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Algebraic Correspondences between Algebraic Varieties

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Let $\boldsymbol{V}^n$ be a complete Variety without multiple Point in the algebraic geometry with the universal domain of all complex numbers. Let k be the smallest field of definition for $\boldsymbol{V}$, then to every $\boldsymbol{V}$-divisor $\boldsymbol{Y}$ is attached the smallest extension $k(\boldsymbol{P})$ of k over which $\boldsymbol{Y}$ is rational. In view of Zariski's results it would not be too restrictive to assume that $\boldsymbol{P}$ is a generic Point of a suitable complete Variety $\boldsymbol{U}^m$ without multiple Point, which is defined over the algebraic closure K of k in $k(\boldsymbol{P})$. There exists then a $(\boldsymbol{U}\times\boldsymbol{V})$-divisor $\boldsymbol{X}$, which is rational over K, such that

$$\boldsymbol{X}\cdot(\boldsymbol{P}\times\boldsymbol{V})=\boldsymbol{P}\times\boldsymbol{Y}.$$

We call such a $\boldsymbol{X}$ a *correspondence between* $\boldsymbol{U}$ *and* $\boldsymbol{V}$, since it is a correspondence in the classical sense if both $\boldsymbol{U}$ and $\boldsymbol{V}$ are curves. In the above connection a problem concerning $\boldsymbol{V}$-divisors can be translated into a problem on correspondences and vice versa. The $\boldsymbol{V}$-divisor $\boldsymbol{Y}$ varies in a linearly equivalent system for the variable Point $\boldsymbol{P}$ when and only when $\boldsymbol{X}$ is of the form

$$\boldsymbol{X}=\boldsymbol{Y}_1\times\boldsymbol{V}+\boldsymbol{U}\times\boldsymbol{Y}_2+(\varphi),$$

where $\boldsymbol{Y}_1$ is a $\boldsymbol{U}$-divisor, $\boldsymbol{Y}_2$ a $\boldsymbol{V}$-divisor and φ a function on $\boldsymbol{U}\times\boldsymbol{V}$. We call such a $\boldsymbol{X}$ a *correspondence with valence zero* in agreement with the case of curves. Since such correspondences form a submodule in the module of all correspondences, we can consider their residue-class module. We call this module the *module of correspondences* and we denote it by $C(\boldsymbol{U},\boldsymbol{V})$.

In this paper we shall assume that both $\boldsymbol{U}$ and $\boldsymbol{V}$ have non-singular projective models, which, we hope, may not be a restriction. Let then

$$\Phi_{\alpha i}\ (1\leqq i\leqq q_\alpha)$$

be a base of the Picard differentials of the first kind, and let

$$\gamma_{\alpha i}\ (1\leqq i\leqq 2q_\alpha)$$

We shall use freely the results and terminologies in Weil's book: Foundations of algebraic geometry, Am. Math. Soc. Colloq., Vol. 29 (1946). About the present paper the author has received kind remarks from Prof. Weil to whom he express his best thanks.

be a base of the topological 1-cycles over rationals on $\boldsymbol{U}$ and on $\boldsymbol{V}$ respectively for $\alpha=1,2$. Then the period matrices

$$\Omega_\alpha = \left(\int_{\gamma_{\alpha j}} \Phi_{\alpha i}\right) \quad (\alpha=1,2)$$

are Riemann matrices and therefore are attached to Abelian Varieties $\boldsymbol{A}$ and $\boldsymbol{B}$. If we denote by $H(\boldsymbol{A}, \boldsymbol{B})$ the module of homomorphisms from $\boldsymbol{A}$ into $\boldsymbol{B}$, our *main result* is

$$C(\boldsymbol{U}, \boldsymbol{V}) \cong H(\boldsymbol{A}, \boldsymbol{B}),^{1)}$$

which is wellknown in the case of curves. We note that the module $H(\boldsymbol{A}, \boldsymbol{B})$ depends only on the categories[2] of $\boldsymbol{A}$ and $\boldsymbol{B}$; hence is the same if we take for $\boldsymbol{A}$ the *Albanese Variety* of $\boldsymbol{U}$ and for $\boldsymbol{B}$ the *Picard Variety* of $\boldsymbol{V}$. If we consider the "true content" of our theorem, this formulation seems to be natural; and the theorem itself could be proved in this form from another aspect.

As far as the author is aware, "correspondences" between algebraic Varieties are studied only in the case when two Varieties and correspondences are of the same dimension. Such correspondences[3] coincide with ours if and only if the Varieties are curves. We were led to our correspondences in our previous paper on Picard Varieties[4] in a natural way and our theorem threw a light on the theory of Picard Varieties in some point.

Now we shall prove our theorem directly in the case when $\boldsymbol{U}$ and $\boldsymbol{V}$ are non-singular projective models; we denote them as $\boldsymbol{M}_1^{n_1}$ and $\boldsymbol{M}_2^{n_2}$ respectively. In this case by prop. 1 in (P) the generic linear section $\boldsymbol{W}_\alpha^1$ of $\boldsymbol{M}_\alpha$ is a non-singular curve on $\boldsymbol{M}_\alpha$ for $\alpha=1,2$. Since $\boldsymbol{M}_\alpha$ is an orientable manifold we can find by prop. 3 in (P) a base

$$\Gamma_{\alpha i} \quad (1 \leqq i \leqq 2q_\alpha)$$

of $(2n_\alpha-1)$-cycles over integers such that

1) In this paper we shall consider these modules abstractly; they are isomorphic if they have the same rank.

2) See A. Weil, Variétés Abeliennes et courbes algébriques, Act. Sc. et Ind. (1948).

3) See O. Zariski, Algebraic surfaces, Erg. d. Math. (1935). See also W. V. D. Hodge, Algebraic correspondences between surfaces, Proc. London Math. Soc., Vol. 44 (1938).

4) On the Picard varieties attached to algebraic varieties, Appendix II, to appear in the Amer. J. of Math. We cite this paper as (P).

$$\Gamma_{\alpha i}\cdot W_{\alpha i}\sim\gamma_{\alpha i}\quad(1\leqq i\leqq 2q_\alpha)$$

over rationals for $\alpha=1,2$. It is wellknown that every $2(n_1+n_2-1)$-dimensional integral cycle X on $M_1\times M_2$ can be written uniquely in the form

$$X\sim\Gamma_1\times M_2+M_1\times\Gamma'_2+\sum_{i,j}s_{ij}(\Gamma_{1i}\times\Gamma_{2j})$$

over integers, where Γ_α is a $2(n_\alpha-1)$-dimensional integral cycle on M_α for $\alpha=1,2$ and where

$$S=(s_{ij})$$

is an integral matrix of type $(2q_1, 2q_2)$. We shall find the conditions under which X is "algebraic", or X is homologous to some $(M_1\times M_2)$-divisor over integers. Here the theory of "harmonic integrals" enter in. We know that there exists an elementary transformation of the doubly projective space into an ordinary projective space, which is everywhere biregular[5]. On the other hand the projective space can be considered in a natural way as a Kähler manifold[6]. In this way the Varieties M_α and $M_1\times M_2$ are Kähler manifolds with the Kähler metrics ω_α and ω respectively. Fortunately it holds thereby

$$\omega=\omega_1+\omega_2.$$

We shall use the same letter when we consider them as differential forms of degree two. It is probably wellknown and can be proved easily either analytically or algebraically that the differential forms Ψ of degree two of the first kind on $M_1\times M_2$ can be written uniquely in the form

$$\Psi=\Psi_1+\Psi_2+\sum_{i,j}a_{ij}\,\Phi_{1i}\wedge\Phi_{2j},$$

where Ψ_α is a similar form on M_α for $\alpha=1,2$ and where a_{ij} are constants. Now a deep result of Lefschetz and Hodge in (H), § 51. 2 tells us that X is algebraic if and only if we have

$$\int_X\Psi\wedge\underbrace{\omega\wedge\cdots\wedge\omega}_{n_1+n_2-2}=0$$

for every Ψ. This equation splits into the following two types of equations

5) See v. d. Waerden, Einführung in die algebraische Geometrie, Springer (1939), § 4.

6) For this see W. V. D. Hodge, The theory and applications of harmonic integrals, Cambridge Univ. Press (1941). We cite this book as (H).

$$\int_{\Gamma_\alpha} \Psi_\alpha \wedge \underbrace{\omega_\alpha \wedge \cdots \wedge \omega_\alpha}_{n_\alpha - 2} = 0 \quad (\alpha = 1, 2),$$

$$\sum_{i,j} s_{ij} \int_{\Gamma_{1i}} \Phi_{1a} \wedge \underbrace{\omega_1 \wedge \cdots \wedge \omega_1}_{n_1 - 1}$$

$$\cdot \int_{\Gamma_{2j}} \Phi_{2b} \wedge \underbrace{\omega_2 \wedge \cdots \wedge \omega_2}_{n_2 - 1} = 0$$

$$(1 \leqq a \leqq q_1, \; 1 \leqq b \leqq q_2).$$

The first type of equations is precisely the condition under which Γ_α is algebraic for $\alpha = 1, 2$. Moreover the second type of equations is transformed into

$$\sum_{i,j} s_{ij} \int_{\gamma_{1i}} \Phi_{1a} \cdot \int_{\gamma_{2j}} \Phi_{2b} = 0,$$

or in matrix notation

$$\Omega_1 S\,{}^t\Omega_2 = 0.$$

We have thus found that X is algebraic if and only if Γ_α are algebraic and if S satisfies the above equation, which is familiar in the classical theory[7]. On the other hand if a correspondence $\boldsymbol{X}$ is homologous to zero over integers, $\boldsymbol{X}$ can be considered as a character of the 1-dimensional integral Betti group of $\boldsymbol{M}_1 \times \boldsymbol{M}_2$. We can find by th. 5 in (P) a $\boldsymbol{M}_\alpha$-divisor $\boldsymbol{Y}_\alpha$, which induces the same character on the 1-dimensional integral Betti group of $\boldsymbol{M}_\alpha$ as $\boldsymbol{X}$ for $\alpha = 1, 2$. Since the Betti group under consideration of $\boldsymbol{M}_1 \times \boldsymbol{M}_2$ is a direct sum of that of $\boldsymbol{M}_\alpha$, we see that $\boldsymbol{X}$ is linearly equivalent to $\boldsymbol{Y}_1 \times \boldsymbol{M}_2 + \boldsymbol{M}_1 \times \boldsymbol{Y}_2$; or what is thes ame thing $\boldsymbol{X}$ is of valence zero. We have thus proved that the mapping

$$\boldsymbol{X} \to S$$

induces an isomorphism of $C(\boldsymbol{M}_1, \boldsymbol{M}_2)$ onto the module of the integral matrix S satisfying the previous equation. However it is wellknown that

6) This reasoning is due essentially to Lefschetz. See his memorable paper: Correspondences between algebraic curves, Ann. of Math., Vol. 28 (1927).

the module of such S is isomorphic with the module of " multiplications " (Λ, L) of Ω_1 to Ω_2, where Λ is a complex matrix of type (q_2, q_1) and L an integral matrix of type $(2q_2, 2q_1)$ satisfying the equation

$$\Lambda\,\Omega_1 = \Omega_2\,L.$$

Finally the module of such multiplications is isomorphic with the module $H(\boldsymbol{A}, \boldsymbol{B})$. Therefore we have an isomorphism between $C(\boldsymbol{M}_1, \boldsymbol{M}_2)$ and $H(\boldsymbol{A}, \boldsymbol{B})$.

In the general case if $\boldsymbol{U}$ and $\boldsymbol{V}$ are birationally equivalent with $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively, we see readily that

$$C(\boldsymbol{U}, \boldsymbol{V}) \cong C(\boldsymbol{M}_1, \boldsymbol{M}_2).$$

On the other hand since $H(\boldsymbol{A}, \boldsymbol{B})$ is itself birationally invariant, we have our theorem. It would be interesting to obtain a generalization of this result for arbitary characteristic and for complete Varieties, which contain no multiple Subvarieties of lower dimension by one.

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