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On a System of Differential Equations

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1. Establishment of Problem.

Notation: A great roman letter means a matrix of the type (m, n).
Problem: Let D be a domain with the boundary Γ in the space R of variable point $(x_1, x_2, \cdots)$. We want to find a real continuous function-matrix U satisfying the following conditions;

$$\Delta U+KU=0 \quad \text{in } D,$$
$$\frac{dU}{dn}+UH=0 \quad \text{on } \Gamma$$

where $\Delta=\sum_i \frac{\partial^2}{\partial x_i^2}$ and $\frac{d}{dn}$ normal derivation, each applying to every element of U, $K^{(m)}$ is a constant symmetric matrix and $H^{(n)}$ is a constant positive definite symmetric matrix.*)

Such a function-matrix is called a harmonic function-matrix in D.

By two matrices U, V holds Green's formula,

$$(1) \quad \int_\Gamma U\frac{dV'}{dn}dw=\int_D U\Delta V'dv+\int_D \sum_i \frac{\partial U}{\partial x_i}\frac{\partial V'}{\partial x_i}dv,$$

$$(2) \quad \int_\Gamma \left(U\frac{dV'}{dn}-\frac{dU}{dn}V'\right)dw=\int_D (U\Delta V'-\Delta U\cdot V')dv,$$

where $'$ means transposition of a matrix.

When U is harmonic, from (1) follows

$$\int_\Gamma U\frac{dU'}{dn}dw=\int_D U\Delta U'dv+\int_D \sum_i \frac{\partial U}{\partial x_i}\frac{\partial U'}{\partial x_i}dv,$$

$$\left(\int_D UU'dv\right)K=\int_\Gamma UHU'dw+\int_D \sum_i \frac{\partial U}{\partial x_i}\frac{\partial U'}{\partial x_i}dv.$$

Now put for arbitrary two matrices U, V,

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*) As the content of the present problem is of rather formal interest, we may make, suitable assumptions about domains, existence of derivatives and continuity as it needs.

$$[U, V]=\int_D UV'dv, \qquad [U, U]=[U],$$

$$\{U, V\}=\int_\Gamma UHV'dw+\int\sum_i \frac{\partial U}{\partial x_i}\frac{\partial V'}{\partial x_i}dv, \quad \{U, U\}=\{U\}.$$

When U is harmonic,

$$K=[U]^{-1}\{U\}.$$

Minimal problem:

Under the contition $[U]=E$ we make $Sp\{U\}$ minimum.*)

As $[U]\neq 0$, $U\neq 0$, it follows that $Sp\{U\}>0$. For if $\{U\}=0$, each integral of $\{U\}$ must be zero. Let $u_1, u_2, \cdots, u_m$ be rows of U, then $Sp\int_\Gamma UHU'dw\cdot =\sum_k\int_\Gamma u_kHu_k'dw=0$, so that $\int_\Gamma u_kHu_k'dw=0$, $u_kHu_k'=0$, $u_k=0$, that is, $U=0$ on Γ. On ther other hand from $Sp\int_D\sum_i\frac{\partial U}{\partial x_i}\frac{\partial U'}{\partial x_i}dv=\sum_k\int_D\sum_i\frac{\partial u_k}{\partial x_i}\times\frac{\partial u_k'}{\partial x_i}dv=0$, follows $\int_D\sum_i\frac{\partial u_k}{\partial x_i}\frac{\partial u_k'}{\partial x_i}dv=0$, $\frac{\partial u_k}{\partial x_i}\frac{\partial u_k'}{\partial x_i}=0$, $\frac{\partial u_k}{\partial x_i}=0$. Hence u_k is a constant row, that is, U is a constant matrix. As U is continuous, $U=0$ in D.

We will reduce the existence of minimum to the ordinary case of one function in the next article. So we assume here existence and will find its minimum and the extreme function-matrix U by calculous of variation.

We make $Sp(\{U\}-L[U])$ minimum, where $L^{(m)}$ means an unknown constant matrix. The condition $[U]=E$ implies $\frac{m(m+1)}{2}$ conditions. Let $[U]=(u_k^j)$, $L=(l_k^j)$, then $SpL[U]=\sum_{k,p} l_k^p u_p^k$, and as $u_k^p=u_p^k$, we must take $l_k^p=l_p^k$, that is L symmetric.

Now

$$\delta Sp\{U\}=Sp\Big(\int_\Gamma(\delta UHU'+UH\delta U')dw+$$

$$\int_D\sum_i\Big(\frac{\partial\delta U}{\partial x_i}\frac{\partial U'}{\partial x_i}+\frac{\partial U}{\partial x_i}\frac{\partial\delta U'}{\partial x_i}\Big)dv\Big)$$

$$=2Sp\Big(\int_\Gamma UH\delta U'dw+\int_D\sum_i\frac{\partial U}{\partial x_i}\frac{\partial\delta U'}{\partial x_i}dv\Big).$$

*) E means unit matrix and S_pT means the trace of a matrix T.

By Green's formula (1)

$$\int_\Gamma \frac{dU}{dn}\,\delta U'dw=\int_D \Delta U\delta U'dv+\int_D\sum_i \frac{\partial U}{\partial x_i}\frac{\partial\delta U'}{\partial x_i}\,dv\,,$$

$$\delta Sp\{U\}=2\,Sp\Big(\int_\Gamma\Big(\frac{dU}{dn}+UH\Big)\delta U'dw-\int_\Gamma \Delta U\delta U'dv\Big).$$

$$\delta SpL[U]=SpL\delta[U]=Sp\Big(L\Big(\int_D(\delta UU'+U\delta U')\,dv\Big)\Big)$$

$$=2Sp\Big(\int_D LU\delta U'dv\Big).$$

$$\delta Sp(\{U\}-L[U])=2Sp\Big(\int_\Gamma\Big(\frac{dU}{dn}+UH\Big)\delta U'dw$$

$$-\int_D(\Delta U+LU)\,\delta U'\,dv\Big)=0\,.$$

Hence we get

$$\Delta U+LU=0 \qquad \text{in } D,$$

$$\frac{dU}{dn}+UH=0 \qquad \text{on } \Gamma.$$

In this extreme case, by Green's formula

$$\{U\}=\int_\Gamma U\Big(\frac{dU'}{dn}+HU'\Big)dw-\int_D U\Delta U'dv=-\int_D U\Delta U'dv$$

$$=\int_D UU'Ldv=[U]\,L=L.$$

Hence L is the extreme value of $\{U\}$.

We denote it by K_1 and call the 1st proper value and the extreme function-matrix by U_1 called a proper function-matrix to K_1.

We proceed to get the next proper value.

Under the conditions

$$[U_1, U]=0,\quad [U]=E,$$

we make $Sp\{U\}$ minimum.

Let $L^{(m)}$, $M^{(m)}$ be constant unknown matrices, moreover L is symmetric.
From

$$\delta(Sp(\{U\}-L[U]-2M[U_1, U])$$

$$=2Sp\left(\int_\Gamma\left(\frac{dU}{dn}+UH\right)\delta U'dw-\int_D(\varDelta U+LU+MU_1)\,\delta U'\,dv\right)=0\,,$$

follows

$$\varDelta U+LU+MU_1=0 \qquad \text{in } D\,,$$

$$\frac{dU}{dn}+UH=0 \qquad \text{on } \Gamma.$$

Here $M=0$, because from Green's formula (2)

$$\int_\Gamma\left(U_1\frac{dU'}{dn}-\frac{dU_1}{dn}U'\right)dw=\int_D(U_1\varDelta U'-\varDelta U_1U')\,dv$$

$$\int_\Gamma(U_1HU'-U_1HU')dw\doteq\int_D(K_1U_1U'-U_1(U'L+U_1'M'))\,dv$$

$$0=-[U_1]\,M'.$$

As $[U_1]=E$, $M=0$.
Hence the conditions become

$$\varDelta U+LU=0 \qquad \text{in } D\,,$$

$$\frac{dU}{dn}+UH=0 \qquad \text{on } \Gamma.$$

L is the value of $\{U\}$ in this extreme case and is denoted by K_2 called the second proper value to which corresponds the proper function U_2. etc.

In this way we have the sequence of proper values $K_1, K_2, \cdots$ and the sequence of the corresponding proper function-matrices $U_1, U_2, \cdots$.

Remark: In the minimul problem we can take H as a diagonal matrix, because by a suitable orthgonal transfoımation 0 we make H diagonal

$$0'H0=\varLambda=\begin{bmatrix} h_1 & & 0 \\ & h_2 & \\ & & \ddots \\ 0 & & h \end{bmatrix}, \quad \text{where } 0<h_1\leqq h_2\leqq\cdots\leqq h_n.$$

Put $U0=U^*$, then $[U]=E\to[U^*]=E$, $[V,U]=0\to[V^*,U^*]=0$, where $V^*=V0$.

$$\{U\}_H=\int_\Gamma U0\Lambda 0'U'dw+\int_D\sum_i\frac{\partial U0}{\partial x_i}\frac{\partial 0'U'}{\partial x_i}dv=\{U^*\}_\Lambda .$$

Hereafter we asumme

$$H=\begin{bmatrix} h_1 & & 0 \\ & h_2 & \\ & & \ddots \\ 0 & & h_n \end{bmatrix},\quad 0<h_1\leqq h_2\leqq\cdots\leqq h_n .$$

2. Construction, Existence of Minimum and Expansion-Theorem.

Construction of proper values and proper function-matrices.
Let $u=(u^1, u^2, \cdots, u^n)$ be a row.
We make $\{u\}$ minimum under the condition $[u]=1$. The solution u_1 satisfies the equations

$$\left.\begin{aligned} &\Delta u_1+k_1u_1=0 \quad \text{in } D \\ &\frac{du_1}{dn}+u_1H=0 \quad \text{on } \Gamma \end{aligned}\right.,\qquad k_1=\{u_1\} .$$

k_1 is the 1st proper value and u_1 the 1st proper function-row. Next we make $\{u\}$ minimum under the conditions $[u]=1$, $[u_1, u]=0$. The solution u_2 satisfies the equations

$$\left.\begin{aligned} &\Delta u_2+k_2u_2=0 \quad \text{in } D \\ &\frac{du_2}{dn}+u_2H=0 \quad \text{on } \Gamma \end{aligned}\right.,\qquad k_2=\{u_2\} . \qquad \text{etc.}$$

Thus we have successive proper values $k_1, k_2, \cdots$ and the corresponding proper function-rows $u_1, u_2, \cdots$.

We proceed to determine these quantities and functions.

Now

$$[u]=\sum_{j=1}^n[u^j]$$

$$\{u\}=\sum_{j=1}^n\{u^j\},\quad\text{where}\quad \{u^j\}=h_j\int_\Gamma(u^j)^2dw+\int_D\sum_i\left(\frac{\partial u^j}{\partial x_i}\right)^2dv .$$

Hence the problem to make $\{u\}$ minimum under the condition $[u]=1$ is reduced that to make $\{u\}$ minimum under the conditions

$$[u^j]=\gamma_j\,, \qquad \sum_{j=1}^{n}\gamma_j=1\,, \qquad \gamma_j\geqq 0\,, \qquad i=1,2,\cdots,n\,.$$

Let k_1^j be the minimum of $\{u^j\}$ under the condition $[u^j]=1$ and u_1^j the proper function, then the minimum of $\{u^j\}$ under the condition $[u^j]=\gamma_j$ is $k_1^j\gamma_j$ with the proper function $u_1^j\sqrt{\gamma_j}$. Hence the problem is reduced to that to make $k_1^1\gamma_1+k_1^2\gamma_2+\cdots+k_1^n\gamma_n$ minimum under the conditions $\sum_{j=1}^{n}\gamma_j=1$, $\gamma_j\geqq 0$ $j=1,2,\cdots,n$.

As $0<h_1\leqq h_2\leqq\cdots\leqq h_n$, we have $k_1^1\leqq k_1^2\leqq\cdots\leqq k_1^n$. Hence the minimum is k_1^1 and the proper function-row is $u_1=(u_1^1,0,0,\cdots,0)$.

Let $k_1^j, k_2^j,\cdots$ be the successive proper values and $u_1^j, u_2^j,\cdots$ be the corresponding proper functions of the minimum-problem of $\{u^j\}$ under the condition $[u^j]=1$.

Now the 2nd minimum-problem is that to make $\sum_{j=1}^{n}\{u^j\}$ minimum under the conditions $[u_1^1,u^1]=0$, $\sum_{j=1}^{n}[u^j]=1$.

That is to make $k_2^1\gamma_1+k_1^2\gamma_2+\cdots+k_1^n\gamma_n$ minimum under the conditions

$$\sum_{j=1}^{n}\gamma_j=1, \quad \gamma_j\geqq 0.$$

The minimum is therefore the least value of $k_2^1, k_1^2,\cdots,k_1^n$ and the proper function-row u_2 has 0 components except the l-th, where l is the upper index of the minimum value. The l-th component is the corresponding proper function, etc.

The above process can be stated in the following form. Let

$$(1)\quad \left.\begin{matrix} k_1^1, & k_2^1, & \cdots \\ k_1^2, & k_2^2, & \cdots \\ \cdots & \cdots & \cdots \\ k_1^n, & k_2^n, & \cdots \end{matrix}\right\}\ \text{be the successive proper values corresponding to}\ \left\{\begin{matrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{matrix}\right.$$

$$\text{and}\quad \left.\begin{matrix} u_1^1, & u_2^1, & \cdots \\ u_1^2, & u_2^2, & \cdots \\ \cdots & \cdots & \cdots \\ u_1^n, & u_2^n, & \cdots \end{matrix}\right\}\ \text{be the corresponding proper functions.}$$

The 1st proper value is the least value in the table (1), that is k_1^1, and the 1st proper function-row is $u_1=(u_1^1,0,0,\cdots,0)$. The 2nd proper value is the least value k^j in (1) except k_1^1, and the 2nd proper function-row u_2 has 0 components except jth component which is the corresponding proper function u^j to k^j.

We omit k_1^1 and this k^j in (1) and continue the same process. We get the next proper value and its proper function-row. etc.

Put $U_1=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$ and $K_1=\{U_1\}$; $U_2=\begin{bmatrix} u_{m+1} \\ u_{m+2} \\ \vdots \\ u_{2m} \end{bmatrix}$ and $K_2=\{U_2\}$. etc.

Then $K_1=\{U_1\}=(\{u_k, u_j\})$, where

$$\{u_k, u_j\}=\int_\Gamma u_k H u_j' dw+\int_D \sum_i \frac{\partial u_k}{\partial x_i} \frac{\partial u_j'}{\partial x_i} dv$$

$$=\int_\Gamma\left(\frac{du_k}{dn}+u_k H\right) u_j' dw-\int_D \Delta u_k u_j' dv$$

$$=k_k\ [u_k, u_j] .$$

As $[U_1]=E$ means $[u_k, u_j]=\delta_{jk}$, we have

$$K_1=(k_k \delta_{kj})=\begin{bmatrix} k_1 & & 0 \\ & k_2 \ddots & \\ 0 & & k_m \end{bmatrix}.$$

Likewise we have

$$K_2=\begin{bmatrix} k_{m+1} & & 0 \\ & k_{m+2} \ddots & \\ 0 & & k_{2m} \end{bmatrix}. \quad \text{etc.}$$

U_i has such a structure that each row has 0 components except one and if the proper value to the qth row is k^j the corresponding proper function u^j occupies the position (q, j).

Now let $F=(f_{kj})$, $U=(u_{kj})$, then (k, j) component of the matrix $[F, U]\,U$ is $\sum_{p,l}[f_{kp}, u_{lp}]\,u_{lj}$.

When U is any one of U_i, then its l-th row has only one non-zero component. Let $u_{lj}\neq 0$, then $p=j$. Hence (k, j) component is $\sum_l [f_{kj}, u_{lj}]\,u_{lj}$. Therefore the (k, j) component of the matrix $\sum_{i=1}^{\infty}[F, U_i]\,U_i$

is $\sum_{i=1}^{\infty}[f_{kj}, u_i^j]u_i^j$. This is the expansion of f_k^j by the functions $u_1^j, u_2^j, \cdots$.

When we know that a function f, satisfying the equation $\frac{df}{dn}+fh_j=0$ on Γ, can be expanded in the form $f=\sum_{i=1}^{\infty}[f, u_i^j]u_i^j$, so far we have the

Expansion-theorem: A function-matrix F, which satisfies the condition $\frac{dF}{dn}+FH=0$ on Γ, can be expanded in the form

$$(1)\qquad F=\sum_{i=1}^{\infty}[F, U_i]U_i.$$

Remark: The formula is also holds when F is a row.

Now we return to the original problem and prove that $K_1, K_2, \cdots$ and the corresponding matrices $U_1, U_2, \cdots$ constructed by us are really a system of successive proper values and the corresponding proper matrices of the original problem.

For this purpose we prove the

Theorem: The minimum of $Sp\{V\}$ under the condition $[V]=E$ is $k_1+k_2+\cdots+k_m$. K_1 is one of the 1st proper values and U_1 is the 1st proper function-matrix

Proof: Let $V=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$, then $[V]=([v_i, v_k])$, $\{V\}=(\{v_i, v_k\})$

and $$Sp\{V\}=\sum_{i=1}^{m}\{v_i\}.$$

A matrix V which make $Sp\{V\}$ minimum must satisfy the condition $\frac{dV}{dn}+VH=0$ on Γ, so we can expand v_i by u_j

$$v_i=\sum_{j=1}^{\infty}x_i^j u_j$$

Then the condition $[V]=E$ becomes $\sum_{j=1}^{\infty}x_i^j x_k^j=\delta_{ik}$ and

$$Sp\{V\}=\sum_{i=1}^{m}\sum_{j=1}^{\infty}k_j(x_i^j)^2.$$

Put

$$F=\sum_{j=1}^{\infty}k_j\left(\sum_{i=1}^{m}(x_i^j)^2\right)+\sum_{i,k}l_{ik}\left(\sum_{j=1}^{\infty}x_i^j x_k^j\right), \quad \text{where} \quad l_{ik}=l_{ki}.$$

Then

$$(1)\quad \frac{1}{2}\frac{\partial F}{\partial x_i^j}=k_j x_i^j+\sum_k l_{ik}x_k^j=0\,.$$

Multiplying x_i^j and summing with j, we have

$$l_{ii}+\sum_{j=1}^{\infty}k_j(x_i^j)^2=0\,.$$

Hence we have

$$Sp\{V\}=\sum_{i=1}^{m}\{v_i\}=-\sum_{i=1}^{m}l_{ii}\,.$$

As the coefficient-matrix of expansion

$$X=\begin{bmatrix} x_1^1 & x_1^2 & \cdots \\ x_2^1 & x_2^2 & \cdots \\ \cdots & \cdots & \cdots \\ x_m^1 & x_m^2 & \cdots \end{bmatrix}$$

has orthonormalized rows, the rank of the matrix X is m. Hence there are at least m linearly independent columns; the number of these columns is $j_1, j_2, \cdots, j_m$. They are non-zero columns. For such j the coefficient-determinant of the system of equations (1) is zero

$$(2)\quad \begin{vmatrix} k_j+l_{11}, & l_{12}, & \cdots, & l_{1m} \\ l_{21}, & k_j+l_{22}, & \cdots, & l_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ l_{m1}, & l_{m2}, & \cdots, & k_j+l_{mm} \end{vmatrix}=0\,.$$

$-k_{j_1}, -k_{j_2}, \cdots, -k_{j_m}$ form the system of proper values of the matrix $L=(l_{ik})$. Hence we have

$$k_{j_1}+k_{j_2}+\cdots+k_{j_m}=-SpL$$

As $Sp\{V\}=-SpL$ is minimum and $k_1\leqq k_2\leqq\cdots$, we must have

$$Sp\{V\}=k_1+k_2+\cdots+k_m\,.$$

Remark: (2) can not hold for $k_j>k_m$; hence the jth column is zero, namely

$$X=\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^p \\ x_2^1 & x_2^2 & \cdots & x_2^p \\ \cdots & \cdots & \cdots & \cdots \\ x_m^1 & x_m^2 & \cdots & x_m^p \end{bmatrix}\ 0\,, \qquad \text{where}\quad k_p=k_m\,,\quad k_{p+1}>k_m\,.$$

Theorem 2. The minimum of $Sp\{V\}$ under the conditions $[U_1, V]=0$, $[V]=E$ is $k_{m+1}+k_{m+2}+\cdots+k_{2m}$. K_2 is one of the 2nd proper values and U_2 is the corresponding proper function-matrix.

Proof: The condition $[U_1, V]=0$ means $[u_i, v_k]=0$, that is, $x_k{}^i=0$ for $i, k=1, 2, \cdots, m$. Hence the coefficient-matrix of expansion of V is

$$X=\begin{bmatrix} 0 & 0 & \cdots & 0 & x_1^{m+1} & x_1^{m+2} & \cdots \\ 0 & 0 & \cdots & 0 & x_2^{m+1} & x_2^{m+2} & \cdots \\ \cdots & & & & & & \\ 0 & 0 & \cdots & 0 & x_m^{m+1} & x_m^{m+2} & \cdots \end{bmatrix}.$$

In this case we make

$$F=\sum_{j=m+1}^{\infty} k_j\Big(\sum_{i=1}^{m} (x_i^j)^2\Big)+\sum_{i,k} l_{ik}\Big(\sum_{j=m+1}^{\infty} x_i^j x_k^j\Big)$$

minimum under the condition $\sum_{j=m+1}^{\infty} x_i^j x_k^j=\delta_{ik}$. In the same way as in the proof of the theorem 1, we have

$$Sp\{V\}=\sum_{i=1}^{m} \{v_i\}=k_{m+1}+k_{m+2}+\cdots+k_{2m}.$$

Remark: The coefficient-matrix of expansion becomes

$$X=\begin{bmatrix} & x_1^{m+1} & x_1^{m+2} & \cdots & x_1^q & \\ 0 & x_2^{m+1} & x_2^{m+2} & \cdots & x_2^q & 0 \\ & \cdots & & & & \\ & x_m^{m+1} & x_m^{m+2} & \cdots & x_m^q & \end{bmatrix}, \quad \text{where} \quad k_q=k_{2m},\ k_{q+1}>k_{2m}. \quad \text{etc.}$$

Thus $K_1, K_2, \cdots$; $U_1, U_2, \cdots$ is one system of successive proper values and the corresponding proper function-matrices.

We call $K_1, K_2, \cdots$ normal proper values and $U_1, U_2, \cdots$ normal proper function-matrices.

The above analysis shows that when we know the existence of minimum in the ordinary case of one function, so far we have the

Existence-theorem; The minimums of the problem in the general case really exist.

The expansion-theorem in the general case essentially only holds for the normal proper function-matrices.

3. Orthogonal Relation and its Completeness.

Definition. Two matrices F, G are said to be orthogonal if $[F, G]=0$.

When $[F, G]=0$, $[F, G]'=[G, F]=0$. Therefore the orthogonal relation is mutual.

Definition. A system of matrices $W_1, W_2, \cdots, W_p$ is said to form a normalized orthogonal system when $[W_i, W_k]=\delta_{ik} E$.

$C_i=[F, W_i]$ is called W_i-component of F.

Put $F_l=F-\sum_{i=1}^{l} C_i W_i$, then $[F_l, W_k]=[F, W_k]-\sum_{i=1}^{l} C_i [W_i, W_k]$. Hence we have

$$[F_l, W_k]=0 \quad \text{for} \quad k=1, 2, \cdots, l.$$

Now

$$\{F_l\}=\{F-\sum_{i=1}^{l} C_i W_i, \quad F-\sum_{k=1}^{l} C_k W_k\}$$

$$=\{F\}+\sum_{i,k} C_i\{W_i, W_k\}C_k'-\sum_{i=1}^{l} C_i\{W_i, F\}-\sum_{k=1}^{l}\{F, W_k\}C_k'.$$

Now for a proper function-matrix U_k

$$\int_\Gamma F\frac{dU_k'}{dn}dw=\int_D\sum_i\frac{\partial F}{\partial x_i}\frac{\partial U_k'}{\partial x_i}dv+\int_D F\Delta U_k'dv,$$

$$-\int_\Gamma FHU_k'dw=\int_D\sum_i\frac{\partial F}{\partial x_i}\frac{\partial U_k'}{\partial x_i}dv-\int_D FU_k'K_k dv.$$

Hence we get

$$\{F, U_k\}=C_k K_k, \quad \{U_i, U_k\}=\delta_{ik}K_k.$$

If we take U_k for W_k we have

$$\{F_l\}=\{F\}+\sum_{i=1}^{l} C_iK_iC_i'-\sum_{i=1}^{l} C_iK_iC_i'-\sum_{i=1}^{l} C_iK_iC_i'$$

and get the relation

$$\{F_l\}=\{F\}-\sum_{i=1}^{l} C_iK_iC_i'.$$

Completeness.

Put $[F_l]=M_lTM_l'$, where $T=\begin{pmatrix}E_r & \\ & 0\end{pmatrix}$.

If $r=0$, then $[F_l]=0$, so that $F_l=0$. We assume that $r>0$. Let m_1 be the first row of M_l^{-1} and put $\varphi=m_1F_l$, then we have

$$[\varphi]=\int_D m_1F_lF_l'm_1'dv=m_1[F_l]m_1'=1,$$

$$[\varphi, U_k]=m_1[F_l, U_k]=0 \text{ for } k=1,2,\cdots,l.$$

Hence we have for the proper value k_{ml+1} in the case of type (1, n)

(1) $\{\varphi\}\geqq k_{ml+1}$

$$\{\varphi\}=\{m_1F\}-\sum_i m_1^{-1}C_iK_iC_i'm_1'.$$

As $K_i>0$, we have

(2) $\{\varphi\}<\{m_1F\}$

As $[F_l]\geqq 0$, by an orthogonal transformation O, we can make $O'[F_l]O$ diagonal.

$$O'[F_l]O\begin{bmatrix} P_1^2 & \\ & 0\end{bmatrix}, \quad \text{where } P=\begin{vmatrix} p_1 & & 0 \\ & p_2 & \\ & & \ddots \\ 0 & & p_r\end{vmatrix}, \quad p_i>0\,.$$

We can also range $p_1, p_2, \cdots, p_r$ in any order by suitable choice of O. We choose O such that $\underset{i=1,2,\cdots,m}{\mathrm{Max}}\, p_i=p_1$.

Put

$$M_l=O\begin{bmatrix} P_1 & \\ & E_{m-r}\end{bmatrix}, \text{ so that}$$

$$M_l^{-1}=\begin{bmatrix} P_1^{-1} & \\ & E_{m-r}\end{bmatrix}O'.$$

Let the first row of O' be v_1, then we have

$$m_1=p_1^{-1}v_1,$$

and (2) becomes

$$p_1^{-2}v_1\{F\}v_1'>\{\varphi\}.$$

When $l\to\infty$, $k_{ml+1}\to\infty$. Therefore $\{\varphi\}\to\infty$ by (1), so that

$$p_1\to 0, \text{ that is, } N[F_l]=p_1^2\to 0\,.$$

Hence we have the
Theorem: $[F_l]\to 0$, when $l\to\infty$.

4. Maximum-Minimum Property of the Proper Values.

Let $F_1, F_2, \cdots, F_r$ be a sequence of function-matrices and $A_1^{(m)}, A_2^{( )}, \cdots, A_r^{(m)}$ be constant matrices.

Put $F=\sum_{i=1}^{r} A_iF_i$, then

$$[F]=\sum_{k,j} A_k\,[F_k, F_j]\,A_j' \text{ and}$$

$$\{F\}=\int_\Gamma (\sum_k A_kF_k)H(\sum_j A_jF_j)'dw+\int_D \sum_i\Big(\sum_{k=1}^r A_k\frac{\partial F_k}{\partial x_i}\Big)\Big(\sum_{j=1}^r \frac{\partial F_j'}{\partial x_i}A_j'\Big)dv$$

$$=\sum_{k,j} A_k\Big(\int_\Gamma F_kHF_j'dw\Big)A_j'+\sum_{k,j} A_k\Big(\int_D\sum_i \frac{\partial F_k}{\partial x_i}\frac{\partial F_j'}{\partial x_i}dv\Big)A_j',$$

$$\{F\}=\sum_{k,j} A_k\,\{F_k, F_j\}\,A_j'.$$

Let $U_1, U_2, \cdots, U_r$ be proper function-matrices each corresponding to the successive proper values $K_1, K_2, \cdots, K_r$ respectively, and put $F_i=U_i$ $(i=1, 2, \cdots, r)$.

Then we have

$$[F]=\sum_{j=1}^{r} A_jA_j',$$

$$\{F\}=\sum_{j=1}^{r} A_jK_jA_j'.$$

Let $V_1, V_2, \cdots, V_{r-1}$ be arbitary given function-matrices and determine the coefficient-matrices $A_1, A_2, \cdots, A_r$ from the conditions

$$(1)\qquad \begin{aligned}&[V_i, F]=0 \qquad i=1, 2, \cdots, r-1\\ &[F]=E.\end{aligned}$$

As the number of conditions in (1) is $(r-1)m^2+\frac{m(m+1)}{2}$, $\frac{m(m-1)}{2}$ others determine $A_1, A_2, \cdots, A_r$ completely. We add the condition,

(2) A_r is symmetric.

As

$$K_j=\begin{bmatrix} k_{(j-1)m+1} & & 0 \\ & k_{(j-1)m+2} & \\ & & \ddots \\ 0 & & k_{jm} \end{bmatrix},$$ it follows that

$$Sp\{F\}=Sp(\sum_{j=1}^{r} A_j K_j A_j')\leqq Sp(\sum_{j=1}^{r-1} A_j A_j' k_{(r-1)m+1}+A_r K_r A_r)$$

$$\leqq Sp((\sum_{j=1}^{r} A_j A_j')K_r)=Sp([F]K_r),$$

$$Sp\{F\}\leqq SpK_r.$$

The minimum of $Sp\{F\}$ with the condition (2) is not smaller than that without the condition (2), so we have the

Theorem: The minimum value of $Sp\{F\}$ under the condition (1) is not greater than SpK_r.