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A Generalization of Laguerre Geometry, II.

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§ 8. Group of holonomy.

We can define the group of holonomy in our space, by developing a tangent hypersphere space along a closed curve as in the case of Riemannian space. Let us consider the relations between the group of holonomy and the structure of our space. In the first place we study the case in which the group of holonomy fixes a hypersphere. Next we consider the case in which the group of holonomy fixes two independent hyperspheres.

(A) The case in which the group of holonomy fixes the hypersphere of the form

$$V^\lambda=\begin{cases} V^i & (\lambda=i) \\ 0 & (\lambda=0). \end{cases}$$

In this case we have

$$\delta V^\lambda+dx^\lambda=dV^\lambda+\Gamma^{\cdot\lambda}_{\mu k}V^\mu dx^k+dx^\lambda=0. \tag{8·1}$$

That is

$$\partial V^i/\partial x^k+\Gamma^i_{jk}V^j+\delta^i_k=0, \quad \Gamma^0_{jk}V^j=0. \tag{8·2}$$

Hence the vector V^i forms so-called* *concurrent vector field.*

(B) The case in which the group of holonomy fixes the hypersphere of the form

$$V^\lambda=\begin{cases} 0 & (\lambda=i) \\ V^0 & (\lambda=0). \end{cases}$$

In this case we have

$$\delta V^\lambda+dx^\lambda=\begin{cases} \Gamma^i_{ok}V^0dx^k+dx^i=0, & (\lambda=i) \\ dV^0=0 & (\lambda=0). \end{cases} \tag{8·3}$$

Since dx^i are arbitrary, we get

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* K. Yano: Sur le parallélisme et la concourance dans l'espace de Riemann. Proc. Imp. Acad. Tokyo, 19 (1943), pp. 189–197.

$$(8\cdot4)\qquad V^0=\text{const},$$

$$(8\cdot5)\qquad \Gamma^i_{ok}=-\delta^i_k/V^0=C\delta^i_k$$

(C) The case in which the group of holonomy fixes the hypersphere of the form

$$V^\lambda=\begin{cases}V^i & (\lambda=i)\\ V^0 & (\lambda=0).\end{cases}$$

In this case we have

$$(8\cdot6)\qquad \delta V^\lambda+dx^\lambda=\begin{cases}dV^i+\Gamma^i_{jk}V^jdx^k+\Gamma^i_{ok}V^0dx^k+dx^i=0,\\ dV^0+\Gamma^o_{ij}V^idx^j=0.\end{cases}$$

Since dx^i are arbitrary, we obtain

$$(8\cdot7)\qquad \begin{cases}\partial V^i/\partial x^k+\Gamma^i_{jk}V^j+\Gamma^i_{ok}V^0+\delta^i_k=0,\\ \partial V^0/\partial x^k+\Gamma^o_{jk}V^j=0.\end{cases}$$

Putting $V_i=g_{ia}V^a$, we have

$$(8\cdot8)\qquad (\partial V_i/\partial x^k-\Gamma^a_{ik}V_a)+\Gamma^o_{ik}V^0+g_{ik}=0.$$

Hence it follows

$$(8\cdot9)\qquad \partial V_i/\partial x^k-\partial V_k/\partial x^i=0.$$

Therefore V_i must be a gradient of a scalar φ; i.e.,

$$(8\cdot10)\qquad V_i=\partial\varphi/\partial x^i.$$

Then (8·8) gives

$$(8\cdot11)\qquad V^j(\partial^2\varphi/\partial x^i\partial x^j-\Gamma^a_{ij}\partial\varphi/\partial x^a)+\Gamma^o_{ij}V^0V^j+\partial\varphi/\partial x^i=0.$$

On the other hand, we have from (8·7)

$$(8\cdot12)\qquad \frac{1}{2}\partial(V^0)^2/\partial x^i+\Gamma^o_{ij}V^0V^j=0.$$

By comparing (8·11) and (8·12) we obtain

$$(8\cdot13)\qquad \frac{1}{2}\frac{\partial}{\partial x^k}\left(g^{ab}\frac{\partial\varphi}{\partial x^a}\frac{\partial\varphi}{\partial x^b}\right)+\frac{\partial}{\partial x^k}\left(\varphi-\frac{1}{2}V^0V^0\right)=0.$$

Hence we have

(8·14) $$V^0=\sqrt{2\varphi+g^{ab}(\partial\varphi/\partial x^a)(\partial\varphi/\partial x^b)},$$

that is

(8·15) $$g_{\lambda\mu}V^\lambda V^\mu=g_{ij}V^iV^j-V^0V^0=-2\varphi.$$

Above relation shows that the tangential distance between the fixed hypersphere V^λ and each point of a hypersurface (φ=const.) must be constant.

In this case Γ^o_{ij} takes the form

(8·16) $$\Gamma^o_{ij}=-\frac{\{g_{ij}+(\partial^2\varphi/\partial x^i\partial x^j-\Gamma^a_{ij}\partial\varphi/\partial x^a)\}}{\sqrt{2\varphi+g^{ab}(\partial\varphi/\partial x^a)(\partial\varphi/\partial x^b)}}.$$

§ 9. The case in which the group of holonomy fixes two hyperspheres.

In the first place we study the case in which the group of holonomy fixes two hyperspheres

(9·1) $$V^\lambda=\begin{cases}V^i & (\lambda=i)\\ V^0 & (\lambda=0),\end{cases}\qquad W^\lambda=\begin{cases}0 & (\lambda=i)\\ W^0 & (\lambda=0).\end{cases}$$

Since W^λ is fixed, we have, as in § 8, Case (B),

(9·2) $$\Gamma^o_{ij}=Cg_{ij},\quad (C=\text{const.})$$

Therefore, we have from (8·7)

(9·3) $$\partial V^i/\partial x^k+\Gamma^i_{jk}V^j+(CV^0+1)\delta^i_k=0,\ \partial V^0/\partial x^k+Cg_{ik}V^i=0,$$

(9·4) $$V^i=g^{ia}\partial\varphi/\partial x^a,$$

φ being a certain scalar. Let us consider the properties of the hypersurface φ=const. Let $x^i=x^i(u^A)$, $(A=\dot{1}, \dot{2}, \cdots, \dot{n}-\dot{1})$ be a parametric representation for such hypersurface. By (9·4) we can put as follows:

(9·5) $$V^i=\rho n^i,$$

where n^i denotes the unit normal to the hypersurface. Then we have from (9·3)

(9·6) $$d(\rho n^i)+\Gamma^i_{jk}\rho n^j dx^k+(CV^0+1)dx^i=0,$$

that is

$$(9\cdot7)\quad \{n^i(\partial\rho/\partial u^A)+\rho(-\partial x^i/\partial u^B)H^{B}_{\cdot A}+(CV^0+1)\partial x^i/\partial u^A\}du^A=0,$$

where $H^{B}_{\cdot A}$ denotes the second fundamental tensor.

Since du^A are arbitrary, we have

$$(9\cdot8)\quad \partial\rho/\partial u^A=0,\quad \rho=\text{const.}$$

$$(9\cdot9)\quad H^{B}_{\cdot A}=\frac{CV^0+1}{\rho}\delta^B_A.$$

On the other hand we have from (9·3)

$$(9\cdot10)\quad \{\partial V^0/\partial u^A+C(\partial\varphi/\partial x^i)(\partial x^i/\partial u^A)\}du^A=0.$$

Since du^A are arbitrary and $\partial\varphi/\partial x^i\cdot\partial x^i/\partial u^A=0$, we have

$$(9\cdot11)\quad \partial V^0/\partial u^A=0,\quad V^0=\text{const.}$$

We find from (9·9), (9·10), (9·11) that $H^{B}_{\cdot A}$ takes the form :

$$(9\cdot12)\quad H^{B}_{\cdot A}=C'\delta^B_A.\quad (C'=\text{const.})$$

Thus the hypersurface ($\varphi=$const.) are totally umbilical and of constant mean curvature. Their orthogonal trajectries are given by

$$(9\cdot13)\quad dx^i/dt=V^i.$$

We have from (9·3), (9·13)

$$(9\cdot14)\quad d^2x^i/dt^2+\Gamma^i_{jk}(dx^j/dt)(dx^k/dt)+(CV^0+1)(dx^i/dt)=0.$$

Therefore they are geodesics.

Theorem. If the group of holonomy fixes two hyperspheres (V^i, V^0), (O, W^0), R_n *contains* ∞^1 *totally umbilical hypersurfaces with constant mean curvature, whose orthogonal trajectries are geodesics.*

§ 10. The case in which the group of holonomy fixes two independent hyperspheres V^λ, W^λ.

In this case we have

$$(10\cdot1)\quad \partial V^i/\partial x^j+\Gamma^i_{aj}V^a+\Gamma^i_{oj}V^0+\delta^i_j=0,\quad \partial V^0/\partial x^j+\Gamma^o_{aj}V^a=0,$$

$$(10\cdot2)\quad \partial W^i/\partial x^j+\Gamma^i_{aj}W^a+\Gamma^i_{oj}W^0+\delta^i_j=0,\quad \partial W^0/\partial x^j+\Gamma^o_{aj}W^a=0.$$

Let us consider a hypersphere of the form, (10·3) $$X^\lambda = AV^\lambda + (1-A)W^\lambda,$$

where A is an arbitrary constant. We find from (10·1), (10·2) that the group of holonomy fixes X^λ. That is

(10·4) $$\partial X^\lambda/\partial x^k + \Gamma^\lambda_{\mu k} X^\mu + \delta^\lambda_k = 0.$$

Therefore we find, as in § 8, Case (C), that X^i is a gradient of a scalar Ω

(10·5) $$X^i = g^{ia}\partial\Omega/\partial x^a,$$

and that

(10·6) $$g_{\lambda\mu}X^\lambda X^\mu = -2\Omega.$$

When A moves over all real numbers, (10·3) generates ∞^1 hyperspheres. The group of holonomy fixes them all. In order that $X^\lambda = AV^\lambda + (1-A)W^\lambda$ passes a point x^i, it is necessary and sufficient that the equation with respect to A

(10·7) $$0 = g_{\lambda\mu}(x)X^\lambda X^\mu = A^2(g_{\lambda\mu}V^\lambda V^\mu - 2g_{\lambda\mu}V^\lambda W^\mu + g_{\lambda\mu}W^\lambda W^\mu) + 2A(g_{\lambda\mu}V^\lambda W^\mu - g_{\lambda\mu}W^\lambda W^\mu) + g_{\lambda\mu}W^\lambda W^\mu$$

has real roots. That is

(10·8) $$(g_{\lambda\mu}V^\lambda W^\mu - g_{\lambda\mu}W^\lambda W^\mu)^2 \geqq (g_{\lambda\mu}W^\lambda W^\mu)(g_{\lambda\mu}V^\lambda V^\mu - 2g_{\lambda\mu}V^\lambda W^\mu + g_{\lambda\mu}W^\lambda W^\mu).$$

When (10·7) is satisfied by certain A, (10·7) gives a hypersurface. From (10·6) and (10·7), Ω must be constant along the hypersurface. From (10·5) the vector X^i must be orthogonal to the hypersurface. Therefore the hypersphere $X^\lambda = AV^\lambda + (1-A)W^\lambda$ touches the hypersurface (10·7).

Definition. When a hypersphere which tourches a hypersurface is fixed by the group of holonomy along the hypersurface, we call it W-hypersurface. Let us find the conditions for W-hypersurface.

Let $x^i = x^i(u^A)$, $(A = \dot{1}, \dot{2}, \cdots, \dot{n}-\dot{1})$ be a parametric representation for the hypersurface. A hypersphere which touches the hypersurface takes the form

$$(10\cdot 9) \qquad V^\lambda = \begin{cases} \rho n^i & (\lambda = i) \\ \varepsilon\rho & (\lambda = 0), \end{cases} \qquad (\varepsilon = \pm 1)$$

where n^i is the unit normal and ρ is a scalar. Since V^λ is fixed by the group of holonomy along the hypersurface, we have

$$(10\cdot 10) \qquad \delta V^\lambda + dx^\lambda = \begin{cases} d(\rho n^i) + \Gamma^i_{jk}\rho n^j dx^k + \Gamma^i_{ok}(\varepsilon\rho)\, dx^k + dx^i = 0, \\ d(\varepsilon\rho) + \Gamma^o_{ij}\rho n^i dx^j = 0, \end{cases}$$

that is

$$(10\cdot 11) \qquad \begin{cases} [n^i(\partial\rho/\partial u^A + \varepsilon\rho\Gamma^a_{ok} t_a dx^k/\partial u^A) \\ \qquad + \partial x^i/\partial u^B\{\rho(-H^{\cdot B}_{\cdot A} + \varepsilon\Gamma^i_{ok}\xi^B_i\, \partial x^k/\partial u^A) + \delta^B_A\}]du^A = 0 \\ (\varepsilon\partial\rho/\partial u^A + \Gamma^o_{ij}\rho n^i\, \partial x^j/\partial u^A)\, du^A = 0, \end{cases}$$

where

$$\xi^B_i = g^{BA} g_{ij}\partial x^j/\partial u^A,\quad g_{ij} n^i = t_j,\quad g_{AB} = g_{ij}(\partial x^i/\partial u^A)(\partial x^j/\partial u^B),\quad g^{AC} g_{BC} = \delta^A_B.$$

Since du^A are arbitrary, we have

$$(10\cdot 12) \qquad \begin{cases} \varepsilon(\partial\rho/\partial u^A) + \Gamma^o_{ij}\rho n^j \partial x^i/\partial u^A = 0, \\ \rho(-H^{\cdot B}_{\cdot A} + \varepsilon\Gamma^i_{ok}\xi^B_i\, \partial x^k/\partial u^A) + \delta^B_A = 0. \end{cases}$$

Especially, if $\Gamma^i_{ok} = C\delta^i_k$, we have

$$(10\cdot 13) \qquad \partial\rho/\partial u^A = 0, \quad H^{\cdot B}_{\cdot A} = (\rho^{-1} + \varepsilon C)\delta^B_A.$$

Therefore, if $\Gamma^i_{ok} = C\delta^i_k$, W—hypersurface becomes totally umbilical one, and if especially C is constant, it becomes totally umbilical hypersurface of constant mean curvature. If (10·8) is satisfied in a domain of R_n, such a domain contains ∞^1 W—hypersurfaces. Their orthogonal trajectries are given by

$$(10\cdot 14) \qquad dx^i/ds = \{AV^i + (1-A)W^i\}/\sqrt{AV^0 + (1-A)W^0},$$

where s denotes the arc length. On the other hand we have from (10·1), (10·2)

$$(10\cdot 15) \qquad \begin{cases} dV^i/ds + \Gamma^i_{jk}V^j dx^k/ds + \Gamma^i_{ok}V^0 dx^k/ds + dx^i/ds = 0, \\ dW^i/ds + \Gamma^i_{jk}W^j dx^k/ds + \Gamma^i_{ok}W^0 dx^k/ds + dx^i/ds = 0. \end{cases}$$

From (10·14), (10·15) we have

$$(10\cdot16)\quad \delta^3x^i/ds^3+P\delta^2x^i/ds^2+Qdx^i/ds+\Gamma^i_{ok}(Rdx^k/ds+S\delta^2x^k/ds^2)$$
$$+T\Gamma^i_{ok;l}\frac{dx^k}{ds}\frac{dx^l}{ds}=0,$$

where P, Q, R, S, T, are certain functions of s, and

$$\delta^2x^i/ds^2=d^2x^i/ds^2+\Gamma^i_{jk}(dx^j/ds)(dx^k/ds),$$
$$\delta^3x^i/ds^3=d/ds(\delta^2x^i/ds^2)+\Gamma^i_{jk}(\delta^2x^j/ds^2)(dx^k/ds)$$
$$\Gamma^i_{ok;l}=\partial\Gamma^i_{ok}/\partial x^l+\Gamma^a_{ok}\Gamma^i_{al}-\Gamma^i_{oa}\Gamma^a_{kl}.$$

Especially, if $\Gamma^i_{ok}=C\delta^i_k$, (10·16) becomes

$$(10\cdot17)\qquad (\delta^3x^i/ds^3)+\bar{P}\delta^2x^i/ds^2+\bar{Q}dx^i/ds=0,$$

where $\bar{P}$, $\bar{Q}$ are certain functions of s. Thus we have the theorem.

Theorem. If the group of holonomy fixes two independent hyperspheres V^λ, W^λ, *and if* (10·8) *is satisfied in a domain of* R_n, *such a domain contains* ∞^1 *W-hypersurfaces, whose orthogonal trajectries are given by* (10·16). *If especially* $\Gamma^i_{ok}=C\delta^i_k$, *it contains* ∞^1 *totally umbilical hypersurfaces whose orthogonal trajectries are given by* (10·17).