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On the Sequence of Additive Set Functions

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In this note we shall discuss three problems on the sequence of additive set fuuctions. In § 1 we prove the Vital-Hahn-Saks theorem in the space with infinite measure and, as its application, Schur's theorem on the equivalence of the strong and the weak convergence in the space (l). We shall remark some convergence theorems on the sequence of Riemann integrals in § 2 and Lebesgue integrals in § 3.

1. The Vitali-Hahn-Saks theorem.

The Vitali-Hahn-Saks theorem [5] is formulated in the following form.

Theorem 1. *Let $\mathfrak{M}$ be the family of all measurable sets E of an abstract space M with total measure $\mu(M)<\infty$. If $\{F_n(E)\}$ is completely additive (c.a.) and absolutely continuous (a. c.) with respect to $\mu(E)$ and* $\lim_{n\to\infty} F_n(E) = F(E)$ *for all $E \in \mathfrak{M}$, then $\{F_n(E)\}$ is uniformly absolutely continuous and $F(E)$ is c.a. and a.c.*

This theorem is also valid in the case $\mu(M)=\infty$, that is,

Theorem 2. *If there is a sequence $\{M_n\}$ wth finite measure such as $M=\bigcup_{n=1}^{\infty} M_n$, then the conclusion of Theorem 1 is valid, i.e., if $\{F_n(E)\}$ is c.a. and a.c. and* $\lim_{n\to\infty} F_n(E)=F(E)$ *for all $E \in \mathfrak{M}$, then for any positive ε, there are $\delta(\varepsilon)$, $m_0(\varepsilon)$ and $n_0(\varepsilon)$ such that for $\mu(E\cap M_i)<\delta$ $(i=1,2,\cdots,m_0)$, we have $F_n(E)<\varepsilon$ for all $n>n_0$.*

Proof. We have $E=\bigcup_{n=1}^{\infty}(E\cap M_n)$ *for any* $E \in \mathfrak{M}$. If we put

$$\nu(E)=\sum_{n=1}^{\infty}\frac{\mu(E\cap M_n)}{2^n\{\mu(M_n)+1\}},$$

then $\nu(M)<\infty$ and $\nu(E)$ is c.a. and a.c. with respect to $\mu(E)$. Since $F_n(E)$ is c.a. and a.c. with respect to $\nu(E)$, applying Theorem 1 to $F_n(E)$ and $\nu(E)$, we get the theorem.

As an application of Theorem 2, we can prove Schur's theorem. In Banach's book [1], the theorems of linear transformation of infinite sequences are proved from a general theorem (the Banach-Steinhaus theorem which is essentially a category theorem), but Schur's theorem is only proved by the direct calculation. Since Theorem 1 is proved by a category theorem, our way of establishing Schur's theorem may be of some interest.

Theorem 3. (I. Schur [6] cf. Banach [1] p. 137.). *In the space* (l), *the weak convergence is equivalent to the strong convergence.*

Proof. By the Banach-Steinhaus theorem a necessary and sufficient condition for the weak convergeece of the sequence $\{x^{(n)}\} \in (l)$ where

$$x^{(n)} = \{a_1^{(n)},\ a_2^{(n)}, \cdots, a_i^{(n)}, \cdots\}$$

is that

(1) $$\sum_{i=1}^{\infty} |a_i^{(n)}| \leqq M, \text{ for all } n,$$

(2) $\lim\limits_{n\to\infty} \sum\limits_{i \in E} a_i^{(n)}$ exists where E is any subset of natural numbers. If we give measure 1 for any natural number, then the set M of all natural numbers has an infinite measure and $M = \cup M_i$, where M_i is a set consisting of a natural number and $\mu(M_i)=1$. Put $F_n(E) = \sum\limits_{i \in E} a_i^{(n)}$, then $\{F_n(E)\}$ converges for all E by (2). From Theorem 2, we get for $n > n_0$,

(3) $$\sum_{i=n_0}^{\infty} |a_i^{(n)}| < \varepsilon/2.$$

If $\{a_i^{(n)}\} \equiv x^{(n)}$ converges weakly to 0, then we have evidently

(4) $$\lim_{n\to\infty} a_i^{(n)} = 0 \qquad (i = 1, 2, \cdots, n_0).$$

From (3) and (4), we get

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} |a_i^{(n)}| = 0,$$

this means that $x^{(n)} \equiv \{a_i^{(n)}\}$ converges strongly to 0. The case where the weak limit is $s \neq 0$, is reduced to the above case.

Theorem 4. *When the limit* $F_n(E)$ *exists for all* $E \in \mathfrak{M}$, *the limit function* $\lim\limits_{n\to\infty} F_n(E) = F(E)$ *is c.a. and a.c., even if* $\mu(M) = \infty$ *and* M *is not an enumerable sum of sets with finite measure.*

(If $\mu(M) < \infty$ or $\mu(M) = \infty$ and M is an enumerable sum of sets with finite measure, the theorem is above proved.)

Proof. If we put

$$\nu(E) = \sum_{n=1}^{\infty} \frac{V_n(E)}{2^n [V_n(M) + 1]}$$

where $V_n(E)$ denotes the total variation of $F_n(E)$, then $\nu(E)$ is c.a., $\nu(E) < \infty$ and $\nu(E)$ is a.c. with respect to $\mu(E)$. Applying Theorem 1 to $\{F_n(E)\}$ and $\nu(E)$, we get the theorem.

Theorem 5. *If $\mu(M)=\infty$, $\int_E f_n(x)d\mu(x)$ converges as $n\to\infty$ for all $E \in \mathfrak{M}$, then the limit function is the indefinite integral.*

Proof. For our purpose it is sufficient to observe the set

$$E=\left\{x \middle| \sup_n |f_n(x)| \neq 0\right\}.$$

Since $\cup E$ satisfies the condition of Theorem 2, we get the theorem.

N.B. If $\mu(M)=\infty$ and M is not an enumerable sum of sets with finite measure, then $F_n(E)$ are not necessarily indefinite integrals. So Theorem 4 and Theorem 5 is not equivalent.

2. Helly's theorem for separable metric space.

Helly's theorem has been extended to n-dimensional Euclidean space by Bochner [2] and Frostmann [3]. Kryloff-Bogoliouboff [4] have discused the convergency of $\int_M f(x)d\mu_n(x)$ where M is a compact metric space. We shall consider this theorem from the convergency of the sequence $\{\mu_n(E)\}$, after Helly.

Definition 1. *Let $\mu(E)$ be a c.a. set-function and E_f and E_0 be closure and interior of E, respectively. If $\mu(E_f-E_0)=0$, then E is said to be a continuous set of μ. If $\mu_n(E)$ converges to $\mu(E)$ for all continuous set of μ, then $\mu_n(E)$ is said to converge to $\mu(E)$.*

Definition 2. *We shall call $\mathfrak{M}$ a net in a metric space M provided that $\mathfrak{M}$ consists of finite or enumerable sets measurable $(\mathfrak{B})$, mutually exclusive and covering the space M. The sets constituting a net will be called its meshes. A sequence $\{\mathfrak{M}_n\}$ of nets will be termed regular, if each mesh of $\mathfrak{M}_{n+1}$ (where $n>0$) is contained in a mesh of $\mathfrak{M}_n$ and further the maximum of diameter of meshes of $\mathfrak{M}_n$ converges to 0 as $n\to\infty$.*

Then it is easy to see that there exists a regular sequence of nets in a separable metric space, by Lindelöf's convering theorem. Further we shall denote meshes of nets by I.

Theorem 6. *Let M be a separable metric space. If $0\leq\mu_n(E)\leq K$ for $n=1,2,\cdots$, and sets measurable $(\mathfrak{B})$ are all μ-measurable, then we can select $\mu_{n_k}(E)$ such that $\mu_{n_k}(E)\to\mu(E)$ as $n\to\infty$.*

Proof. Since meshes are enumerable in all, we can select μ_{n_k} such that $\mu_{n_k}(I_o)$ and $\mu_{n_k}(I_f)$ exist for all I. Let O and F be arbitrary open and closed sets respectively, and put

$$\mu(O)=\underset{I_f\subset O}{\text{l.u.b}}\left\{\lim_{k\to\infty}\mu_{n_k}(I_f)\right\},$$

$$\mu(F)=\underset{I_0\supset F}{\text{g.l.b}}\left\{\lim_{k\to\infty}\mu_{n_k}(I)\right\}.$$

Then we have

$\mu(O)\leqq\varliminf_{k\to\infty}\mu_{n_k}(O)$, $\mu(F)\geqq\varlimsup_{k\to\infty}\mu_{n_k}(F)$. It is also easy to see that μ are monotone and finitely additive functions of any O and F, respectively. Relation between $\mu(O)$ and $\mu(F)$ is given by

$$\mu(O)=\underset{F\subset O}{\text{l.u.b}}\{\mu(F)\},\quad \mu(F)=\underset{O\subset F}{\text{g.l.b}}\{\mu(O)\}.$$

For any set E, we put $\underline{\mu}(E)=\underset{F\subset E}{\text{l.u.b}}\{\mu(F)\}$, $\overline{\mu}(E)=\underset{O\supset F}{\text{g.l.b}}\{\mu(O)\}$, which are termed inner and outer measures of E. If $\underline{\mu}(E)=\overline{\mu}(E)$, then E is called to be measurable. Then evidenly any Borel set is measurable. If E is any continuous set, then we have

$$\mu(E)=\mu(E_o)\leqq\varliminf_{k\to\infty}\mu_{n_k}(E_o)\leqq\varliminf_{k\to\infty}\mu_{n_k}(E)\geqq\varlimsup_{k\to\infty}\mu_{n_k}(E)$$

$$\geqq\varlimsup_{k\to\infty}\mu_{n_k}(E_f)\leqq\mu(E_f)\leqq\mu(E).$$

Thus $\mu_{n_k}(E)\to\mu(E)$ as $k\to\infty$, which proves the theorem.

Theorem 7. *If M is a compact metric space and $\mu_n(E)\to\mu(E)$ as $n\to\infty$, then, for any continuous function $f(x)$,*

$$\lim_{n\to\infty}\int_M f(x)d\mu_n(x)=\int_M f(x)\ d\mu(x),$$

where the integral is taken in the Riemann sense.

Proof. Riemann integral can be approximated by Riemann sums where the sets of division are continuous sets.

Theorem 8. (Kryloff-Bogoliouboff [4]). *Let M be a compact metric space. If $0\leqq\mu_n(E)\leqq K$ for $n=1,2,\cdots$, then we can select $\mu_{n_k}(E)$ such that for any continuous $f(x)$,*

$$\lim_{k\to\infty}\int_M f(x)\ d\mu_{n_k}(x)=\int_M f(x)\ d\mu(x),$$

where the integral is taken in the Riemann sense.

Proof. This theorem is immediate from the above two theorems.

3. Convergence of the seqnence of Lebesgue integral.

Theorem 9. *Let $f(x)$ be bounded and Borel-measurable function, and $\{\mu_n(E)\}$ be a sequence of c.a measure functions (positive or not) such that the set measurable ($\mathfrak{B}$) is μ_n-measurable. If $\int_M |d\ (\mu_m - \mu_n)| \to 0$ as $m,n \to \infty$, then there exists a c.a. set function $\mu(E)$ such that*

$$\lim_{n\to\infty}\int_M f(x)\ d\mu_n(x) = \int_M f(x)\ d\mu(x),$$

where the integral is taken in the Lebesgue sense.

Proof. Since the sequence $\{\mu_n(E)\}$ converges for all Borel sets, $\mu(E)$ is c.a. by Theorem 1. Since

$$\left|\int_M f(x)\ d\mu_n(x) - \int_M f(x)d\ (\mu_n - \mu)\right| \leqq \left|\int_M f(x)d\ (\mu_n - \mu)\right|$$

$$\leqq \int_M |f(x)||d(\mu_n - \mu)| \leqq K\int_M |d\ (\mu_n - \mu)| \to 0,$$

we get the theorem.

Theorem 10. *Let $\{P_n(E) | E \in \mathfrak{B}\}$ be a sequence of c.a. measure functions such as*

$$\sum_{n=1}^{\infty}\int_M |dP_n| < \infty,$$

then $\sum_{n=1}^{\infty} P_n(E)$ converges to a measure function $P(E)$. Further if $f(x)$ is bounded and Borel-measusable, then

$$\sum_{n=1}^{\infty}\int_M f(x)d\ P_n(x) = \int_M f(x)d\ P(x).$$

Proof. Let us put

$$\sum_{i=1}^{\infty} p_i(E) = \mu_n(E),$$

then

$$\int_M |d(\mu_n - \mu_m)| = \int_M |d(P_{n+1} + \cdots + P_m)|$$

$$\leqq \int_M \sum_{i=n+1}^{m} |dp_i| \to 0, \text{ as } m,n \to \infty.$$

Thus the theorem is reduced to the above theorem.

This theorem is proved by F. Yagi [7], in case M is the one-dimensional Euclidean space.

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