is a complex variable with positive real part and we take the principal branch as the determination of $\sqrt{z}$.

The formula (1) was used by Hardy-Ramanujan, and also by Rademacher subsequently, in their famous researches [2] and [4] on the function $p(n)$.

It is the main object of this paper to give a proof of (1) which is directly based on the following well known transformation formula in the theory of elliptic theta-functions:

$$(2)\qquad \sum_{n=-\infty}^{\infty} e^{-(n+\alpha)^{2}\pi t}=\frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty} e^{-\frac{n^{2}\pi}{t}}\cos(2\pi n\alpha),$$

where α is real and $t>0$.

Let us begin by defining a branch of the logarithm of $f(x)$ for $|x|<1$ by

$$F(x)=\log f(x)=\sum_{n=1}^{\infty}\log\frac{1}{1-x^n}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{mn}}{m},$$

where we take

$$\log\frac{1}{1-x^n}=\sum_{m=1}^{\infty}\frac{x^{mn}}{m}.$$

We put

$$x=e^{\frac{2\pi ih}{k}-\frac{2\pi z}{k}},\qquad \tilde{x}=e^{\frac{2\pi iH}{k}-\frac{2\pi}{kz}}.$$

Then, to prove (1), it suffices to show that

$$F(x)=\Omega i+\frac{1}{2}\log z+\frac{\pi}{12kz}-\frac{\pi z}{12k}+F(\tilde{x}), \tag{3}$$

$$\Omega=\Omega_{h,k}=\pi\sum_{m=1}^{k-1}\frac{m}{k}\left(\frac{hm}{k}-\left[\frac{hm}{k}\right]-\frac{1}{2}\right), \tag{4}$$

where in (3) we take the principal branch as the determination of $\log z$.

We may assume $z>0$, for the general case of z with positive real part is reduced to this case by the theory of analytic continuation. We shall prove (3) by decomposing it into its real and imaginary parts:

$$\Re F(x)=\frac{1}{2}\log z+\frac{\pi}{12kz}-\frac{\pi z}{12k}+\Re F(\tilde{x}), \tag{5}$$

$$\Im F(x)=\Omega+\Im F(\tilde{x}). \tag{6}$$

To establish (5), we start from the integral

$$\int_0^{\infty}e^{-t^2-\frac{a^2}{t^2}}\,dt=\frac{\sqrt{\pi}}{2}e^{-2a}\qquad(a>0),$$

which is an easy consequence of

$$\int_0^{\infty}e^{-t^2}\,dt=\frac{\sqrt{\pi}}{2}.$$

We transform the above integral into the following form:

$$\int_0^\infty e^{-a^2t^2-\frac{b^2}{t^2}}dt=\frac{\sqrt{\pi}}{2a}e^{-2ab} \qquad (a>0,\ b>0).$$

Putting $a=\sqrt{\pi}\,mz/k$ and $b=\sqrt{\pi}\,n$, where m and n are positive integers, we find

$$\frac{1}{m}e^{-\frac{2\pi mnz}{k}}=\frac{2z}{k}\int_0^\infty e^{-\frac{\pi m^2z^2t^2}{k^2}-\frac{\pi n^2}{t^2}}dt. \tag{7}$$

Hence, by absolute convergence,

$$F(x)=\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{1}{m}e^{mn\left(\frac{2\pi ih}{k}-\frac{2\pi z}{k}\right)}=\frac{2z}{k}\int_0^\infty\sum_{m=1}^\infty\sum_{n=1}^\infty e^{\frac{2\pi ihmn}{k}-\frac{\pi m^2z^2t^2}{k^2}-\frac{\pi n^2}{t^2}}dt,$$

whence follows

$$\begin{aligned}\Re F(x)&=\frac{2z}{k}\int_0^\infty\sum_{m=1}^\infty\sum_{n=1}^\infty e^{-\frac{\pi m^2z^2t^2}{k^2}-\frac{\pi n^2}{t^2}}\cos\frac{2\pi hmn}{k}\,dt\\&=\frac{z}{k}\int_0^\infty\sum_{m=1}^\infty e^{-\frac{\pi m^2z^2t^2}{k^2}}\left(\sum_{n=-\infty}^\infty e^{-\frac{\pi n^2}{t^2}}\cos\frac{2\pi hmn}{k}-1\right)dt\\&=\frac{z}{k}\int_0^\infty\sum_{m=1}^\infty e^{-\frac{\pi m^2z^2t^2}{k^2}}\left(\sum_{r=1}^k\cos\frac{2\pi hmr}{k}\sum_{l=-\infty}^\infty e^{-\frac{\pi}{t^2}(kl+r)^2}-1\right)dt.\end{aligned}$$

Now we have, by (2),

$$\begin{aligned}\sum_{l=-\infty}^\infty e^{-\frac{\pi}{t^2}(kl+r)^2}&=\sum_{l=-\infty}^\infty e^{-\frac{\pi k^2}{t^2}\left(l+\frac{r}{k}\right)^2}=\frac{t}{k}\sum_{n=-\infty}^\infty e^{-\frac{\pi n^2t^2}{k^2}}\cos\frac{2\pi nr}{k}\\&=\frac{t}{k}\left(2\sum_{n=1}^\infty e^{-\frac{\pi n^2t^2}{k^2}}\cos\frac{2\pi nr}{k}+1\right),\end{aligned}$$

and further

$$\sum_{r=1}^k\cos\frac{2\pi hmr}{k}=\Re\sum_{r=0}^{k-1}e^{\frac{2\pi ihmr}{k}}=\begin{cases}k, & \text{if } k\,|\,m,\\ 0, & \text{otherwise.}\end{cases}$$

Thus we have

$$\sum_{m=1}^\infty e^{-\frac{\pi m^2z^2t^2}{k^2}}\sum_{r=1}^k\cos\frac{2\pi hmr}{k}=k\sum_{m=1}^\infty e^{-\pi m^2z^2t^2},$$

$$\begin{aligned}\Re F(x)=\frac{z}{k}\int_0^\infty\Bigg(&\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{2t}{k}e^{-\frac{\pi m^2z^2t^2}{k^2}-\frac{\pi n^2t^2}{k^2}}\sum_{r=1}^k\cos\frac{2\pi hmr}{k}\cos\frac{2\pi nr}{k}\\&+t\sum_{m=1}^\infty e^{-\pi m^2z^2t^2}-\sum_{m=1}^\infty e^{-\frac{\pi m^2z^2t^2}{k^2}}\Bigg)dt.\end{aligned} \tag{8}$$

Since we defined

$$x=e^{\frac{2\pi ih}{k}-\frac{2\pi z}{k}}, \qquad \tilde{x}=e^{\frac{2\pi iH}{k}-\frac{2\pi}{kz}},$$

it follows from (8) that

$$\Re F(\tilde{x})=\frac{1}{kz}\int_0^\infty\Big(\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{2t}{k}e^{-\frac{\pi m^2t^2}{k^2z^2}-\frac{\pi n^2t^2}{k^2}}\sum_{r=1}^k\cos\frac{2\pi Hmr}{k}\cos\frac{2\pi nr}{k}$$

$$+t\sum_{m=1}^\infty e^{-\frac{\pi m^2t^2}{z^2}}-\sum_{m=1}^\infty e^{-\frac{\pi m^2t^2}{k^2z^2}}\Big)dt$$

$$=\frac{z}{k}\int_0^\infty\Big(\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{2t}{k}e^{-\frac{\pi m^2t^2}{k^2}-\frac{\pi n^2z^2t^2}{k^2}}\sum_{r=1}^k\cos\frac{2\pi Hmr}{k}\cos\frac{2\pi nr}{k}$$

$$+t\sum_{n=1}^\infty e^{-\pi n^2t^2}-\frac{1}{z}\sum_{n=1}^\infty e^{-\frac{\pi n^2t^2}{k^2}}\Big)dt.$$

We find here, since $hH\equiv -1 \pmod{k}$,

$$\sum_{r=1}^k\cos\frac{2\pi Hmr}{k}\cos\frac{2\pi nr}{k}=\sum_{r=1}^k\cos\frac{2\pi Hm(hr)}{k}\cos\frac{2\pi n(hr)}{k}$$

$$=\sum_{r=1}^k\cos\frac{2\pi mr}{k}\cos\frac{2\pi hnr}{k}.$$

Inserting this in the above result and interchanging the indices m and n, we obtain

$$(9)\quad \Re F(\tilde{x})=\frac{z}{k}\int_0^\infty\Big(\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{2t}{k}e^{-\frac{\pi m^2z^2t^2}{k^2}-\frac{\pi n^2t^2}{k^2}}\sum_{r=1}^k\cos\frac{2\pi hmr}{k}\cos\frac{2\pi nr}{k}$$

$$+t\sum_{m=1}^\infty e^{-\pi m^2t^2}-\frac{1}{z}\sum_{m=1}^\infty e^{-\frac{\pi m^2t^2}{k^2}}\Big)dt.$$

Subtracting (9) from (8) side-by-side, we get

$$\Re F(x)-\Re F(\tilde{x})$$

$$=\frac{z}{k}\int_0^\infty\Big(t\sum_{n=1}^\infty e^{-\pi n^2z^2t^2}-\sum_{n=1}^\infty e^{-\frac{\pi n^2z^2t^2}{k^2}}-t\sum_{n=1}^\infty e^{-\pi n^2t^2}+\frac{1}{z}\sum_{n=1}^\infty e^{-\frac{\pi n^2t^2}{k^2}}\Big)dt$$

$$=\frac{z}{k}\int_0^\infty t\sum_{n=1}^\infty e^{-\pi n^2z^2t^2}\,dt-\frac{z}{k}\int_0^\infty t\sum_{n=1}^\infty e^{-\pi n^2t^2}\,dt$$

$$+\frac{1}{k}\int_0^\infty\Big(\sum_{n=1}^\infty e^{-\frac{\pi n^2t^2}{k^2}}-z\sum_{n=1}^\infty e^{-\frac{\pi n^2z^2t^2}{k^2}}\Big)dt.$$

We have here

$$\frac{z}{k}\int_0^\infty t\sum_{n=1}^\infty e^{-\pi n^2z^2t^2}\,dt=\frac{1}{2kz}\int_0^\infty\sum_{n=1}^\infty e^{-\pi n^2u}\,du$$

$$=\frac{1}{2kz}\sum_{n=1}^\infty\int_0^\infty e^{-\pi n^2u}\,du=\frac{1}{2\pi kz}\sum_{n=1}^\infty\frac{1}{n^2}=\frac{\pi}{12kz},$$

$$\frac{z}{k}\int_0^\infty t\sum_{n=1}^\infty e^{-\pi n^2t^2}\,dt=\frac{z}{2k}\int_0^\infty\sum_{n=1}^\infty e^{-\pi n^2u}\,du=\frac{z}{2k\pi}\sum_{n=1}^\infty\frac{1}{n^2}=\frac{\pi z}{12k},$$

$$\frac{1}{k}\int_0^\infty\Big(\sum_{n=1}^\infty e^{-\frac{\pi n^2t^2}{k^2}}-z\sum_{n=1}^\infty e^{-\frac{\pi n^2z^2t^2}{k^2}}\Big)dt=\int_0^\infty\Big(\sum_{n=1}^\infty e^{-\pi n^2u^2}-z\sum_{n=1}^\infty e^{-\pi n^2z^2u^2}\Big)du$$

$$=\lim_{\alpha\to+0}\Big(\int_\alpha^\infty\sum_{n=1}^\infty e^{-\pi n^2u^2}\,du-z\int_\alpha^\infty\sum_{n=1}^\infty e^{-\pi n^2z^2u^2}\,du\Big)$$

$$=\lim_{\alpha\to+0}\int_\alpha^{\alpha z}\sum_{n=1}^\infty e^{-\pi n^2u^2}\,du.$$

Since we have

$$\sum_{n=1}^\infty e^{-\pi n^2t}=\frac{1}{\sqrt{t}}\sum_{n=1}^\infty e^{-\frac{\pi n^2}{t}}-\frac{1}{2}+\frac{1}{2\sqrt{t}}$$

as the special case $\alpha=0$ of (2), we get

$$\lim_{\alpha\to+0}\int_\alpha^{\alpha z}\sum_{n=1}^\infty e^{-\pi n^2u^2}\,du=\lim_{\alpha\to+0}\int_\alpha^{\alpha z}\Big(\frac{1}{u}\sum_{n=1}^\infty e^{-\frac{\pi n^2}{u^2}}-\frac{1}{2}+\frac{1}{2u}\Big)du$$

$$=\lim_{\alpha\to+0}\Big(\frac{1}{2}\log z-\frac{\alpha z-\alpha}{2}+\int_\alpha^{\alpha z}\frac{1}{u}\sum_{n=1}^\infty e^{-\frac{\pi n^2}{u^2}}\,du\Big)=\frac{1}{2}\log z.$$

Thus we find finally

$$\Re F(x)-\Re F(\tilde{x})=\frac{\pi}{12kz}-\frac{\pi z}{12k}+\frac{1}{2}\log z,$$

which is equivalent to (5).

We now turn to the proof of (6). We shall first show that $\Omega=\Im F(x)-\Im F(\tilde{x})$ is a constant depending on k and h only, and then determine the value of Ω explicitly, using this constancy of Ω.

We have

$$\frac{d}{dz}\Omega=\Im\frac{d}{dz}F(x)-\Im\frac{d}{dz}F(\tilde{x})=\Im\Big(F'(x)\frac{dx}{dz}\Big)-\Im\Big(F'(\tilde{x})\frac{d\tilde{x}}{dz}\Big)$$

$$= -\mathfrak{J}\left(\frac{2\pi}{k}xF'(x)\right) - \mathfrak{J}\left(\frac{2\pi}{kz^2}\tilde{x}F'(\tilde{x})\right).$$

Hence, to prove the constancy of Ω, it suffices to show that

$$z^2\mathfrak{J}G(x) + \mathfrak{J}G(\tilde{x}) = 0,$$

where we define, for $|w| < 1$,

$$G(w) = wF'(w) = w\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} nw^{mn-1} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} nw^{mn}.$$

We start from

$$ne^{-\frac{2\pi mnz}{k}} = \frac{2z}{k}\int_0^\infty mne^{-\frac{\pi m^2z^2t^2}{k^2} - \frac{\pi n^2}{t^2}}\,dt,$$

which is equivalent to (7). We find, by absolute convergence,

$$G(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} ne^{mn\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k}\right)} = \frac{2z}{k}\int_0^\infty \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mne^{\frac{2\pi ihmn}{k} - \frac{\pi m^2z^2t^2}{k^2} - \frac{\pi n^2}{t^2}}\,dt,$$

and hence

$$\mathfrak{J}G(x) = \frac{2z}{k}\int_0^\infty \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mne^{-\frac{\pi m^2z^2t^2}{k^2} - \frac{\pi n^2}{t^2}} \sin\frac{2\pi hmn}{k}\,dt$$

$$= \frac{z}{k}\int_0^\infty \sum_{m=1}^{\infty}\left(me^{-\frac{\pi m^2z^2t^2}{k^2}} \sum_{n=-\infty}^{\infty} ne^{-\frac{\pi n^2}{t^2}} \sin\frac{2\pi hmn}{k}\right)dt$$

$$= \frac{z}{k}\int_0^\infty \sum_{m=1}^{\infty} me^{-\frac{\pi m^2z^2t^2}{k^2}} \sum_{r=1}^{k} \sin\frac{2\pi hmr}{k} \sum_{l=-\infty}^{\infty}(kl+r)e^{-\frac{\pi}{t^2}(kl+r)^2}\,dt.$$

Now, differentiating (2) with respect to α, we obtain

$$\sum_{n=-\infty}^{\infty}(n+\alpha)e^{-(n+\alpha)^2\pi t} = \frac{2}{t\sqrt{t}}\sum_{n=1}^{\infty} ne^{-\frac{\pi n^2}{t}}\sin(2\pi n\alpha),$$

and hence

$$\sum_{l=-\infty}^{\infty}(kl+r)e^{-\frac{\pi}{t^2}(kl+r)^2} = k\sum_{l=-\infty}^{\infty}\left(l+\frac{r}{k}\right)e^{-\frac{\pi k^2}{t^2}\left(l+\frac{r}{k}\right)^2}$$

$$= \frac{2t^3}{k^2}\sum_{n=1}^{\infty} ne^{-\frac{\pi n^2t^2}{k^2}}\sin\frac{2\pi nr}{k},$$

$$(10)\quad \mathfrak{J}G(x) = \frac{2z}{k^3}\int_0^\infty t^3\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mne^{-\frac{\pi m^2z^2t^2}{k^2} - \frac{\pi n^2t^2}{k^2}}\sum_{r=1}^{k}\sin\frac{2\pi hmr}{k}\sin\frac{2\pi nr}{k}\,dt.$$

Changing here z to $1/z$ and h to H, we derive

$$\mathfrak{Z}G(\tilde{x})=\frac{2}{k^3 z}\int_0^\infty t^3\sum_{m=1}^\infty\sum_{n=1}^\infty mne^{-\frac{\pi m^2t^2}{k^2z^2}-\frac{\pi n^2t^2}{k^2}}\sum_{r=1}^k \sin\frac{2\pi Hmr}{k}\sin\frac{2\pi nr}{k}\,dt$$

$$=\frac{2z^3}{k^3}\int_0^\infty t^3\sum_{m=1}^\infty\sum_{n=1}^\infty mne^{-\frac{\pi m^2t^2}{k^2}-\frac{\pi n^2z^2t^2}{k^2}}\sum_{r=1}^k \sin\frac{2\pi Hmr}{k}\sin\frac{2\pi nr}{k}\,dt.$$

We find here, since $hH\equiv -1 \pmod{k}$,

$$\sum_{r=1}^k \sin\frac{2\pi Hmr}{k}\sin\frac{2\pi nr}{k}=\sum_{r=1}^k \sin\frac{2\pi Hm(hr)}{k}\sin\frac{2\pi n(hr)}{k}$$

$$=-\sum_{r=1}^k \sin\frac{2\pi mr}{k}\sin\frac{2\pi hnr}{k}.$$

Inserting this in the above result and interchanging the indices m and n, we get

$$\mathfrak{Z}G(\tilde{x})=-\frac{2z^3}{k^3}\int_0^\infty t^3\sum_{m=1}^\infty\sum_{n=1}^\infty mne^{-\frac{\pi m^2z^2t^2}{k^2}-\frac{\pi n^2t^2}{k^2}}\sum_{r=1}^k \sin\frac{2\pi hmr}{k}\sin\frac{2\pi nr}{k}\,dt.$$

A comparison of the last equality with (10) yields

$$\mathfrak{Z}G(\tilde{x})=-z^2\mathfrak{Z}G(x),$$

which was to be proved.

We shall now deduce the formula (4) for Ω. The following proof of (4) is substantially that given in Dedekind [1]. Dedekind omits, however, the verification of the uniform convergence of a certain series which, though easy to establish, forms the kernel of his proof; we shall give, in the following, a detailed discussion on this uniformity.

We may assume $k>1$. For if $k=1$ both x and $\tilde{x}$ are real and we have

$$\Omega=\mathfrak{Z}F(x)-\mathfrak{Z}F(\tilde{x})=0-0=0,$$

while the expression (4) also vanishes for $k=1$.

We have, by (6), assuming $z>0$ as hitherto,

(11) $$\lim_{z\to 0}\mathfrak{Z}F(x)=\Omega+\lim_{z\to 0}\mathfrak{Z}F(\tilde{x})=\Omega+\mathfrak{Z}F(0)=\Omega.$$

Let us put

$$R=e^{-\frac{2\pi z}{k}}\ (z>0), \qquad \rho=e^{\frac{2\pi ih}{k}},$$

so that $x=R\rho$. Then

$$F(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{mn}}{m}=\sum_{n=1}^{\infty}\frac{1}{n}\frac{x^n}{1-x^n},$$

(12) $$\mathfrak{Z}F(x)=\sum_{n=1}^{\infty}\frac{1}{n}\mathfrak{Z}\left(\frac{x^n}{1-x^n}\right)=\sum_{n=1}^{\infty}\frac{1}{n}\mathfrak{Z}\left(\frac{1}{1-x^n}\right)=\sum_{n=0}^{\infty}A_n(R),$$

where we define, for $n=0,1,2,\cdots$,

$$A_n(R)=\sum_{r=1}^{k-1}\frac{1}{r+kn}\mathfrak{Z}\left(\frac{1}{1-\rho^r R^{r+kn}}\right).$$

We put further, for $n=1,2,3,\cdots$,

(13) $$B_n(R)=\sum_{r=1}^{k-1}\frac{1}{kn}\mathfrak{Z}\left(\frac{1}{1-\rho^r R^{r+kn}}\right).$$

Then

$$|A_n(R)-B_n(R)|\leqq\sum_{r=1}^{k-1}\frac{r}{kn(r+kn)}\left|\mathfrak{Z}\left(\frac{1}{1-\rho^r R^{r+kn}}\right)\right|$$

$$\leqq\sum_{r=1}^{k-1}\frac{r}{kn(r+kn)}\frac{1}{|1-\rho^r R^{r+kn}|^2}.$$

We now define, for real positive y and natural numbers $r<k$,

$$\lambda(y)=|1-\rho^r y|^2=1-2y\cos 2\theta+y^2=(y-\cos 2\theta)^2+\sin^2 2\theta,$$

where we use the abbreviation $\theta=\pi hr/k$. We find then, if $\cos 2\theta\leqq 0$,

$$\lambda(y)\geqq 1+y^2>1,$$

and if $\cos 2\theta>0$, then we have $k\geqq 4$ (since $k=2$ or $k=3$ would imply $\cos 2\theta<0$ against the hypothesis), and hence

$$\lambda(y)\geqq\sin^2 2\theta\geqq\sin^2\frac{2\pi}{k}\geqq\sin^2\frac{\pi}{k}>\frac{1}{k^2}.$$

We thus derive in both cases

(14) $$|1-\rho^r y|^2>\frac{1}{k^2},$$

and this combined with the above inequality gives

(15) $$|A_n(R)-B_n(R)|\leqq\frac{k^2}{n^2}\qquad(n=1,2,3,\cdots).$$

Let us define further

$$C_n(R)=\sum_{r=1}^{k-1}\frac{1}{kn}\,\Im\left(\frac{1}{1-\rho^r R^{kn}}\right) \qquad (n=1,2,3,\cdots). \tag{16}$$

Then we have

$$C_n(R)=\sum_{r=1}^{k-1}\frac{1}{kn}\,\Im\left(\frac{1}{1-\rho^{k-r}R^{kn}}\right)=\sum_{r=1}^{k-1}\frac{1}{kn}\,\Im\left(\frac{1}{1-\rho^{-r}R^{kn}}\right).$$

Adding this to (16) side-by-side, we find

$$2C_n(R)=\sum_{r=1}^{k-1}\frac{1}{kn}\,\Im\left(\frac{1}{1-\rho^r R^{kn}}+\frac{1}{1-\rho^{-r}R^{kn}}\right)=0;$$

therefore, subtracting (16) from (13) and noting (14), we obtain

$$|B_n(R)|\leqq\sum_{r=1}^{k-1}\frac{1}{kn}\left|\frac{1}{1-\rho^r R^{r+kn}}-\frac{1}{1-\rho^r R^{kn}}\right|$$

$$\leqq\sum_{r=1}^{k-1}\frac{1}{kn}\,\frac{R^{kn}(1-R^r)}{|1-\rho^r R^{r+kn}|\cdot|1-\rho^r R^{kn}|}$$

$$\leqq\frac{k}{n}\,R^{kn}(1-R)\sum_{r=1}^{k-1}\sum_{m=0}^{r-1}R^m\leqq\frac{k^3}{n}\,R^{kn}(1-R).$$

Since the unique root of

$$\frac{d}{dy}\,y^{kn}(1-y)=kny^{kn-1}-(kn+1)y^{kn}=0$$

in the interval $0<y<1$ is given by $y=kn/(kn+1)$, at which point the function $y^{kn}(1-y)$ takes its maximum in $0<y<1$, we have

$$|B_n(R)|\leqq\frac{k^3}{n}\,\frac{1}{kn+1}<\frac{k^2}{n^2}.$$

It follows from (15) and the last inequality that

$$|A_n(R)|<\frac{2k^2}{n^2} \qquad (n=1,2,3,\cdots).$$

Hence $\sum_{n=0}^{\infty}A_n(R)$ is uniformly convergent with respect to R in the interval $0<R<1$. We have therefore, noting (11) and (12),

$$\Omega=\lim_{R\to 1}\sum_{n=0}^{\infty}A_n(R)=\sum_{n=0}^{\infty}A_n(1)$$

(17) $$=\sum_{n=0}^{\infty}\sum_{r=1}^{k-1}\frac{1}{r+kn}\,\mathfrak{Z}\left(\frac{1}{1-\rho^r}\right)=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{r=1}^{k-1}\frac{1}{r+kn}\cot\frac{\pi hr}{k}\,.$$

We find, on the other hand, since

$$t-[t]-\frac{1}{2}=-\sum_{n=1}^{\infty}\frac{\sin 2\pi nt}{\pi n}\qquad (t \text{ real and not integer}),$$

$$\sum_{m=1}^{k-1}2m\sin\theta\sin 2m\,\theta=\sum_{m=1}^{k}2m\sin\theta\sin 2m\,\theta\qquad (\theta=\pi hr/k)$$

$$=\sum_{m=1}^{k}\left(m\cos(2m-1)\,\theta-m\cos(2m+1)\theta\right)=-k\cos\theta\,,$$

the following result:

$$\pi\sum_{m=1}^{k-1}\frac{m}{k}\left(\frac{hm}{k}-\left[\frac{hm}{k}\right]-\frac{1}{2}\right)=-\pi\sum_{m=1}^{k-1}\frac{m}{k}\sum_{n=1}^{\infty}\frac{1}{\pi n}\sin\frac{2\pi hmn}{k}$$

$$=-\sum_{n=1}^{\infty}\frac{1}{kn}\sum_{m=1}^{k-1}m\sin\frac{2\pi hmn}{k}=-\sum_{n=0}^{\infty}\sum_{r=1}^{k-1}\frac{1}{k(r+kn)}\sum_{m=1}^{k-1}m\sin\frac{2\pi hmr}{k}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{r=1}^{k-1}\frac{1}{r+kn}\cot\frac{\pi hr}{k}\,,$$

and comparing this with (17), we find the required formula for the case $k>1$.

The rest of the present paper is devoted to the study of two properties of Ω.

First, we continue from (17) and derive for $k>1$

$$\Omega=-\frac{1}{2}\sum_{n=1}^{\infty}\sum_{r=1}^{k-1}\frac{1}{kn-r}\cot\frac{\pi hr}{k}\,,$$

which added to (17) side-by-side yields

$$2\Omega=\sum_{r=1}^{k-1}\frac{1}{2r}\cot\frac{\pi hr}{k}-\sum_{r=1}^{k-1}\cot\frac{\pi hr}{k}\sum_{n=1}^{\infty}\frac{r}{k^2n^2-r^2}$$

$$=\frac{1}{2}\sum_{r=1}^{k-1}\left(\cot\frac{\pi hr}{k}\right)\left(\frac{1}{r}-\sum_{n=1}^{\infty}\frac{2r}{k^2n^2-r^2}\right).$$

Since

(18) $$\cot s=\frac{1}{s}-\sum_{n=1}^{\infty}\frac{2s}{\pi^2n^2-s^2}\qquad (|s|<\pi)\,,$$

we have

$$\cot\frac{\pi r}{k}=\frac{k}{\pi}\left(\frac{1}{r}-\sum_{n=1}^{\infty}\frac{2r}{k^2n^2-r^2}\right),$$

and hence we get

(19) $$\Omega_{h,k}=\frac{\pi}{4k}\sum_{m=1}^{k-1}\cot\frac{\pi hm}{k}\cot\frac{\pi m}{k}\qquad(k>1),$$

which also holds for $k=1$, if an empty sum signifies zero. We have thus found two different expressions (4) and (19) for Ω.

Secondly, our arithmetic function $\Omega_{h,k}$ is known to satisfy a curious equality called the reciprocity formula for Dedekind sums, which asserts that, for coprime positive integers h and k,

(20) $$\sum_{m=1}^{k-1}\frac{m}{k}\left\{\frac{hm}{k}\right\}+\sum_{n=1}^{h-1}\frac{n}{h}\left\{\frac{kn}{h}\right\}=\frac{h^2+k^2+1}{12hk}-\frac{1}{4},$$

where we use the abbreviation $\{t\}=t-[t]-\dfrac{1}{2}$ for real t.

Rademacher and Whiteman [5] gave a simple arithmetic proof for (20). We shall give, in the following, a simple contour-integration proof for

(21) $$\frac{1}{k}\sum_{m=1}^{k-1}\cot\frac{\pi hm}{k}\cot\frac{\pi m}{k}+\frac{1}{h}\sum_{n=1}^{h-1}\cot\frac{\pi kn}{h}\cot\frac{\pi n}{h}=\frac{h^2+k^2+1}{3hk}-1,$$

which is equivalent to (20) on account of (19) and (4).

We put, for a complex variable $z=x+iy$ (x and y real),

$$g(z)=\cot z\cot\frac{z}{k}\cot\frac{hz}{k}.$$

We take $\alpha>\dfrac{1}{2}$ and define four paths of integration as follows: The path A begins at αi, thence goes vertically down to $(1/2)i$, then runs along the semi-circle $z=(1/2)e^{i\theta}$, $(1/2)\pi\leqq\theta\leqq(3/2)\pi$ with origin the centre counter-clockwise, and finally descends from $-(1/2)i$ down to $-\alpha i$; the path B starts at $-\alpha i$, proceeds horizontally to the right and ends at $k\pi-\alpha i$; we name C the path which begins at $k\pi-\alpha i$, ascends vertically up to $k\pi-(1/2)i$, then runs clockwise along the semi-circle $z=k\pi+(1/2)e^{i\theta}$, $(3/2)\pi\geqq\theta\geqq(1/2)\pi$, and finally proceeds from $k\pi+(1/2)i$ vertically upwards, terminating at $k\pi+\alpha i$; the last path D originates at $k\pi+\alpha i$, goes horizontally to the left and ends at αi.

We name K the contour obtained by successive junctions of the paths just obtained. Then, integrating once round K in the positive sense, we get

$$\int_K g(z)dz=\int_A+\int_B+\int_C+\int_D,$$

which equals $2\pi i$ times the sum of the residues of $g(z)$ at the poles inside K.

Since $g(z)$ has a period $k\pi$, we have

$$\int_A+\int_C=0.$$

We find also, uniformly in x,

$$\cot z\to\mp i \qquad (y\to\pm\infty),$$

and hence, as α tends to infinity,

$$\int_B+\int_D\to\int_0^{k\pi}(-i)dx-\int_0^{k\pi}idx=-2k\pi i.$$

Now we have, by (18) for instance, the expansion

$$z\cot z=1-\frac{1}{3}z^2+\cdots \qquad (|z|<\pi),$$

and hence, for small $|z|$,

$$(z\cot z)\left(\frac{z}{k}\cot\frac{z}{k}\right)\left(\frac{hz}{k}\cot\frac{hz}{k}\right)=1-\frac{h^2+k^2+1}{3k^2}z^2+\cdots.$$

Hence the residue of $g(z)$ at the origin is $-(h^2+k^2+1)/3h$. Further, all the poles of $g(z)$ inside K and different from the origin are $\pi m\ (m=1,2,\cdots,k-1)$ in the case $k>1$, and $(k/h)\pi n(n=1,2,\cdots,h-1)$ in the case $h>1$, with the residues

$$\cot\frac{\pi hm}{k}\cot\frac{\pi m}{k} \qquad \text{and} \qquad \frac{k}{h}\cot\frac{\pi kn}{h}\cot\frac{\pi n}{h}$$

respectively. Hence we find, making $\alpha\to\infty$,

$$\sum_{m=1}^{k-1}\cot\frac{\pi hm}{k}\cot\frac{\pi m}{k}+\frac{k}{h}\sum_{n=1}^{h-1}\cot\frac{\pi kn}{h}\cot\frac{\pi n}{h}=\frac{h^2+k^2+1}{3h}-k,$$

which is equivalent to (21).

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