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On Riemann Surfaces, on which no Bounded Harmonic Function Exists

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Let F be a Riemann surface spread over the z-plane, on which *no one-valued, bounded and non-constant harmonic function exists.* If F possesses no Green's function, the above condition is satisfied as Myrberg proved[1]. Let F_ρ be a connected piece of F lying above an open disc K: $|z-z_0|<\rho$, which is cut off from F by the circumference $|z-z_0|=\rho$. By a function $z=z_\rho(x)$, we map the universal covering surface $F_\rho^{(\infty)}$ of F_ρ conformally on $|x|<1$. Then, we shall prove:

Theorem 1. *The function* $(z_\rho(x)-z_0)/\rho$ *belongs to* U-*class in Seidel's sense.*[2]

By Frostman's theorem[3] on functions belonging to U-class, we have immediately the following

Theorem 2. F_ρ *covers every point in* K *except possibly a set of logarithmic capacity zero.*

In other words, *if a connected piece above a disc does not cover a set of positive capacity, then there exists a one-valued, bounded and non-constant harmonic function on the original Riemann surface.*

Some consequences of this theorem will be stated later.

For the proof we use the following extension of Löwner's theorem.

Lemma. (*Kametani-Ugaheri*[4]). *Let* $w=w(x)$ *be regular in* $|x|<1$ *aud* $w(0)=0$, $|w(x)|<1$, *and let* e *be an arbitrary set of points* $e^{i\theta}$ *on* $|x|=1$, *such that* $w(e^{i\theta})=\lim_{r\to 1} w(re^{i\theta})$ *exists and* $|w(e^{i\theta})|=1$. *Further, let* E *be the set of* $w(e^{i\theta})=e^{i\varphi}$ *on* $|w|=1$ *for* $e^{i\theta}\in e$. *Then, we have* $m_*e\leqq m^*E$, *where* m_* *and* m^* *denote the inner and outer linear measure of the sets respectively.*

Proof of Theorem 1.

Let it be remarked before the proof: we can assume that F does not cover three points a, b, c lying outside K on the z-plane. In fact, we can exclude, if necessary, all the points lying above a, b, c which are isolated points on F and have no influence on the existence of bounded harmonic function on F.

Since $z=z_\rho(x)$ is regular and bounded in $|x|<1$, $\lim_{r\to 1} z_\rho(re^{i\theta})=z_\rho(e^{i\theta})$ exists for almost all $e^{i\theta}$ on $|x|=1$. Let e_K be the set of $x=e^{i\theta}$, such that

$z_\rho(e^{i\theta}) \in K$ i. e. $|z_\rho(e^{i\theta})-z_0|<\rho$. Since $z_\rho(e^{i\theta})$ is measurable in θ, and since K is an open set, e_K is a measurable set. *Under the assumption that* $me_K>0$, *we shall construct a one-valued, bounded and non-constant harmonic function on* F.

First, we divide the open disc K into a countable number of half-closed rectangles $Q_1, Q_2, \cdots$, whose sides are parallel to the coordinate axes of the z-plane. Let e_n be the set of $x=e^{i\theta}$, such that $z_\rho(e^{i\theta}) \in Q_n$. Then, since $\sum_{n=1}^{\infty} e_n=e_K$, there exists an index n, for which $me_n>0$. Suppose that, for any such division, $me_n>0$ would hold for only one index n corresponding to one rectangle Q_n. Then, by repeated subdivision of Q_n, we see easily that there would exist a point z_1 in K, such that $z_\rho(e^{i\theta})=z_1$ for almost all $e^{i\theta} \in e_K$. Then, by Lusin-Priwaloff's theorem[5], e_K must be of measure zero, which is a contradiction. Hence, dividing K suitably, we can find two rectangles Q, Q' ($Q, Q' \subset K$, $QQ'=0$) satisfying the condition: the sets e_Q, $e_{Q'}$ of $x=e^{i\theta}$, such that $z_\rho(e^{i\theta}) \in Q$, $\in Q'$ respectively, are both of positive measure.

Let $e^{i\theta}$ be a point of e_Q, then, since $\lim_{r\to 1} z_\rho(re^{i\theta})=z_\rho(e^{i\theta}) \in Q$, the curve $z=z_\rho(re^{i\theta})$ ($e^{i\theta} \in e_Q$, $0\leq r<1$) on $F_\rho^{(\infty)}$ defines an accessible boundary point $\Omega(x=e^{i\theta}; Q)$ of $F_\rho^{(\infty)}$, whose projection belongs to Q. Let $F^{(\infty)}$ be the universal covering surface of F, so that $F_\rho^{(\infty)}$ is a connected piece of $F^{(\infty)}$ above the disc K. Then, $\Omega(x=e^{i\theta}; Q)$ is, at the same time, an accessible boundary point of $F^{(\infty)}$.

Since F does not cover three points a, b, c on the z-plane, $F^{(\infty)}$ is of hyperbolic type and can be mapped conformally on $|w|<1$ by $z=z(w)$, $w=w(z)$, so that the point $z=z_\rho(0)$ on $F^{(\infty)}$ corresponds to $w=0$. $z=z(w)$ is meromorphic and $\neq a$, $\neq b$, $\neq c$ in $|w|<1$, and is automorphic with respect to a Fuchsian group G, whose fundamental domain corresponds to F in one-to-one manner.

Consider the function $w=w_\rho(x)=w(z_\rho(x))$. When w moves along the curve $w=w_\rho(re^{i\theta})$ ($e^{i\theta} \in e_Q$, $0\leq r<1$), $z=z(w)$ tends to $\Omega(x=e^{i\theta}; Q)$ as is readily seen. Hence, this curve must end at a point $w_\rho(e^{i\theta})=e^{i\varphi}$ on $|w|=1$. In fact, otherwise, $z=z(w)$ would reduce to a constant by the well-known Gross-Koebe's theorem. Let E_Q be the set of $w_\rho(e^{i\theta})=e^{i\varphi}$ for $e^{i\theta} \in e_Q$, then, by the lemma, $0<me_Q\leq m^*E_Q$. Further, by Iversen-Lindelöf's theorem, $\lim_{R\to 1} z(Re^{i\varphi})=z(e^{i\varphi}) \in Q$ exists for any $e^{i\varphi} \in E_Q$.

Let M_Q be the set of all the points $e^{i\varphi}$ on $|w|=1$, such that

$\lim_{R\to 1} z(R^{i\varphi}) = z(e^{i\varphi})$ exists and $\in Q$. It is easily seen that M_Q is measurable. Since $E_Q \subset M_Q$, we have $0 < m^* E_Q \leqq m M_Q$. Further, M_Q is invariant by the Fuchsian group G, as is seen by Iversen-Lindelöf's theorem.

Starting from the set $e_{Q'}$ on $|x|=1$, we obtain similary another set $M_{Q'}$ on $|w|=1$, such that $mM_{Q'}>0$ and $\lim_{r\to 1} z(Re^{i\varphi}) = z(e^{i\varphi}) \in Q'$ for and $e^{i\varphi} \in M_{Q'}$. Since Q and Q' are disjoint, so are the sets M_Q and $M_{Q'}$, and from $mM_Q>0$ and $mM_{Q'}>0$ we obtain $0<mM_Q<2\pi$.

Then, the harmonic function defined by the Poisson integral

$$u(w) = \frac{1}{2\pi}\int_{M_Q} \frac{1-R^2}{1+R^2-2R\cos(\psi-\varphi)} d\psi \quad (w=Re^{i\varphi})$$

is $\not\equiv$ const., and is automorphic with respect to the Fuchsian group G, since M_Q is invariant by G. Hence, $u(z)=u(w(z))$ is a one-valued, bounded and non-constant harmonic function on F. Thus Theorem 1 is proved.

The following corollaries are derived from Theorem 2.

Corollary 1. *For any $z \in K$, let $0 \leqq n(z) \leqq \infty$ denote the number of sheets of F_ρ above z, and let Γ be the set of points $z \in K$, such that $n(z) < N = \sup_{z\in K} n(z) \leqq \infty$. Then, Γ is of (inner) capacity zero.*

The same was proved by Y. Nagai[6] and M. Tsuji[7] under the more restrictive assumption that F possesses no Green's function.

To deduce Corollary 1, it suffices to prove:

Lemma. *If* Cap. $\Gamma>0$, *we can find an open disc K_1 contained in K, such that a connected piece of F_ρ above K_1 does not cover a set of positive capacity in K_1.*

Proof. For any integer n, we denote by Γ_n the set of points $z \in K$, such that $n(z) \leqq n$. Then, Γ_n is closed with respect to K and $\Gamma_{n-1} \subset \Gamma_n$, $\sum_{n<N} \Gamma_n = \Gamma$. Hence, for a value of $n<N$ we have Cap. $\Gamma_n>0$. Let m be the smallest of such indices. Since $m<N$, $K-\Gamma_m$ is a non-empty open set. Hence, the boundary set B_m of Γ_m with respect to K is not empty and Cap. $B_m>0$. Then, since $B_m = B_m(\Gamma_m - \Gamma_{m-1}) + B_m\Gamma_{m-1}$ and Cap. $B_m\Gamma_{m-1} \leqq$ Cap. $\Gamma_{m-1}=0$, we have Cap. $B_m(\Gamma_m-\Gamma_{m-1})>0$. Hence, we can find a point $z_1 \in B_m(\Gamma_m - \Gamma_{m-1})$, such that, for any small disc K_1 about z_1, Cap. $K_1 B_m(\Gamma_m - \Gamma_{m-1})>0$ and consequently Cap. $K_1\Gamma_m>0$. Since $z_1 \in \Gamma_m - \Gamma_{m-1}$, F_ρ has exactly m discs above K_1, if K_1 is sufficiently small (ν-sheeted disc counted as ν discs). Besides these m discs, F_ρ has at least one connected piece above K_1. In fact, since $z_1 \in B_m$, K_1 contains points z, such that $n(z)>m$. Since any point of K_1 is already covered by the mentioned

m discs, this connected piece does not cover the set $K_1\Gamma_m$ of positive capacity, q. e. d.

Corollary. 2. *The set Γ_Ω of the projections of direct accessible boundary points Ω of F is of (inner) capacity zero.*

A direct accessible boundary point is, by definition, an accessible boundary point Ω, such that, for sufficiently small $\rho>0$, the ρ-neighbourhood of Ω does not cover the projection of Ω. From this it is easily seen that *F possesses Iversen's property.*[8]

Corollary 2 contains the following Kametani-Tsuji-Noshiro's theorem[9]: *Let $z=f(w)$ be k-valued algebroidal outside a bounded closed set of capacity zero on the w-plane, and $w=\varphi(z)$ be its inverse function. Then, the set of projections on the z-plane of the direct transcendental singularities of $w=\varphi(z)$ is of capacity zero.* In fact, the Riemann surface of $z=f(w)$ spread over the w-plane, which is conformally equivalent to that of $w=\varphi(z)$ spread over the z-plane, possesses no Green's function, as can be seen easily.

To deduce Corollary 2, it suffices to prove:

Lemma. *If* Cap. $\Gamma_\Omega>0$, *we can find a disc K on the z-plane, such that a connected piece of F above K does not cover a set of positive capacity in K.*

Proof. Let $\{z_\lambda\}$ $(\lambda=1,2,\cdots)$ be a sequence of all the rational points on the z-plane, and $K_{\lambda\mu}$ $(\mu=1,2,\cdots)$ be the disc $|z-z_\lambda|<1/\mu$. We denote by $F_{\lambda\mu\nu}$ $(\nu=1,2,\cdots)$ the connected pieces of F above $K_{\lambda\mu}$. Further, let $\Gamma_{\lambda\mu\nu}$ be the set of points in $K_{\lambda\mu}$, which are not covered by $F_{\lambda\mu\nu}$. By the definition of direct accessible boundary points, we see easily that $\sum_{\lambda,\mu,\nu}\Gamma_{\lambda\mu\nu}\supset\Gamma_\Omega$, so that Cap. $(\sum_{\lambda,\mu,\nu}\Gamma_{\lambda\mu\nu})>0$. Since $\Gamma_{\lambda\mu\nu}$ are Borel sets, it follows that Cap.$\Gamma_{\lambda\mu\nu}>0$ for certain values of λ, μ and ν, q. e. d.

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