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On the Class Field Theory on Algebraic Number Fields with Infinite Degree

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By the celebrated Takagi's class field theory a finite normal abelian extension field K_0 over an algebraic number field k_0 with finite degree is completely characterized by the corresponding ideal group $H(K_0/k_0)$ mod. $\mathfrak{f}$. (Cf. Takagi [9]). Using the notion of "idèle" Chevalley has reformed the class field theory so that we can characterize the Galois group $G(\tilde{k}_0/k_0)$ of the maximal abelian extension $\tilde{k}_0$ of k_0 by a suitable factor group of the group $J(k_0)$ of all the idèles. (Cf. Chevalley [1], [2], Weil [11]).

On the other hand the ideal theory of an algebraic number field k with infinite degree was investigated by many authors (Herbrand [4], Krull [6], Moriya [7] and others). Especially Moriya [8] has extended the Takagi's class field theory on such field k. Nevertheless the idèle theory on such field k does not yet appear in the literature. The aim of this note is to extend the Chevalley's idèle theory on algebraic number fields with infinite degree and to reform the class field theory established by Moriya. Our chief method is to consider the inductive limit group of the idèle groups $J(k_\lambda)$ of algebraic number fields $k_\lambda \subset k$ with finite degree.

1. Let P be the rational number field, and k be an algebraic number field over P with infinite degree. We shall denote by $k_\lambda (\lambda \in \Lambda)$ the fields which are subfields of k and have finite degree over P. We have then $k = \cup_{\lambda \in \Lambda} k_\lambda$.

Now we shall define a semi-order

$$\lambda < \mu \qquad \text{for } k_\lambda \subset k_\mu \tag{1}$$

in Λ, then Λ becomes a directed set.

By a prime divisor $\mathfrak{p}$ of k we shall mean as usual an equivalence class of valuations of k. A valuation of k induces a valuation of $k_\lambda \subset k$, and so a prime divisor $\mathfrak{p}$ of k determines a unique prime divisor $\mathfrak{p}_\lambda$ of $k_\lambda \subset k$. We shall denote it by

$$\mathfrak{p}_\lambda = \pi_\lambda \mathfrak{p} \qquad (\lambda \in \Lambda). \tag{2}$$

If $\lambda < \mu$, then $\mathfrak{p}_\mu$ of k_μ is an extension of $\mathfrak{p}_\lambda$ of k_λ. We shall denote then

$$\mathfrak{p}_\lambda = \pi_\lambda^\mu \mathfrak{p}_\mu \qquad (\lambda < \mu). \tag{3}$$

We have then evidently

$$\pi_\lambda^\mu \cdot \pi_\mu^\nu = \pi_\lambda^\nu \qquad (\lambda < \mu < \nu). \tag{4}$$

(1·1) *If a set of prime divisors* $\mathfrak{p}_\lambda$ *of* $k_\lambda (\lambda \in \Lambda)$ *satisfies the condition* (3) *then there exists one and only one prime divisor* $\mathfrak{p}$ *of* k *such that* (2) *holds.*

We shall denote this prime divisor $\mathfrak{p}$ of k by

$$\mathfrak{p} = \lim_\lambda \mathfrak{p}_\lambda. \tag{5}$$

(1·2) *Let* $\mathfrak{p}_\lambda$ *be a prime divisor of* k_λ, *then there exists at least a prime divivisor* $\mathfrak{p}$ *of* k *with* (2).

Let $\mathfrak{R}(k)$ and $\mathfrak{R}(k_\lambda)$ be the set of all the prime divisors of k or k_λ respectively, then

$$\mathfrak{R}(k_\lambda) = \pi_\lambda \mathfrak{R}(k), \quad \mathfrak{R}(k_\lambda) = \pi_\lambda^\mu \mathfrak{R}(k_\mu) \quad (\lambda < \mu). \tag{6}$$

By (1·1) we have also

(1·3) $\qquad \mathfrak{R}(k) = \text{proj.} \lim_\lambda \mathfrak{R}(k_\lambda)$.

Remark. We can select a confinal sequence $P \subset k_1 \subset k_2 \subset \dots, k = \bigcup_{n=1}^{\infty} k_n$ of Λ and so (1·3) becomes a projective limit of a sequence: $\mathfrak{R}(k) = \text{proj.} \lim_n \mathfrak{R}(k_n)$.

2. Let $k(\mathfrak{p})$ ($k_\lambda(\mathfrak{p}_\lambda)$) be the completion of k (k_λ) w. r. t. $\mathfrak{p}$ ($\mathfrak{p}_\lambda$) and $k^*(\mathfrak{p})$ ($k_\lambda^*(\mathfrak{p}_\lambda)$) be the multiplicative group of all the non-zero elements of $k(\mathfrak{p})$ ($k_\lambda(\mathfrak{p}_\lambda)$). As usual we can consider $k_\lambda(\mathfrak{p}_\lambda) \subset k(\mathfrak{p})$ for $\mathfrak{p}_\lambda = \pi_\lambda \mathfrak{p}$ and $k_\lambda(\mathfrak{p}_\lambda) \subset k_\mu(\mathfrak{p}_\mu)$ for $\mathfrak{p}_\lambda = \pi_\lambda^\mu \mathfrak{p}_\mu (\lambda < \mu)$.

Now we shall put $\tilde{J}(k)$ as the direct product group of all the $k^*(\mathfrak{p})$ ($\mathfrak{p} \in \mathfrak{R}(k)$):

$$\tilde{J}(k) = \Pi_{\mathfrak{p} \in \mathfrak{R}(k)} k^*(\mathfrak{p}). \tag{7}$$

So an element $\mathfrak{a} \in \tilde{J}(k)$ is representable as

$$\mathfrak{a} = \{\mathfrak{a}(\mathfrak{p}) \mid \mathfrak{a}(\mathfrak{p}) \in k^*(\mathfrak{p}),\ \mathfrak{p} \in \mathfrak{R}(k)\}.$$

We shall call $\mathfrak{a}(\mathfrak{p})$ the $\mathfrak{p}$-*component* of $\mathfrak{a}$.

For the field k_λ an element $\mathfrak{a}_\lambda \in \tilde{J}(k_\lambda) = \Pi_{\mathfrak{p}_\lambda \in \mathfrak{R}(k_\lambda)} k_\lambda^*(p_\lambda)$ is called by Chevalley [2] an *idèle* if almost all the components $\mathfrak{a}_\lambda(\mathfrak{p}_\lambda)$ are units in $k_\lambda^*(\mathfrak{p}_\lambda)$. The group $J(k_\lambda)$ of all the idèles of k_λ is called the *fundamental group* of k by him.

Now let an idèle $\mathfrak{a}_\lambda \in J(k_\lambda)$ be given. Then by the mapping $\pi_\lambda^\mu(\mathfrak{R}(k_\mu)) = \mathfrak{R}(k_\lambda)\ (\lambda < \mu)$ we can define the dual mapping ψ_μ^λ:

$$\mathfrak{a}_\mu = \psi_\mu^\lambda(\mathfrak{a}_\lambda) \in J(k_\mu) \quad \text{for } \lambda < \mu \tag{9}$$

by

$$\psi_\mu^\lambda(\mathfrak{a}_\lambda)(\mathfrak{p}_\mu) = \mathfrak{a}_\lambda(\pi_\lambda^\mu(\mathfrak{p}_\mu)) \quad \mathfrak{p}_\mu \in \mathfrak{R}(k_\mu). \tag{10}$$

It is easy to see that the mapping ψ_μ^λ:

$$\psi_\mu^\lambda(J(k_\lambda)) \subset J(k_\mu) \qquad (\lambda < \mu) \tag{11}$$

is an isomorphism of $J(k_\lambda)$ into $J(k_\mu)\ (\lambda < \mu)$. We have also

$$\psi_\mu^\lambda \cdot \psi_\nu^\mu = \psi_\nu^\lambda \quad \text{for } \lambda < \mu < \nu. \tag{12}$$

Quite analogously by the mapping $\pi_\lambda(\mathfrak{R}(k)) = \mathfrak{R}(k_\lambda)$ we can define the dual mapping ψ^λ of $J(k_\lambda)$ into $\tilde{J}(k)$:

$$\psi^\lambda(\mathfrak{a}_\lambda) = \mathfrak{a} \in \tilde{J}(k) \tag{13}$$

by

$$\mathfrak{a}(\mathfrak{p}) = \mathfrak{a}_\lambda(\pi_\lambda(\mathfrak{p})) \quad \mathfrak{p} \in \mathfrak{R}(k). \tag{14}$$

It is also an isomorphism of $J(k_\lambda)$ into $\tilde{J}(k)$ and we have

$$\psi^\mu \cdot \psi_\mu^\lambda = \psi^\lambda \quad (\lambda < \mu). \tag{15}$$

From (15) follows then

$$\psi^\lambda(J(k_\lambda)) \subset \psi^\mu(J(k_\mu)) \quad \text{for } \lambda < \mu. \tag{16}$$

Now we shall define the *fundamental group* $J(k)$ of k by

$$J(k) = \cup_{\lambda \in \Lambda} \psi^\lambda(J(k_\lambda)) \subset \tilde{J}(k) \tag{17}$$

and we shall call the elements of $J(k)$ the *idèles* of k. Hence an idèle $\mathfrak{a}$ of k can be represented for some $\lambda \in \Lambda$ as

$$\mathfrak{a}=\psi^{\lambda}(\mathfrak{a}_{\lambda}) \quad \mathfrak{a}_{\lambda}\in J(k_{\lambda}), \tag{18}$$

and then also for all $\mu>\lambda$

$$\mathfrak{a}=\psi^{\mu}(\mathfrak{a}_{\mu}) \text{ for } \mathfrak{a}_{\mu}=\psi^{\lambda}_{\mu}(\mathfrak{a}_{\lambda})\in J(k_{\mu}). \tag{19}$$

An idèle $\mathfrak{a}_{\lambda}\in J(k_{\lambda})$ is called a *principal idèle* by Chevalley [2] if we have for an element $a_{\lambda}\in k_{\lambda}^{*}$ $\mathfrak{a}_{\lambda}(\mathfrak{p}_{\lambda})=a_{\lambda}$ for all $\mathfrak{p}_{\lambda}\in\mathfrak{R}(k_{\lambda})$. We shall denote by $P(k_{\lambda})$ the group of all the principal idèles. It is clear that $\psi^{\lambda}_{\mu}(P(k_{\lambda}))\subset P(k_{\mu})$ for $\lambda<\mu$. In the same way we shall define the *principal idèle* $\mathfrak{a}$ of k by $\mathfrak{a}(\mathfrak{p})=a\in k^{*}$ for every $\mathfrak{p}\in\mathfrak{R}(k)$ with a fixed element $a\in k^{*}$, and denote by $P(k)$ the group of all the principal idèles of k. We have then

$$P(k)=\cup_{\lambda\in\Lambda}\psi^{\lambda}(P(k_{\lambda}))\subset J(k).^{(1)} \tag{20}$$

Now we shall consider a topology of $J(k)$. For an algebraic number field k_{λ} with finite degree a convenient topology, which is slightly different from the topology introduced by Chevalley [2], is defined by Weil [11], K. Iwasawa[2] and others. By this topology $\psi^{\lambda}_{\mu}(J(k_{\lambda}))\subset J(k_{\mu})$ for $\lambda<\mu$ is a homeomorphism of $J(k_{\lambda})$ onto a closed subgroup of $J(k_{\mu})$.

In general, let be given a system of topological groups $G_{\lambda}(\lambda\in\Lambda)$ with a directed set Λ and for each pair $\{G_{\lambda}, G_{\mu}\}$ $(\lambda<\mu)$ an isomorphic and homeomorphic mapping ψ^{λ}_{μ} of G_{λ} onto a closed subgroup $\psi^{\lambda}_{\mu}(G_{\lambda})$ of G_{μ} such that $\psi^{\lambda}_{\mu}\cdot\psi^{\mu}_{\nu}=\psi^{\lambda}_{\nu}$ for $\lambda<\mu<\nu$ holds. Then we can define a topological group G, which is called the inductive limit group of $G: G=\text{ind.}\lim_{\lambda}G_{\lambda}$ (Cf. Weil [10] p. 109, Freudenthal [3]).

In our case, for the group $J(k)$ defied by (17) we can introduce the topology as

$$J(k)=\text{ind.}\lim_{\lambda}J(k_{\lambda}). \tag{22}$$

Namely a set $O\subset J(k)$ is open if and only if $O\cap\psi^{\lambda}(J(k_{\lambda}))$ is the image by ψ^{λ} of an open set in $J(k_{\lambda})$ for every $\lambda\in\Lambda$. This is equivalent to the fact that O has the form

$$O=\cup_{\lambda\in\Lambda}\psi^{\lambda}(O_{\lambda}) \qquad O_{\lambda}: \text{ open subsets of } J(k_{\lambda}).^{(3)} \tag{23}$$

3. Let k be an algebraic number field with infinite degree and K a finite extension field over k. Let the degree of K/k be $n: n=[K:k]$. K is a simple extension of k: $K=k(\theta)$, and θ is a root of an irreducible polynomial $f(X)\in k[X]$ with degree n. Let $k_{\alpha}(\alpha=\alpha(K))$ be the field which is obtained by adjunction of all the coefficients of $f(X)$ fo P. Let us take then

$$\Lambda'=\{\lambda \mid \lambda\in\Lambda,\ \lambda>\alpha\} \tag{24}$$

Namely, Λ' is the residual set of Λ w.r.t. $\alpha=\alpha(K)$. Let us put $K_{\lambda}=k_{\lambda}(\theta)$ for $\lambda\in\Lambda'$, then we have $[K_{\lambda}:k_{\lambda}]=n$ and $k=\cup_{\lambda\in\Lambda'}k_{\lambda}$, $K=\cup_{\lambda\in\Lambda'}K_{\lambda}$.

Let a prime divisor $\mathfrak{p}$ of k be extended to g prime divisors $\mathfrak{P}^{(1)},\dots,\mathfrak{P}^{(g)}$ of K. We shall define the *norm* of an idèle $\mathfrak{A}\in J(K)$:

$$N_{K/k}\mathfrak{A}=\mathfrak{a}\in\tilde{J}(k) \tag{25}$$

by

$$\mathfrak{a}(\mathfrak{p})=\Pi_{i=1}^{g}N_{K(\mathfrak{P}^{(i)})/k(\mathfrak{p})}(\mathfrak{A}(\mathfrak{P}^{(i)}))\quad \mathfrak{p}\in\mathfrak{R}(k). \tag{26}$$

Instead of π, ψ etc. for k we shall denote by Π, Ψ etc. the corresponding mappings for K. Then for some $\lambda\in\Lambda'$ we have $\mathfrak{A}=\Psi^{\lambda}(\mathfrak{A}_{\lambda})$, with $\mathfrak{A}_{\lambda}\in J(K_{\lambda})$. Let be $\mathfrak{p}_{\lambda}=\pi_{\lambda}\mathfrak{p}$, $\mathfrak{P}_{\lambda}^{(i)}=\Pi_{\lambda}\mathfrak{P}^{(i)}(i=1,\dots,g)$, then $\mathfrak{P}_{\lambda}^{(i)}(i=1,\dots,g)$ are all extensions of $\mathfrak{p}_{\lambda}$ to K_{λ}. Let $\mathfrak{Q}_{\lambda}^{(1)},\dots,\mathfrak{Q}_{\lambda}^{(h)}(h\leqq g)$ be the set of all the different ones within $\mathfrak{P}_{\lambda}^{(1)},\dots,\mathfrak{P}_{\lambda}^{(g)}$. For example, $\mathfrak{P}_{\lambda}^{(1)}=\dots=\mathfrak{P}_{\lambda}^{(a)}=\mathfrak{Q}_{\lambda}^{(1)}$, $\mathfrak{P}_{\lambda}^{(a+1)}=\dots=\mathfrak{Q}_{\lambda}^{(2)},\dots$, then we have $[K_{\lambda}(\mathfrak{Q}_{\lambda}^{(1)}):k_{\lambda}(\mathfrak{p}_{\lambda})]=\sum_{i=1}^{a}[K(\mathfrak{P}^{(i)}):k(\mathfrak{p})]$ and $\mathfrak{A}(\mathfrak{P}^{(1)})=\dots=\mathfrak{A}(\mathfrak{P}^{(a)})=\mathfrak{A}_{\lambda}(\mathfrak{Q}_{\lambda}^{(1)})$ etc. It is then easy to see by the well-known methods in the theory of algebra that

$$\Pi_{i=1}^{a}N_{K(\mathfrak{P}^{(i)})/\mathfrak{p}(k)}(\mathfrak{A}(\mathfrak{P}^{(i)}))=N_{K_{\lambda}(\mathfrak{Q}_{\lambda}^{(1)})/k_{\lambda}(\mathfrak{p}_{\lambda})}(\mathfrak{A}_{\lambda}(\mathfrak{Q}_{\lambda}^{(1)})).$$

On the other hand the norm of an idèle $\mathfrak{A}_{\lambda}\in J(K_{\lambda})$:

$$N_{K_{\lambda}/k_{\lambda}}(\mathfrak{A}_{\lambda})=\mathfrak{a}_{\lambda}\in J(k_{\lambda})$$

is defined by Chevalley [2] by

$$\mathfrak{a}_{\lambda}(\mathfrak{p}_{\lambda})=\Pi_{i=1}^{h}N_{K_{\lambda}(\mathfrak{Q}_{\lambda}^{(i)})/k_{\lambda}(\mathfrak{p}_{\lambda})}(\mathfrak{A}_{\lambda}(\mathfrak{Q}_{\lambda}^{(i)})).$$

Hence we have $N_{K/k}\mathfrak{A}=\mathfrak{a}=\psi^{\lambda}(\mathfrak{a}_{\lambda})=\psi^{\lambda}(N_{K_{\lambda}/k_{\lambda}}\mathfrak{A}_{\lambda})$ for $\lambda\in\Lambda'$; this proves the following:

(3.1) *The norm $N_{K/k}\mathfrak{A}=\mathfrak{a}$ of an idèle $\mathfrak{A}\in J(K)$ is an idèle of k and*

$$N_{K/k}(\Psi^{\lambda}\mathfrak{A}_{\lambda})=\psi^{\lambda}(N_{K_{\lambda}/k_{\lambda}}\mathfrak{A}_{\lambda})\quad \textit{for } \lambda\in\Lambda'. \tag{28}$$

From (28) we have also

$$N_{K/k}J(K)=\cup_{\lambda\in\Lambda}\psi^{\lambda}(N_{K_\lambda/k_\lambda}J(K_\lambda)). \tag{29}$$

Since N_{K_λ/k_λ} is a continuous mapping of $J(K_\lambda)$ into $J(k_\lambda)$ for every $\lambda\in\Lambda$ we can easily prove by the definition of the topology of $J(k)$ the following:

(3.2) *$N_{K/k}$ is a continuous mapping of $J(K)$ into $J(k)$.*

4. Let k be an algebraic number field with infinite degree. By the *absolute degree* $N(k)$ of k we shall mean as in Moriya [8] the following Steinitz's G-number: let $k=\cup_\lambda k_\lambda$, $N_\lambda=[k:P]$ and for a prime nnmber p $N_\lambda=p^{r_\lambda}M_\lambda$ $(p, M_\lambda)=1$, then we shall put

$$N(k)=\Pi_p p^r, \quad r=\sup_\lambda r_\lambda. \tag{30}$$

Let us put $N_0(k)=\Pi'_p p^r$ for all the primes p with $r<\infty$ and $N^*(k)=\Pi''_p p^r$ for all the primes p with $r=\infty$. Then

$$N(k)=N^*(k)\cdot N_0(k). \tag{31}$$

We shall call after Moriya [8] $N^*(k)$ the *infinite part* of $N(k)$.

Now let K be a finite normal abelian extension field over k with the Galois group $G(K/k)$ and with degree $n=[K:k]$. Let us put

$$H(K/k)=P(k)\cdot N_{K/k}(J(K))\subset J(k) \tag{32}$$

and

$$\mathfrak{H}(K/k)=J(k)/H(K/k), \quad h(K/k)=[J(k):H(K/k)]. \tag{33}$$

By (20) and (29) we have

$$H(K/k)=\cup_{\lambda\in\Lambda}\psi^{\lambda}(H(K_\lambda/k_\lambda)) \tag{34}$$

for the corresponding group $H(K_\lambda/k_\lambda)=P(k_\lambda)\cdot N_{K_\lambda/k_\lambda}(J(K_\lambda))$ $(\lambda\in\Lambda')$.

For an algebraic number field k_λ with finite degree over P we have

$$\mathfrak{H}(K_\lambda/k_\lambda)\cong G(K_\lambda/k_\lambda), \text{ and } h(K_\lambda/k_\lambda)=[K_\lambda:k_\lambda] \tag{35}$$

by the class field theory (Cf. Chevalley [2]), but for our case (35) is not always true. (Cf. Moriya [8]).

(4.1) *For a finite normal abelian extension field K/k we have*

$$h(K/k) \leq [K:k].^{(4)} \tag{36}$$

(Proof) Let a system of representartives of classes $J(k)$ mod. $H(K/k)$ be $\{\mathfrak{a}^{(1)}, \mathfrak{a}^{(2)}, \ldots, \mathfrak{a}^{(r)}, \ldots\}$. For any finite number of them: $\{\mathfrak{a}^{(1)}, \ldots, \mathfrak{a}^{(r)}\}$, there exists $\lambda \in \Lambda'$ such that $\mathfrak{a}^{(i)} = \psi^{\lambda}(\mathfrak{a}_{\lambda}^{(i)})$, $\mathfrak{a}_{\lambda}^{(i)} \in J(k_{\lambda})$, $i=1, \ldots, r$. If for some pair (i,j) $(i \neq j)$ $\mathfrak{a}_{\lambda} = \mathfrak{a}_{\lambda}^{(i)}(\mathfrak{a}_{\lambda}^{(j)})^{-1} = a_{\lambda} \cdot N_{K_{\lambda}/k_{\lambda}}\mathfrak{A}_{\lambda}$ $(a_{\lambda} \in P(k_{\lambda}),\ \mathfrak{A}_{\lambda} \in J(K_{\lambda}))$, then $\psi^{\lambda}(\mathfrak{a}_{\lambda}) = \mathfrak{a}^{(i)}(\mathfrak{a}^{(j)})^{-1} = \psi^{\lambda}(a_{\lambda}) \cdot \psi^{\lambda}(N_{K_{\lambda}/k_{\lambda}}\mathfrak{A}_{\lambda}) = \psi^{\lambda}(a_{\lambda}) \cdot N_{K/k}(\Psi^{\lambda}\mathfrak{A}_{\lambda}) \in H(K/k)$, which is a contradiction. Hence $\mathfrak{a}_{\lambda}^{(1)}, \ldots, \mathfrak{a}_{\lambda}^{(r)}$ belong to r different classes of $J(k_{\lambda})$ mod. $H(K_{\lambda}/k_{\lambda})$, so that $r \leq h(K_{\lambda}/k_{\lambda}) = [K_{\lambda}:k_{\lambda}] = [K:k]$. Since we can take $\{\mathfrak{a}^{(1)}, \ldots, \mathfrak{a}^{(r)}\}$ arbitrarily from $\{\mathfrak{a}^{(1)}, \mathfrak{a}^{(2)}, \ldots\}$, we have (36), q. e. d.

(4.2) *For a finite normal abelian extension K/k $h(K/k)$ is relatively prime to $N^*(k)$.*

(Proof) Let $h(K/k) = p^r t$ $(r>0)$ and p^{∞} is a factor of $N(k)$. Then there exists an idèle $\mathfrak{a} \in J(k)$ such that $\mathfrak{a} \notin H(K/k)$ and $\mathfrak{a}^p \in H(K/k)$. Hence we have for some $\lambda, \mu, \nu \in \Lambda'$, $\mathfrak{a} = \psi^{\lambda}(\mathfrak{a}_{\lambda})$, $\mathfrak{a}_{\lambda} \in J(k_{\lambda})$; $\mathfrak{a}^p = a \cdot N_{K/k}\mathfrak{A}$; $a = \psi^{\mu}(a_{\mu})$, $a_{\mu} \in P(k_{\mu})$; $\mathfrak{A} = \Psi_{\nu}(\mathfrak{A}_{\nu})$, $\mathfrak{A}^{\nu} \in J(K_{\nu})$. Take $\rho > \lambda, \mu, \nu$. Then $\mathfrak{a} = \psi^{\rho}(\mathfrak{a}_{\rho})$, $\mathfrak{a}^p = \psi^{\rho}(a_{\rho}) \cdot N_{K/k}\Psi^{\rho}(\mathfrak{A}_{\rho}) = \psi^{\rho}(\mathfrak{b}_{\rho})$, $\mathfrak{b}_{\rho} = a_{\rho}N_{K_{\rho}/k_{\rho}}\mathfrak{A}_{\rho} \in J(k_{\rho})$. Therefore, we have $\mathfrak{b}_{\rho} \in H(K_{\rho}/k_{\rho})$, $\mathfrak{b}_{\rho} = \mathfrak{a}_{\rho}^p$. Since p^{∞} is a factor of $N(k)$, we can take $\sigma > \rho$ such that $M = [k_{\sigma}:k_{\rho}]$ is divisible by p.

By the theorem of translation in the class field theory (Chevalley [2])

$$H(K_{\sigma}/k_{\sigma}) = \{\mathfrak{a}_{\sigma} \mid N_{k_{\sigma}/k_{\rho}}\mathfrak{a}_{\sigma} \in H(K_{\rho}/k_{\rho})\} \quad (\rho < \sigma). \tag{37}$$

For the idèle $\mathfrak{a}_{\sigma} = \psi_{\sigma}^{\rho}\mathfrak{a}_{\rho}$, $\mathfrak{a}_{\rho} \in J(k_{\rho})$ we have $N_{k_{\sigma}/k_{\rho}}\mathfrak{a}_{\sigma} = \mathfrak{a}_{\rho}^{M} = \mathfrak{b}_{\rho}^{M/p} \in H(K_{\rho}/k_{\rho})$. so that $\psi_{\sigma}^{\rho}(\mathfrak{a}_{\rho}) \in H(K_{\sigma}/k_{\sigma})$ by (37). Hence we have $\mathfrak{a} = \psi^{\sigma}(\psi_{\sigma}^{\rho}(\mathfrak{a}_{\rho})) \in H(K/k)$ by (34), which contradicts with our assumption. Therefore, $h(K/k)$ is relatively prime to $N^*(k)$, q. e. d.

(4.3) *Let K be a finite normal abelian extension over k and $[K:k]$ is relatively prime to $N^*(k)$, then we have*

$$\mathfrak{H}(K/k) \cong G(K/k) \text{ and so } h(K/k) = [K:k]. \tag{38}$$

(Proof) By our assumption we can take $\beta = \beta(K) \in \Lambda'$ such that $M = [k_{\lambda}:k_{\beta}]$ is relatively prime to $n = [K:k]$ for $\lambda > \beta$.

Let $\mathfrak{a}_{\lambda}^{(1)}, \ldots, \mathfrak{a}_{\lambda}^{(n)}$ be a complete system of representatives of classes of $J(k_{\lambda})$ mod. $H(K_{\lambda}/k_{\lambda})$. If $\mathfrak{a}^{(i)} = \psi^{\lambda}(\mathfrak{a}_{\lambda}^{(i)})$ $(i=1, \ldots, n)$ belong to different classes mod. $H(K/k)$ in $J(k)$, then we see by (4.1) that $\{\mathfrak{a}^{(1)}, \ldots \mathfrak{a}^{(n)}\}$ is also a complete system of representatives of $J(k)$ mod. $H(K/k)$. Hence we have

$$H(K/k)\cong H(K_\lambda/k_\lambda)\cong G(K_\lambda/k_\lambda)\cong G(K/k).$$

If this is not the case, there exists some pair $(i,j)\,(i\neq j)$ with $H(K/k)\ni\mathfrak{a}=\mathfrak{a}^{(i)}(\mathfrak{a}^{(j)})^{-1}=\psi^\lambda(\mathfrak{a}_\lambda)$ for $\mathfrak{a}_\lambda=\mathfrak{a}_\lambda^{(i)}(\mathfrak{a}_\lambda^{(j)})^{-1}$. Then by (34) we have for some $\mu>\lambda$ $\mathfrak{a}=\psi^\mu(\mathfrak{a}_\mu)=\psi^\mu(\psi_\mu^\lambda(\mathfrak{a}_\lambda))$ with $\mathfrak{a}_\mu\in H(K_\mu/k_\mu)$. By the theorem of translation we have as in (4.1) $\mathfrak{a}_\lambda^M\in H(K_\lambda/k_\lambda)$ for $M=[k_\mu:k_\lambda]$. Since M and n are relatively prime, this means $\mathfrak{a}_\lambda\in H(K_\lambda/k_\lambda)$, which is a contradiction, q. e. d.

We can easily prove the following by (4.1), (4.2), (4.3):

(4.4) *Let K be a finite normal abelian extension over k and let*

$$n=[K:k]=n^*\cdot n_0 \tag{39}$$

where n^* is a divior of $N^*(k)$ and n_0 is relatively prime to $N^*(k)$. Then we have

$$h(K/k)=n_0. \tag{40}$$

We shall call a finite normal abelian extension K/k a *class field* over k if the equality

$$h(K/k)=[K:k] \tag{41}$$

holds. Then by (4.4) *a necessary and sufficient condition that a finite normal abelian extension K/k is a class field is that $[K:k]$ is relatively prime to $N^*(k)$.*

As in the classical case, we can easily prove:

(4.5) (i) *Let K and K' be class fields over k. Then*

$$K\supset K' \text{ if and only if } H(K/k)\subset H(K'/k). \tag{42}$$

Then we have also

$$[K:K']=[H(K'/k):H(K/k)]. \tag{43}$$

(ii) *Let K and K' be class fields over k, then $K\cup K'$ and $K\cap K'$ are also class fields over k with*

$$H(K\cup K'/k)=H(K/k)\cap H(K'k),$$
$$H(K\cap K'/k)=H(K/k)\cdot H(K'/k). \tag{44}$$

5. Let K be a class field over k. Then the subgroup $H=H(K/k)$ of $J(k)$ satisfies the following four conditions (i)-(iv):

(i) $H \supset P(k)$.

(ii) $[J(k):H]=n<\infty$; *n and $N^*(k)$ are relatively prime.*

(iii) *H is an open and closed subgroup of $J(k)$.*

(Proof) By the class field theory for finite algebraic fields $H(K_\lambda/k_\lambda)$ are open and closed subgroups of $J(k)\,(\lambda\in\Lambda')$ (Cf. Weil [11]). Hence $H(K/k)$ is open by (34) and by the definition of the topolpgy of $J(k)$. Since $[J(k):H(K/k)]$ is finite, $H(K/k)$ is also closed, q. e. d.

We shall denote by H_λ for a subgroup H of $J(k)$ the subgroup of $J(k_\lambda)$ defined by

$$\psi^\lambda(H_\lambda)=\psi^\lambda(J(k_\lambda))\cap H. \tag{45}$$

(iv) *There exists $\beta(H)\in\Lambda$ such that for all $\nu>\mu>\beta(H)$*

$$H_\nu=\{\mathfrak{a}_\nu \mid N_{k_\nu/k_\mu}\mathfrak{a}_\nu\in H_\mu,\ \mathfrak{a}_\nu\in J(k_\nu)\} \tag{46}$$

hold.

(Proof) Take $\beta(H)$ as in the proof of (4.3). We shall see now

$$\psi^\mu(H(K_\mu/k_\mu))=\psi^\mu(J(k_\mu))\cap H(K/k) \quad \text{for } \mu>\beta, \tag{47}$$

then (46) follows from (47) by the theorem of translation. In (47) the inclusion relation $\subset$ holds obviously by (34). Conversely let $\mathfrak{a}=\psi^\mu(\mathfrak{a}_\mu)$, $\mathfrak{a}_\mu\in J(k_\mu)$ belongs to $H(K/k)$, then for some $\nu>\mu$ $\psi^\mu_\nu(\mathfrak{a}_\mu)\in H(K_\nu/k_\nu)$ as in (4.1). By the the theorem of translation we have $\mathfrak{a}_\mu^M=N_{k_\mu/k_\mu}(\psi^\mu_\nu(\mathfrak{a}_\mu))$ $\in H(K_\mu/k_\mu)$ with $M=[k_\nu:k_\mu]$. Since $(M,n)=1$, we have $\mathfrak{a}_\mu\in H(K_\mu/k_\mu)$, i. e. $\mathfrak{a}=\psi^\mu(\mathfrak{a}_\mu)\in\psi^\mu(H(K_\mu/k_\mu))$. This shows (47), q. e. d.

Remark. Just as in (4.2) from (iv) follows that $[J(k):H]$ is relatively prime to $N^*(k)$.

We shall call a subgroup H of $J(k)$ with the propeities (i)-(iv) a *characteristic subgroup.* (This corresponds to the K-Gruppe in Moriya [8]). We shall prove now the converse:

(5.1) *Let H be a characteristic subgroup of $J(k)$, then there exists a class field K over k with $H=H(K/k)$.*

(Proof) Take $\lambda>\beta(H)$ in (iv). Then we have as in the proof of (4.3) $[J(k):H]=[J(k_\lambda):H_\lambda]$. Since H_λ is an open and closed subgroup of $J(k_\lambda)$ with finite index containing $P(k_\lambda)$, there exists a class field

$K_\lambda = k_\lambda(\theta)$ over k_λ with $H_\lambda = H(K_\lambda/k_\lambda)$ (Whaples [12]). But by the property (iv) and by the theorem of translation $H_\mu = H(K_\mu/k_\mu)$ for $\mu > \lambda$ holds for $K_\mu = k_\mu(\theta)$. Hence for the field $K = k(\theta) = \cup_{\lambda \in \Lambda} K$ we have $H(H/k) = \cup_\mu \psi^\mu(H(K_\mu/k_\mu)) = \cup_\mu \psi^\mu(H_\mu) = H$. On the other hand $[K : k] = [K_\lambda : k_\lambda] = [J(k_\lambda) : H_\lambda] = [J(k) : H]$. Hence K is the required class field, q. e. d.

Now let $\boldsymbol{H}(k)$ be the set of all the characteristic subgroups of $J(k)$.

(i) If $H_1, H_2 \in \boldsymbol{H}(k)$, then $H_1 \cap H_2 \in \boldsymbol{H}(k)$

(ii) If $H_1 \in \boldsymbol{H}(k)$ and $H_1 \subset H_2 \subset J(k)$, then $H_2 \in \boldsymbol{H}(k)$

i. e. $\boldsymbol{H}(k)$ makes a lattice with $H_1 \cup H_2 = H_1 \cdot H_2$ and with the set-theoretical intersection $H_1 \cap H_2$. On the other hand let $\boldsymbol{K}(k)$ be the set of all the class fields over k.

(i) If K_1, $K_2 \in \boldsymbol{K}(k)$, then $K_2 \cdot K_2 \in \boldsymbol{K}(k)$.

(ii) If $K_1 \in \boldsymbol{K}(k)$ and $K_1 \supset K_2 \supset k$, then $K_2 \in \boldsymbol{K}(k)$.

i. e. $\boldsymbol{K}(k)$ makes a lattice with the field compositum $K_1 \cup K_2 = K_1 \cdot K_2$ and with the set-theoretical intersection $K_1 \cap K_2$.

Let $\tilde{k}$ be the union of all the class fields over k. Then the Galois-group $\tilde{G} = G(\tilde{k}/k)$ is a compact topological group by the Krull's topology (Cf. Krull [5]). For a finite extension $\tilde{k} \supset K \supset k$ the correspoding group $G(K) = \{\sigma | \sigma \in \tilde{G},\ \sigma(a) = a$ for all $a \in K\}$ is open and closed and has a finite index $[\tilde{G} : G(K)] = [K : k]$.

Let $\boldsymbol{G}(k)$ be the lattice of all the open and closed subgroups G of $\tilde{G}$ with finite indices. By the above considerations and by the Galois theory for infinite algebraic extensions we have

(5.2) *Let $H \in \boldsymbol{H}(k)$. Then there exists a class field $K/k \in \boldsymbol{K}(k)$ with $H = H(K/k)$. To K corresponds $G = G(K) \in \boldsymbol{G}(k)$. This correspondence*

$$\boldsymbol{H}(k) \in H \rightarrow \varphi(H) = G \in \boldsymbol{G}(k)$$

is a one to one correspondence between $\boldsymbol{H}(k)$ and $\boldsymbol{G}(k)$ which is also a lattice-isomorphism :

(i) $H_1 \supset H_2 \rightleftarrows \varphi(H_1) \supset \varphi(H_2)$ *and then* $[H_1 : H_2] = [\varphi(H_1) : \varphi(H_2)]$

(ii) $\varphi(H_1 \cap H_2) = \varphi(H_1) \cap \varphi(H_2)$

(iii) $\varphi(H_1 \cdot H_2) = \varphi(H_1) \cdot \varphi(H_2)$.

Finally let us put

$$H_0 = \cap H \quad \text{for all } H \in \boldsymbol{H}(k). \tag{48}$$

Then H_0 is also a closed subgroup of $J(k)$. We shall put then

$$I(k)=J(k)/H_0. \tag{49}$$

Now we shall give a new topology in $I(k)$ which is in general weaker than the the topology induced from $J(k)$, i. e. we shall take as the basis of neighourhoods of the neutral element of $I(k)$ the set of all the subgroups H/H_0 for $H\in \boldsymbol{H}(k)$. Then $I(k)$ is a topological group which is totally disconnected and totally bounded. Let the completion of $I(k)$ be denoted $\overline{I(k)}$. Then $\overline{I(k)}$ is a compact topological group.

Theorem. Let $\tilde{k}$ be the union of all the class fields over k and let $\tilde{G}=G(\tilde{k}/k)$ be the compact Galois-group of $\tilde{k}/k$. Then

$$\tilde{G}\simeq\overline{I(k)} \quad (\textit{isomorphism and homeomorphism}). \tag{50}$$

(Proof) Put $\tilde{k}=\cup_n K_n$, $k\subset K_1\subset K_2\subset\ldots$, $K_n\in\boldsymbol{K}(k)$, $G_n=G(\tilde{k}/K_n)$, $H_n=H(K_n/k)$ and $I_n=H_n/H_0$. Then $H_1\supset H_2\supset\ldots$ and $H_0=\cap_n H_n$. Hence $\cap_n I_n=\{1\}$. By (4.3) $\tilde{G}/G_n\simeq G(K_n/k)\simeq H(K_n/k)\simeq J(k)/H_n\simeq I(k)/I_n$. Let an isomorphic mapping $I(k)/I_n\simeq\tilde{G}/G_n$ be $\varphi^{(n)}$. Then for $m<n$ $\varphi^{(n)}$ also induces an isomorphic mapping $\varphi_m^{(n)}$ of $I(k)/I_m$. By the well-known diagonal methods we can choose mappings $\varphi_0^{(n)}$ of $I(k)/I_n$ to $\tilde{G}/G_n$ such that $\varphi^{(n)}$ induces $\varphi^{(m)}$ on $I(k)/I_m$ for $m<n$.

Since $\overline{I(k)}$ and $\tilde{G}$ can be defined as the projective limit group of sequences $\{I(k)/I_n\}$ and $\{\tilde{G}/G_n\}$ respectively we can define an isomorphic and homeomorphic mapping φ_0 of $\overline{I(k)}$ onto $\tilde{G}$ such that φ_0 induces $\varphi_0^{(n)}$ on $\overline{I(k)}/\overline{I_n}\simeq I(k)/I_n$ (Cf. Freudenthal [3]), q. e. d.

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(1) We shall call an idèle $\mathfrak{a}\in J(k)$ a unit idèle if the $\mathfrak{p}$-componts $\mathfrak{a}(\mathfrak{p})$ are units in $k(\mathfrak{p})$ for all finite prime divisors. Let $U(k)$ be the group of all the unit idèles, then $\mathfrak{J}(k)=J(k)/U(k)$ is isomorphic to the group of all the "umkehrbare Ideale" of k (Cf. Krull [6]).

(2) K. Iwasawa, On L-functions, to be published elsewhere, which was read at the meeting of the Math. Soc. Japan, May, 1950.

(3) Though $J(k_\lambda)$ are locally compact, $J(k)$ is not always locally compact, and though $P(k_\lambda)$ are discrete subgroups of $J(k_\lambda)$, $P(k)$ is not necessarily discrete. But we can get some properties of $J(k)$ by (22). For example, we can define $V^*(\mathfrak{a})$ for $\mathfrak{a}\in J(k)$ by modifying slightly the definition of $V(\mathfrak{a}_\lambda)$ for $\mathfrak{a}_\lambda\in J(k_\lambda)$, which was introduced by Artin and Whaples (Cf. E. Artin, W. Whaples, Axiomatic characterization of fields by the product formula for valuations, Bull. Amer. Math. Soc., 51(1945), 469–492. Put $J^*(k)=\{\mathfrak{a}\mid V^*(\mathfrak{a})=1\}$. Then $J^*(k)\supset P(k)$, $J^*(k)\supset U(k)$. The structure of the group $J_\lambda(k)/P(k)U(k)$ is important for the analogy with the theory of algebraic functions. For these considerations cf. Y. Kawada, On the class field theory on infinite algebraic number fields, (in Japanese). Math. Reports of Tôdai-Kyôyôgakubu, 1 (1950), 85–100.

(4) (4.1), (4,2), (4.3), (4,4), (5,1) correspond to Satz 4, Satz 5, Satz 6, Satz 7, and Satz 14 in Moriya [8].