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On a Theorem of E. Cartan.

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Introduction

In 1913, E. Cartan proved two remarkable theorems which give a penetrating method for the determination of all irreducible representations of semi-simple complex Lie algebras [2].[1] These theorems can be formulated as follows:

I. *There exists a one-to-one correspondence between the irreducible representations of a given semi-simple Lie algebra* $\tilde{L}$ *and the highest (possible) weights of* $\tilde{L}$.

Thus every highest possible weight ξ is actually a highest weight of some irreducible representation P of $\tilde{L}$ and conversely P is uniquely determined by ξ up to equivalence. (In this sense, we write $P=P_{\xi}$.)

II. *All the highest (possible) weights of* $\tilde{L}$ *form a semi-group, isomorphic to the direct product of* n *semi-groups, each of which is formed of all non-negative integers,* n *denoting the rank of* $\tilde{L}$, *i. e. the (complex) dimension of maximal abelian subalgebras in* $\tilde{L}$.

Thus the sum $\xi+\xi'$ of two highest possible weights ξ and ξ' of $\tilde{L}$ is again a highest possible weight of $\tilde{L}$ and the corresponding irreducible representation $P_{\xi+\xi'}$ is composed from those of ξ and ξ' through a definite process. (We call it the *Cartan composite* of P_{ξ} and $P_{\xi'}$.) In this manner, all the irreducible representations of $\tilde{L}$ can be generated from n of them, called *fundamental*.

The essential part of these theorems consists in the existence of an irreducible representation for every highest possible weight (in I), and in that of n fundamental weights for the semi-group (in II). Cartan's original proof deals separately with the different types of simple algebras. So his proofs of I and II both depend on his former results on the classification of simple Lie algebras [1]. Shortly afterwards, H. Weyl [10] remarked that Theorem I might be obtained without the classification-theory by means of the completeness of prime characters of compact groups, established by Peter and himself [8][2]. Furthermore, he deduced in his lecture [11] also without the classification-theory the existence of *bases* of root systems, essentially equivalent to the Theorem II of Cartan, from the detailed considerations of certain finite transformation groups $\mathfrak{S}$.[3] In the present note we shall give a direct proof of the existence of bases of root systems in simplification of Weyl's method, together with some applications to the related problems. For the sake of completeness we include in the Appendix a proof of Weyl's theorem concerning compact semi-simple Lie groups.[4]

Notations. We denote by E^n an n-dimensional (real) Euclidean space. Its origin (or zero vector) is denoted by 0. E^n is also considered as a linearly ordered vector group, the order being defined lexicographically by

$$x<y \rightleftarrows x^{(1)}=y^{(1)},\cdots, x^{(i-1)}=y^{(i-1)},\ x^{(i)}<y^{(i)},$$
$$\text{for}\quad x=x^{(1)}\theta_1+\cdots+x^{(n)}\theta_n,\ y=y^{(1)}\theta_1+\cdots+y^{(n)}\theta_n,$$

where $\{\theta_1,\cdots,\theta_n\}$ is a fixed base of E^n. (This is the only method to make a vector space into a linearly ordered vector group.) For a subset $\mathfrak{a}$ of E^n, $\{\{\mathfrak{a}\}\}$ and $\{\{\mathfrak{a}\}\}_l$ denote the additive closure of $\mathfrak{a}$ and the linear closure of $\mathfrak{a}$ with real coefficients, respectively.

ξ being any element of E^n and (ξ, x) the inner product of ξ, x in E^n, every character of E^n is given by the formula

$$e^{2\pi\sqrt{-1}(\xi,x)}\quad \text{for } x\in E^n.$$

We shall identify the character $e^{2\pi\sqrt{-1}(\xi,x)}$ with the element ξ, and the character group of E^n with E^n itself. The annihilator of a closed subgroup Γ of E^n in this sense will be denoted by $\Gamma^\wedge$. Then the following formulae are obvious:

$$\begin{aligned}&\Gamma^{\wedge\wedge}=\Gamma,\\ &\Gamma\subseteq\Gamma_1 \text{ implies } \Gamma^\wedge\supseteq\Gamma_1^\wedge \text{ and } \Gamma_1/\Gamma\cong\Gamma^\wedge/\Gamma_1^\wedge.\end{aligned}\tag{0.1}$$

We denote by $S_\xi(k)$ the reflection of E^n with respect to a hyperplane $\pi_\xi(k)=\{x\,;\ x\in E^n, (\xi, x)=k\}$; in particular, we put $\pi_\xi=\pi_\xi(0)$, $S_\xi=S_\xi(0)$. Obviously we have

$$S_\xi x = x - \frac{2(\xi, x)}{(\xi, \xi)}\xi \qquad \text{for } x \in E^n, \tag{0.2}$$

$$S_\xi(k) = T\left(\frac{2k\xi}{(\xi,\xi)}\right)S_\xi, \tag{0.3}$$

where $T(\gamma)$ denotes the translation of E^n defined by the addition of a vector γ in E^n. $\mathfrak{T}(\Gamma)$ denotes the group composed of all $T(\gamma)\,(\gamma \in \Gamma)$.

§ 1. Preliminaries.

We shall give in this section some special notations and theorems on semi-simple complex Lie algebras, which are necessary for our later considerations.

Let $\tilde{L}$ be a semi-simple complex Lie algebra. We take and fix one of its Cartan decomposition

$$\begin{aligned} &\tilde{L} = \tilde{H} + \textstyle\sum_\alpha M_\alpha \qquad (\alpha \in \mathfrak{r}), \\ &(\text{complex}) \dim \tilde{H} = n = \text{rank of } \tilde{L}, \\ &\mathfrak{r} = \textit{root system} \text{ of } \tilde{L}. \end{aligned} \tag{1.1}$$

For convenience, we define an inner product in $\tilde{H}$ by

$$\begin{aligned} &(h, h) = -\frac{1}{(2\pi)^2}\varphi(h) \qquad \text{for } h \in \tilde{H}, \\ &\varphi(h) = \textstyle\sum_\alpha \alpha(h)^2 = \text{fundamental quadratic form of } \tilde{L}, \end{aligned} \tag{1.2}$$

and we always consider the roots of $\tilde{L}$ as vectors in $\tilde{H}$ in the following way:

$$\alpha(h) = 2\pi\sqrt{-1}\,(\alpha, h) \text{ for } h \in \tilde{H}. \tag{1.3}$$

$\mathfrak{r}$ is thus regarded as a subset of $\tilde{H}$.

If we put $\Theta = \{\{\mathfrak{r}\}\}_l$, Θ becomes an n-dimensional (real) Euclidean space according to the above-defined inner product, and $\mathfrak{r}$ satisfies in Θ the following four conditions:

(i) $0 \bar{\in} \mathfrak{r}$,

(ii) when $\alpha \in \mathfrak{r}$, $-\alpha$ is also a vector of $\mathfrak{r}$, and $\{\{\alpha\}\}_l$ contains no other vector of $\mathfrak{r}$,

(iii) for any α, $\beta \in \mathfrak{r}$, $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer,

(iv) when α, $\beta \in \mathfrak{r}$, $S_\alpha\beta=\beta-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ is contained in $\mathfrak{r}$, too.

It is known that, conversely, these conditions are sufficient to characterize root systems.[5)]

Now let P be any (complex) representation of $\tilde{L}$ on the representation module $\mathfrak{M}$. As is well-known, $\mathfrak{M}$ is completely reducible and we may choose a base $\{e_1,\cdots,e_N\}$ of $\mathfrak{M}$ such that

$$he_i=\lambda^i(h)e_i \text{ for } h\in\tilde{H} \qquad (i=1,\cdots,N), \tag{1.4}$$

where λ^i is a linear form on $\tilde{H}$, called a *weight* of this representaion. The root forms (and the zero form) are nothing other than the weights of the adjoint representation of $\tilde{L}$. We shall call the linear form which appears as a weight of some representation of $\tilde{L}$, simply a weight of $\tilde{L}$ It is readily seen that the set of all weights of $\tilde{L}$ forms an additive group Λ of linear forms on $\tilde{H}$.

As in (1.3) we can regard weights of $\tilde{L}$ as vectors contained in $\tilde{H}$. Then it is proved that for any weight λ and any root α, $\frac{2(\alpha,\lambda)}{(\alpha,\alpha)}$ is an integer (this shows that $\lambda\in\Theta$) and that the following forms

$$\lambda,\ \lambda-\varepsilon\alpha,\ \lambda-2\varepsilon\alpha,\cdots,\ \lambda-\frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha(=S_\alpha\lambda), \qquad \varepsilon=\text{sign}(\alpha,\lambda),^{6)} \tag{1.5}$$

are weights of one and the same representation of $\tilde{L}$. In particular $S_\alpha\lambda$ has the same multiplicity as that of λ.

We now fix a linear order in Θ once for all. If we put $\mathfrak{r}^*=\{2\alpha^*\,;\,\alpha^*=\frac{\alpha}{(\alpha,\alpha)},\ \alpha\in\mathfrak{r}\}$, the above-mentioned property of weights may be written simply as

$$\Lambda\subseteq\{\{\mathfrak{r}^*\}\}^\wedge. \tag{1.6}$$

Each element of the right hand side of (1.6) will be called a *possible weight* of $\tilde{L}$. Let $\mathfrak{S}$ be the (finite) group of Euclidean transformations of Θ generated by $S_\alpha(\alpha\in\mathfrak{r})$. If the inequalities

$$S\xi \leq \xi \quad \text{for all } S \in \mathfrak{S} \tag{1.7}$$

are satisfied for a possible weight ξ, we shall call it a *highest possible weight.*

It is convenient to state here the following

Proposition 1. $\Lambda = \{\{\mathfrak{r}^*\}\}^\wedge$.

From this "criterion for weights" follows immediately that every highest possible weight is actually a highest weight of some reprsentation, i. e. the first half of Theorem I in the Introduction. We shall give a proof in the Appendix.[7]

§ 2. Properties of root systems.

In §§ 2, 3 we consider root systems *in abstracto.* So we call here a root system every (finite) set of vectors in E^n with the four properties stated in § 1. The dimensionality, congruence, irreducibility...etc for root systems are defined as usual and all irreducible root systems can be determined explicitly, whence the complete classification of simple Lie algebras is obtained.[8]

Proposition 2. *Let* $\mathfrak{r}$ *be a root system in* E^n. *Then* $\mathfrak{r}^* = \{2\alpha^*; \alpha^* = \frac{\alpha}{(\alpha,\alpha)}, \alpha \in \mathfrak{r}\}$ *is again a root system in* E^n.

Remark, As one readily sees, the factor 2 before α^* is inessential and can be replaced by any other real number $k \neq 0$, but it will be convenient to take $k=2$ in succeding sections. In the next proof we consider the case $k=1$.

Proof. The first two conditions for root systems are clearly satisfied by $\mathfrak{r}^*$. When $\alpha^* = \frac{\alpha}{(\alpha,\alpha)}$, $\beta^* = \frac{\beta}{(\beta,\beta)}$ are in $\mathfrak{r}^*$, we have easily

$$\frac{2(\alpha^*, \beta^*)}{(\alpha^*, \alpha^*)} = \frac{2(\alpha, \beta)}{(\beta, \beta)} \quad (=\text{integer}), \tag{2.1}$$

$$S_{\alpha^*}\beta^* = S_\alpha \beta^* = (S_\alpha \beta)^*, \tag{2.2}$$

which proves (iii) and (iv) respectively, q. e. d.

We shall call $\mathfrak{r}^*$ in Proposition 2 the *inverse system* of $\mathfrak{r}$. Obviously

$$\mathfrak{r}^{**} = \mathfrak{r} \tag{2.3}$$

ane as $S_\alpha = S_{2\alpha\alpha^*}$ the transformation group $\mathfrak{S}$ generated by $S_\alpha(\alpha\in\mathfrak{r})$ coincides with that generated by $S_{2\alpha^*}(2\alpha^*\in\mathfrak{r}^*)$ and it holds by (2.2), $S\beta^*=(S\beta)^*$ for all $S\in\mathfrak{S}$.[7)]

Lemma 1. *Let* $\alpha, \beta\in\mathfrak{r}, \alpha\neq\pm\beta$ (i. e. $\{\{\alpha\}\}_l\neq\{\{\beta\}\}_l$). *Then* $\beta-\varepsilon\alpha\in\mathfrak{r}$, *where* $\varepsilon=\mathrm{sign}(\alpha, \beta)$.

Proof. On account of the condition (ii), we have only to consider the case $(\alpha, \beta)>0$, $(\alpha, \alpha)\geq(\beta, \beta)$. Then as $(\alpha, \alpha)>2(\alpha, \beta)-(\beta, \beta)$, it holds $(\alpha, \alpha)>(\alpha, \beta)>0$. But the inequality $1>\frac{(\alpha, \beta)}{(\alpha, \alpha)}>0$ implies by (iii) $\frac{2(\alpha, \beta)}{(\alpha, \alpha)}=1$. Hence we have from (iv) $\beta-\alpha=S_\alpha\beta\in\mathfrak{r}$, q. e. d.

Lemma 2. *Let* $\{\alpha_i\}$, $\{\xi^i\}(i=1,\cdots,n)$ *be a pair of mutually dual bases of* $\boldsymbol{E}^n$ *such that* $(\alpha_i, \xi_j)=\delta_{ij}$[10)] $(i, j=1,\cdots,n)$. *If* $(\alpha_i, \alpha_j)\leq 0$ *for* $i\neq j$, *then it holds* $(\xi_i, \xi_j)\geq 0$ *for all* $i, j=1,\cdots,n$.

We omit the easy proof by induction with respect to n.

From now on we shall regard $\boldsymbol{E}^n$ as a linearly ordered vector group by means of an arbitrary (but fixed) linear order and consider the connections between a root system and the linear orders.

Proposition 3. *Let* $\mathfrak{r}$ *be an n-dimensional root system in* $\boldsymbol{E}^n$. *If we put* $\mathfrak{r}^+=\{\alpha\,;\alpha\in\mathfrak{r}, \alpha>0\}$, *there exist n basic roots* $\alpha_1,\cdots,\alpha_n$ *in* $\mathfrak{r}^+$ *so that any positive root* α *can be expressed uniquely as follows*

$$\alpha=p_1\alpha+\cdots+p_n\alpha_n \quad (p_i: \textit{non-negative integers}). \tag{2.4}$$

Before the proof we give some remarks. If we assume that the proposition is proved and that α_i are arranged in the increasing order

$$\alpha_1<\alpha_2<\cdots<\alpha_n,$$

then we have naturally

$$\left\{\begin{array}{l} \alpha_1=\mathrm{Min}(\mathfrak{r}^+), \\ \alpha_2=\mathrm{Min}(\mathfrak{r}^+-\{\{\alpha_1\}\}_l)^{11)} \\ \quad\cdots\cdots \\ \alpha_n=\mathrm{Min}(\mathfrak{r}^+-\{\{\alpha_1,\cdots,\alpha_{n-1}\}\}_l). \end{array}\right. \tag{2.5}$$

Thus the set of *basic roots* $\{\alpha_1,\cdots,\alpha_n\}$, which we call simply a *base* of $\mathfrak{r}$, will be determined uniquely for a given linear order in $\boldsymbol{E}^n$.

Further, $\alpha_j-\varepsilon\alpha_i$, $\varepsilon=\mathrm{sign}(\alpha_i, \alpha_j)$ are roots for $i<j$ by Lemma 1. Therefore if $(\alpha_i, \alpha_j)>0$, i. e. $\varepsilon=+1$, we would have $\alpha_j>\alpha_j-\varepsilon\alpha_i\in\mathfrak{r}^+-$

$\{\{a_1,\cdots,a_{j-1}\}\}_l$, which contradicts to (2.5). Hence it holds

$$(a_i, a_j) \leq 0 \text{ for } i \neq j. \tag{2.6}$$

Proof of Proposition 3. We proceed by induction on the dimension n. If $n=1$, then $\mathfrak{r}^+$ contains only one vector, and the proposition is obvious. Let $n>1$. We define $a_1,\cdots,a_n$ by (2.5) and put $\mathfrak{r}'=\mathfrak{r}\cap\{\{a_1,\cdots,a_{n-1}\}\}_l$; clearly it forms an $(n-1)$-dimensional root system and $\mathfrak{r}'^+=\mathfrak{r}^+\subset\{\{a_1,\cdots,a_{n-1}\}\}_l$. We can also verify

$$\begin{cases} a_1=\mathrm{Min}(\mathfrak{r}'^+), \\ a_2=\mathrm{Min}(\mathfrak{r}'^+-\{\{a_1\}\}_l), \\ \quad\cdots\cdots \\ a_{n-1}=\mathrm{Min}(\mathfrak{r}'^+-\{\{a_1,\cdots,a_{n-2}\}\}_l). \end{cases} \tag{2.5$'$}$$

If follows from the assumption of induction that any positive root a' of $\mathfrak{r}'$ may be expressed in the form

$$a'=p_1'a_1+\cdots+p_{n-1}'a_{n-1} (p_i' \text{: non-negative integers}). \tag{2.4$'$}$$

Now we prove the proposition inductively from lower to higher a. When $0<a<a_n$, a being contained in $\mathfrak{r}'^+$, the proposition is valid by (2.4)$'$, and it is also obvious for $a=a_n$. We may assume therefore $a>a_n$ $(>a_{n-1}>\cdots>a_1)$ and we consider n vectors $a-a_i (i=1,\cdots,n)$. We have only to prove that at least one of them is a root; then this vector will be indeed a positive root lower than a, having an expression of type (2.4) by the assumption of second induction, so that the expression of a follows immediately. Suppose, on the contrary, that all $a-a_i$ were not in $\mathfrak{r}$. We would have by Lemma 1, $\varepsilon=\mathrm{sign}(a_i, a)\neq+1$, namely

$$(a_i, a)\leq 0 \quad (i=1,\cdots,n),$$

On the other hand, by (2.6) and Lemma 2

$$(\xi_i, \xi_j)\geq 0 \quad (i,j=1,\cdots,n),$$

for the dual base $\{\xi_i\}$ to $\{a_i\}$. These two systems of inequalities would imply

$$a=\sum_i(a, a_i)\xi_i=\sum_{i,j}(a, a_i)(\xi_i,\xi_j)a_j\leq 0,$$

which contradicts to $a>0$, q. e. d.

In the above proof, the expression $a=\sum_i p_i a_i$ of the root $a>a_n$ has at least two positive coefficients, so that if $(\alpha_{i_0}, \alpha)>0$,

$$S_{\alpha_{i_0}}\alpha=\left(p_{i_0}-\frac{2(\alpha_{i_0}, \alpha)}{(\alpha_{i_0},\alpha_{i_0})}\right)\alpha_{i_0}+\sum_{i\neq i_0} p_i\alpha_i$$

has at least one positive coefficient. Therefore it is also a positive root by Proposition 1 and thus $\alpha> S_{\alpha_{i_0}}\alpha>0$. We shall call for a while a set $\mathfrak{a}$ of roots *closed*, when $\alpha, \beta\in\mathfrak{a}$ implies $S_\alpha\beta\in\mathfrak{a}$; then it follows by the same arguments as in the above proof the following

Corollary 1. *A root system* $\mathfrak{r}$ *is the minimal closed set containing its base* $\{\alpha_1,\cdots, \alpha_n\}$. (Thus a root system is uniquely determined by any one of its bases.)

Corollary 2. *The transformation group* $\mathfrak{S}$ *is generated by n reflections* $S_{\alpha_1},\cdots,S_{\alpha_n}$.[12]

For we have $S_{S_\alpha\beta}=S_\alpha S_\beta S_\alpha^{-1}$.

§ 3. Some applications.

We denote by Π the (open) angular domain of E^n defined by inequalities

$$(\alpha, x)>0 \text{ for all } \alpha\in\mathfrak{r}^+. \tag{3.1}$$

According to Proposition 3, we may replace (3.1) by

$$(\alpha_i, x)>0 \text{ for } i=1,\cdots, n, \tag{3.1$'$}$$

where $\{\alpha_i\}$ is a base of $\mathfrak{r}$. Therefore the angular domain Π is limited by just n faces.

Obviously Π is one of the connected components of $E^n-\cup_\alpha\pi_\alpha$[11] and, since the set $\cup_\alpha\pi_\alpha$ is $\mathfrak{S}$-invariant, it holds

$$S\Pi=\Pi \text{ or } S\Pi\cap\Pi=\phi^{12)} \text{ for any } S\in\mathfrak{S}. \tag{3.2}$$

More precisely, it is known that Π is a "fundamental domain" of $\mathfrak{S}$ and so we have

Lemma 3. $S\Pi\cap\Pi=\phi$ *for every* $S\in\mathfrak{S}$, $S\neq 1$.[14]

The proof will be given in the Appendix.[15]

We now consider the relation between the bases of $\mathfrak{r}$ correspoding to different linear orders of E^n.

Proposition 4. *Let* $\{\alpha_1,\cdots,\alpha_n\}$, $\{\beta_1,\cdots,\beta_n\}$ *be two bases of* $\mathfrak{r}$ *corresponding* *to two linear orders of* E^n. *Then there exists a uniquely determined transformation* S *in* $\mathfrak{S}$ *such that in some arrangement of* $\{\beta_i\}$ *we have*

$$\beta_i = S\alpha_i \quad (i=1,\cdots,n). \tag{3.3}$$

(It follows that the number of various bases of $\mathfrak{r}$ is equal to the order of $\mathfrak{S}$.)

Proof. The angular domain defined by inequalities

$$(\beta_i, x) > 0 \text{ for } i=1,\cdots,n \tag{3.1)''$$

is, like the one defined by (3.1)′, a connected component of $E^n - \cup_\alpha \pi_\alpha$. Hence it coincides with some $S\Pi$, where S is uniquely determined according to Lemma 3, and we have $\pi_{\beta_i} = S\pi_{\alpha_i} = \pi_{S\alpha_i}$ in some arrangement of $\{\beta_i\}$. If follows that $\beta_i = \pm S\alpha_i$, and that for any point x in Π we have

$$(S\alpha_i, Sx) = (\alpha_i, x) > 0, \ (\beta_i, Sx) > 0,$$

whence we have (3.3), q. e. d.

We denote by $\mathfrak{T}$ the group of all Euclidean transformations of E^n which leave $\mathfrak{r}$ invariant, and by $\mathfrak{P}$ the subgroup of $\mathfrak{T}$ which leaves the base $\{\alpha_i\}$ invariant. We have immediately

Corollary. $\mathfrak{S}$ *is a normal subgroup of* $\mathfrak{T}$ *and*

$$\mathfrak{T} = \mathfrak{P}\cdot\mathfrak{S}, \quad \mathfrak{P}\cap\mathfrak{S} = \{1\}, \tag{3.4}$$

where $\mathfrak{P}$ *consists of the so-called "particular rotations", i. e. if* $P \in \mathfrak{P}$ *is regarded as a permutation of* $\mathfrak{r}$, *the sum of the root vectors in each cycle of the permutation* P *does not vanish.*[16]

For if $\alpha \gtrless 0$ we have $P\alpha \gtrless 0$, $P^2\alpha \gtrless 0, \cdots$etc, so that $\alpha + P\alpha + P^2\alpha + \cdots \neq 0$.

We have seen already that a base of a root system satisfies the following three conditions:

(i) $\alpha_1, \cdots, \alpha_n$ are linearly independent,

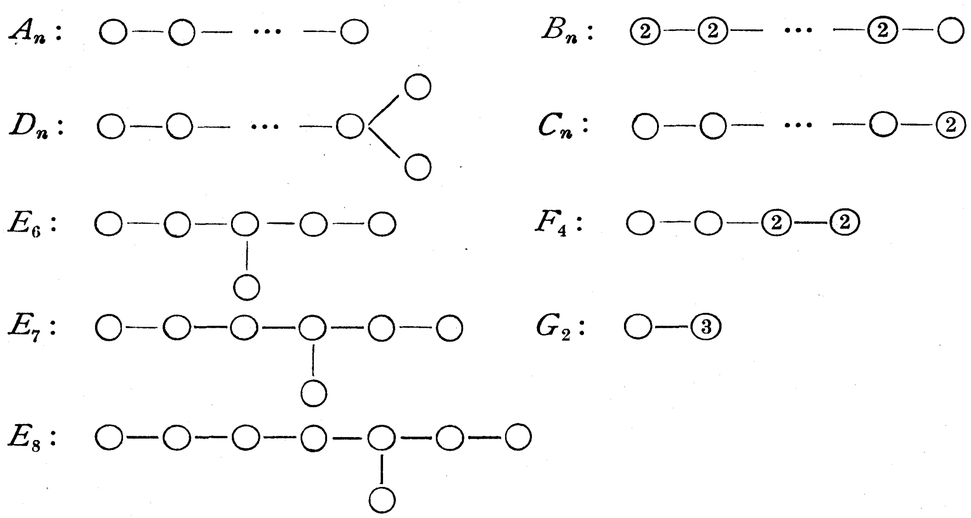
(ii) $\dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ are integers for $i, j = 1, \cdots, n$,

(iii) $(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$,

and, if moreover the root system is an irreducible one, we have by the Cor. 1 to Prop. 3,

(iv) for any α_i, α_j there exists a chain $\alpha_i = \alpha_{i_0}, \alpha_{i_1}, \cdots, \alpha_{i_r} = \alpha_j$ such that $(\alpha_{i_{k-1}}, \alpha_{i_k}) < 0$ $(k=1,\cdots,r)$.

Now we can easily classify the sets of vectors satisfying these conditions, the method being quite similar to that of van der Waerden's classification of the irreducible root systems. We can illustrate the result as follows.[17)]



In these schemata the small circles represent the vectors and the lines connecting two circles show that the corresponding vectors are not orthogonal. The number marked in a circle is the ratio of length of the corresponding vector to that of any unmarked one, the lengths of the vectors corresponding to these unmarked circles being all equal to each other. The angle between two vectors which are not orthogonal is 120°, 135° and 150° according as the ratio of lengths of them (the ratio of the longer to the shorter) is 1, 2 and 3, respectively.

It follows from this result that the conditions (i), (ii), (iii) characterize completely the bases of the root systems. It implies, in particular, together with (2.2) and Cor. 1 to Prop. 3 that if $\{\alpha_1, \cdots, \alpha_n\}$ is a base of $\mathfrak{r}$, $\{2\alpha_1^*, \cdots, 2\alpha_n^*\}$ is a base of the inverse system $\mathfrak{r}^*$ and the schema of the latter is obtained from that of the former by replacing the numbers in the circles by their inverses. Thus B_n and C_n are mutually inverse, while all the other irreducible root spstems are self-inverse.[18)]

On the other hand, if we denote by $A(\tilde{L})$ the group of all automorphisms of $\tilde{L}$ and by $I(\tilde{L})$ the adjoint group, i. e. the group of all inner automorphisms of $\tilde{L}$, it is known after Gantmacher [7] that

$$A(\tilde{L})/I(\tilde{L}) \cong \mathfrak{T}/\mathfrak{S} \cong \mathfrak{P}. \tag{3.5}$$

By the above schemata, $\mathfrak{P}$ can be readily determined, for it can be considered as the group of all permutations of a base, which leave invariant the configuration of the corresponding schema. Thus if $\tilde{L}$ is simple, $\mathfrak{P}$ reduces to the unity group except for A_n, D_n and E_6. It consists of two elements for A_n, $D_n (n \neq 4)$, E_6 and is isomorphic to the symmetric group of three letters for D_4.[16)]

§ 4. Cartan's theorem.

We now prove Theorem II. By (1.6) and (1.7), the highest possible weights of $\tilde{L}$ are characterized by the following two properties:

(i) $\xi \in \{\{\mathfrak{r}^*\}\}^\wedge$,

(ii) $S\xi \leq \xi$ for all $S \in \mathfrak{S}$.

Therefore it is clear that they form an additive semi-group.

Condition (ii) implies in particular $S_\alpha \xi = \xi - \frac{2(\alpha, \xi)}{(\alpha, \alpha)} \alpha \leq \xi$ for all $\alpha \in \mathfrak{r}$, namely

(ii)′ $(\alpha, \xi) \geq 0$ for all $\alpha \in \mathfrak{r}^+$.

These two conditions (ii), (ii)′ are equivalent. For if the (closed) domain defined in Θ by the inequalities (ii) were not identical with that defined by the inequalities (ii)′, some outer point of the former would be contained in the inner part of the latter, namely in $\Pi = \{\theta ; \theta \in \Theta, (\alpha, \theta) > 0$ for all $\alpha \in \mathfrak{r}^+\}$. It would then exist such θ_0 in Π and $S_0 \neq 1$ in $\mathfrak{S}$ that $S\theta_0 \leq S_0\theta_0$ for all $S \in \mathfrak{S}$. This would imply in particular

$$(\alpha, S_0\theta_0) > 0 \text{ for all } \alpha \in \mathfrak{r}^+,$$

and thus $S_0\theta_0 \in S_0\Pi \cap \Pi$, what is impossible by Lemma 3.

By this remark, we may replace the conditions (i), (ii) by the following one

$$(\xi, 2\alpha^*) \text{ are non-negative integers for all } \alpha \in \mathfrak{r}^+. \qquad (4.1)$$

Making use of our Propositions 2 and 3, (4.1) willl be again reduced to the following form:

$$(\xi, 2\alpha_i^*) \text{ are non-negative integers for } i = 1, \cdots, n, \qquad (4.2)$$

where $\{2\alpha_1^*, \cdots, 2\alpha_n^*\}$ is a base of $\mathfrak{r}^*$. Therefore, in order to find the fundamental weights, it is sufficient to take the dual base $\{\xi_1,\cdots,\xi_n\}$ to $\{2a_1^*,\cdots,2a_n^*\}$ in the Euclidean space Θ; for ξ_i obviously satisfy (4.2) and we have for any highest possible weight ξ

$$\xi=\sum_i(\xi,2a_i^*)\xi_i,$$

where $(\xi,2a_i^*)$ are by (4.2) non-negative integers for all i. The Theorem II is thus proved.

Example. The method employed in the above proof is also useful for the actual determination of fundamental weights of a given semi-simple Lie algebra. We indicate it here only for the simple algebras of type A_n.

Let $\theta^0,\theta^1,\cdots,\theta^n$ be the weights of its irreducible representation $SL(n+1,C)$.[19] Then $\{\theta^1,\cdots,\theta^n\}$ is a base of Θ and we have

$$\begin{aligned}&\mathfrak{r}=\{\theta^i-\theta^j\,;\ i\neq j,\ i,j=1,\cdots,n\},\\&\theta^0+\theta^1+\cdots+\theta^n=0.\end{aligned}\tag{4.3}$$

If we take the dual base $\{\theta_1,\cdots,\theta_n\}$ to $\{\theta^1,\cdots,\theta^n\}$, the inner product in Θ will be expressed as follows (*cf.* (1.2), (1.3)):

$$(\theta,\theta)=\sum_\alpha(\alpha,\theta)^2=\sum_{i,j}g_{ij}(\theta^i,\theta)(\theta^j,\theta)=\sum_{i,j}g^{ij}(\theta_i,\theta)(\theta_j,\theta),\tag{4.4}$$

where we can easily show that $g_{ij}=(2n+2)(1+\delta_{ij})$.[10] In order to simplify our calculations, we shall neglect in the following the inessential factor $2n+2$ of g_{ij}. Thus

$$(g_{ij})=\begin{pmatrix}2&1&\cdots&1\\1&2&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&2\end{pmatrix}\quad\text{and}\quad(g^{ij})=(g_{ij})^{-1}=\frac{1}{n+1}\begin{pmatrix}n&-1&\cdots&-1\\-1&n&\cdots&-1\\\vdots&\vdots&\ddots&\vdots\\-1&-1&\cdots&n\end{pmatrix}.$$

Hence, the square of the lengths of the roots being all equal to 2, $\mathfrak{r}=\mathfrak{r}^*$ and the expression of $\mathfrak{r}^*$ in $\{\theta_1,\cdots,\theta_n\}$ is

$$\mathfrak{r}^*=\{\theta_i-\theta_j,\pm\theta_i\,;\ i\neq j,\ i,j=1,\cdots,n\}.\tag{4.5}$$

If we consider the lexicographical linear order of Θ with respect to $\{\theta_1,\cdots,\theta_n\}$, the basic roots of $\mathfrak{r}^*$, arranged in the decreasing order, are

$$\begin{cases} 2a_1^* = \theta_1 - \theta_2, \\ 2a_2^* = \theta_2 - \theta_3, \\ \cdots \\ 2a_{n-1}^* = \theta_{n-1} - \theta_n, \\ 2a_n^* = \theta_n. \end{cases} \tag{4.6}$$

The expression of the fundamental weights in $\{\theta^1, \cdots, \theta^n\}$ can be obtained simply by the computation of the inverse matrix; we have namely

$$\begin{pmatrix} 1 & -1 & & & 0 \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 1 \\ 0 & & & & 1 \end{pmatrix} = (\delta_{ij})$$

and thus

$$\begin{cases} \xi_1 = \theta^1, \\ \xi_2 = \theta^1 + \theta^2, \\ \cdots \\ \xi_n = \theta^1 + \theta^2 + \cdots + \theta^n = -\theta^0. \end{cases} \tag{4.7}$$

[20]

Appendix

We shall give here a proof of Proposition 1 in § 1. To the purpose, we shall recall one of the main results of Weyl's paper [10]. After some preparations in § § 5, 6, we shall obtain in § 7 the main result which contains Weyl's theorem in a sharpened form. A proposition equivalent to Prop. 1 will be obtained by way of the proof of this theorem.

§ 5. Preliminaries

While we have been hitherto exclusively concerned with subjects of purely algebraic character, we must employ analytical methods in the following considerations.[21]

It is well-known that the unitary restriction of the Cartan decomposition (1.1) affords a compact (real) form L of $\tilde{L}$. We have thus a decomposition

$$L=H+\cdots, \qquad H=\tilde{H}\cap L, \tag{5.1}$$

where H is identical with $\theta=\{\{\mathfrak{r}\}\}_l$.

We shall denote generally by $\tilde{\mathfrak{G}}$ a connected semi-simple (complex) Lie group which corresponds to the Lie algebra $\tilde{L}$, and by $\tilde{\mathfrak{H}}$, $\mathfrak{G}$ and $\mathfrak{H}$ the (real) analytic subgroups of $\tilde{\mathfrak{G}}$ corresponding to $\tilde{H}$, L and H, respectively. $\mathfrak{G}$ is called the *compact form* of $\tilde{\mathfrak{G}}$. We denote, in particular, the simplg connected Lie group and the adjoint group of $\tilde{L}$ by $\tilde{\mathfrak{G}}^0$ and $\tilde{\mathfrak{G}}_0$, and their compact forms by $\mathfrak{G}^0$ and $\mathfrak{G}_0$, respectively.[22] Similarly, we shall make use of notations such as Δ^0, Δ, Δ_0 to denote the corresponding notions concerning $\tilde{\mathfrak{G}}^0$, $\tilde{\mathfrak{G}}$, $\tilde{\mathfrak{G}}_0$ respectively.

(After that Weyl's theorem is proved to be valid, it will follow, as we shall show it, that $\mathfrak{G}$ is a maximal compact subgroup of $\tilde{\mathfrak{G}}$, while $\mathfrak{H}$ is a maximal torus subgroup of $\mathfrak{G}$ and is identical with its centralizer $Z(\mathfrak{H})$ in $\mathfrak{G}$, and that the (left) homogeneous space $\tilde{\mathfrak{G}}/\mathfrak{G}$ is homeomorphic to a Euclidean space, while $\mathfrak{G}/\mathfrak{H}$ is a simply connected and compact space.)

As the subalgebra $\tilde{H}$ is abelian, the exponential mapping from $\tilde{H}$ onto $\tilde{\mathfrak{H}}$ gives a group-theoretical homomorphism; we denote by Δ the kernel of this homomorphism. If $\tilde{\mathfrak{G}}_1$ is a homomorphic image of $\tilde{\mathfrak{G}}$, we have $\Delta \subseteq \Delta_1$ and, in particular,

$$\Delta^0 \subseteq \Delta \subseteq \Delta_0. \tag{5.2}$$

(It will be proved in the following section, that Δ is a discrete subgroup of rank n contained in θ.)

On the other hand, let u be an element of the normalizer $N(\mathfrak{H})$ of $\mathfrak{H}$ in $\mathfrak{G}$ and $S(u)$ the Euclidean transformation of θ induced in $\theta=H$ by the adjoint representation of u. The set of all $S(u)\,(u\in N(\mathfrak{H}))$ forms a transformation group $\mathfrak{S}_1$, which is a homomorphic image of $N(\mathfrak{H})$. As the kernel of this homomorphism is obviously $Z(\mathfrak{H})$, it holds the natural isomorphism

$$\mathfrak{S}_1 \cong N(\mathfrak{H})/Z(\mathfrak{H}) \tag{5.3}$$

It should be noted that $\mathfrak{S}_1$ is determined by the Lie algebra L and is independent of the choice of the Lie group $\mathfrak{G}$. (It will be proved in § 7 that $\mathfrak{S}_1$ is identical with $\mathfrak{S}$.)

When $\Delta \subseteq \theta$,[23] Δ is clearly $\mathfrak{S}_1$-invariant, whence follows that the composite $\mathfrak{S}_1(\Delta)$ of the groups $\mathfrak{S}_1$ and $\mathfrak{T}(\Delta)$ splits into the following form:

$$\mathfrak{S}_1(\Delta)=\mathfrak{S}_1\cdot\mathfrak{T}(\Delta),\ \mathfrak{S}_1\cap\mathfrak{T}(\Delta)=\{1\},\quad \mathfrak{T}(\Delta):\text{ normal subgroup of }\mathfrak{S}_1(\Delta). \tag{5.4}$$

If we consider the group of Euclidean transformations generated by all $S_\alpha(k)\,(\alpha\in\mathfrak{r}, k=0, \pm1,\cdots)$, we obtain an algebraic analoge $\mathfrak{S}(\{\{\mathfrak{r}^*\}\})$ of $\mathfrak{S}_1(\Delta)$; we have then from (0.3) and $\mathfrak{S}$-invariance of $\{\{\mathfrak{r}^*\}\}$

$$\mathfrak{S}(\{\{\mathfrak{r}^*\}\})=\mathfrak{S}\cdot\mathfrak{T}(\{\{\mathfrak{r}^*\}\}),\ \mathfrak{S}\cap\mathfrak{T}(\{\{\mathfrak{r}^*\}\})=\{1\},\quad \mathfrak{T}(\{\{\mathfrak{r}^*\}\}):\text{ normal subgroup of }\mathfrak{S}(\{\{\mathfrak{r}^*\}\}). \tag{5.5}$$

§ 6 Relations between the weight group Λ, the kernel Δ and the root system $\mathfrak{r}$.

We denote by Λ the subgroup of Θ composed of all weights of $\tilde{L}$ that appear in some one-valued representations of $\tilde{\mathfrak{G}}$. Accordingly, the group of all weights of $\tilde{L}$, formerly denoted by Λ, will be written now as Λ^0. Since $\{\{\mathfrak{r}^*\}\}^\wedge\supseteq\Lambda\supseteq\{\{\mathfrak{r}\}\}$, it is clear that Λ is a discrete group of rank n.

When $\mathfrak{G}$ is compact,[23] it holds the following

Proposition 5. *Δ is the annihilator of Λ.* (Thus Δ is also a discrete group of rank n contained in Θ.)

Proof. Let $\mathfrak{P}$ be a one-valued representation of $\tilde{\mathfrak{G}}$ and P the representation of $\tilde{L}$ derived from $\mathfrak{P}$. If we choose the base (1.4) in the representation module, any h in $\tilde{H}$ will be represented by a matrix of the form

$$P(h)=\begin{pmatrix} 2\pi\sqrt{-1}\ (\lambda^1, h) & & 0 \\ & \ddots & \\ 0 & & 2\pi\sqrt{-1}\,(\lambda^N, h)\end{pmatrix},$$

where $\lambda^1,\cdots,\lambda^N$ are weights of this representation. Therefore, if $h\in\Delta$, the matrix

$$\mathfrak{P}(\exp h)=\exp(P(h))=\begin{pmatrix} e^{2\pi\sqrt{-1}(\lambda^1, h)} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi\sqrt{-1}(\lambda^N, h)}\end{pmatrix}$$

should be equel to the unit matrix and it follows $h\in\{\{\lambda^1,\cdots,\lambda^N\}\}^\wedge$. If we take a faithful representation $\mathfrak{P}$ of $\tilde{\mathfrak{G}}$, which exists surely since we assumed $\mathfrak{G}$ to be compact,[24] we see immediately that the converse of the above argument is also true. We have thus

$$\Lambda^\wedge \subseteq \{\{\lambda^1,\cdots,\lambda^N\}\}^\wedge = \Delta \subseteq \Lambda^\wedge,$$

which completes the proof.

Corollary 1. *A one-valued representation* $\mathfrak{P}$ *of* $\tilde{\mathfrak{G}}$ *is faithful, if and only if its weights* $\{\lambda^1,\cdots\lambda^N\}$ *generate the whole weight group* Λ *of* $\tilde{\mathfrak{G}}$.

Corollary 2. $\Delta_0 = \{\{\mathfrak{r}\}\}^\wedge = \cap_\alpha \cup_k \pi_\alpha(k)$. $(\alpha \in \mathfrak{r}, k=0, \pm 1,\cdots)$

Corollary 3. $\Delta \supseteq \{\{\mathfrak{r}^*\}\}$.

The second half of Corollary 1 is included in the above proof, while its first half is a consequence of the existence of a faithful representation of $\tilde{\mathfrak{G}}$, as well as the fact that a representation of $\tilde{\mathfrak{G}}$ is uniquely determined by the highest weights of its irreducible parts. Corollaiy 2 is a special case of the proposition, but also evident from the definition of $\pi_\alpha(k)$, and, in fact, is independent of the compactness of $\mathfrak{G}_0$. Corollary 3 comes from (1.6), (0.1) and the above proposition.

By a similar reason as above, we can also conclude from Peter-Weyl's theorem that $\mathfrak{S}$ is contained in $\mathfrak{S}_1$[25]; therefore if $\mathfrak{G}^0$ compact,[23] we have

$$\mathfrak{S} \subseteq \mathfrak{S}_1, \quad T(\{\{\mathfrak{r}^*\}\}) \subseteq T(\Delta^0). \tag{6.1}$$

More precisely, we shall prove in the next section

Proposition 6. $\mathfrak{S} = \mathfrak{S}_1 (\cong N(\mathfrak{H})/\mathfrak{H})$, $\{\{\mathfrak{r}^*\}\} = \Delta^0$.

On the other hand, as we shall have $Z(\mathfrak{H}^0) = \mathfrak{H}^0$, $\mathfrak{H}^0$ is the complete inverse-image of $\mathfrak{H}$ with respect to the covering homomorphism of $\mathfrak{G}^0$ onto $\mathfrak{G}$. Therefore the Poincaré group of $\mathfrak{G}$ (or $\tilde{\mathfrak{G}}$) isomorphic to Δ/Δ^0.[26] It follows from Cor. 2 to Prop. 5, Prop. 6 and (0.1) the following

Corollary. *The Poincaré group of the adjoint group of* $\tilde{L}$ *is isomorphic to* $\{\{\mathfrak{r}\}\}^\wedge/\{\{\mathfrak{r}^*\}\}$. *Thus if the root systems of two semi-simple complex Lie algebras are mutually inverse, the Poincaré groups of their adjoint groups are mutually isomorphic* (*cf.* (2.3)).

Remark. We note here some remarks on the results so far obtained. We have proved, or shall prove, the following three propositions:

(i) $\Lambda^0 = \{\{\mathfrak{r}^*\}\}^\wedge$ (Prop. 1),

(ii) $\Lambda^0 = \Delta^{0\wedge}$ (Prop. 5),

(iii) $\{\{\mathfrak{r}^*\}\} = \Delta^0$ (Prop. 6).

It is easy to see that any one of them is a direct consequence of the other two. Inparticular, proposition 6 is sufficient for the proof of Proposition 1. On the other hand, it is worth noting that (iii) might be verified independently of Proposition 5.[27] Therefore if the Proposition 1 in question could be proved directly, it would follow (ii) and whence we should have a new proof of the faithful representability of connected semi-simple complex Lie groups.

§ 7. Weyl's theorem.

Let us denote by $\mathfrak{G}^{(r)}$ the set of all "regular" elements in $\mathfrak{G}$, i. e. the elements which have the eigen-value 1 only n times in the adjoint representation; we set $\mathfrak{G}^{(s)}=\mathfrak{G}-\mathfrak{G}^{(r)}$,[11] $\mathfrak{H}^{(r)}=\mathfrak{H}\cap\mathfrak{G}^{(r)}$, $\mathfrak{H}^{(s)}=\mathfrak{H}\cap\mathfrak{G}^{(s)}$. The complete inverse-image of $\mathfrak{H}^{(s)}$ in $H=\theta$ by the exponential mapping is equal to the union of $\pi_\alpha(k)$ $(\alpha\in\mathfrak{r},\ k=0,\pm1,\cdots)$, which we denote by $H^{(s)}$; we set $H^{(r)}=H-H^{(s)}$ and denote by Ξ an arbitrary connected component of $H^{(r)}$.[28] It should be noted that the transformations in $\mathfrak{S}_1(\Delta)$ leave $H^{(s)}=\cup_{\alpha,k}\pi_\alpha(k)$ invariant as follows.

$$S(u)\pi_\alpha(k)=\pi_{S(u)\alpha}(k),\quad T(\delta)\pi_\alpha(k)=\pi_\alpha(k+(\alpha,\delta)). \tag{7.1}$$

(An automorphism of L, which leave H invariant, induces in θ a permutation of $\mathfrak{r}$. *Cf.* also (5.2) and Cor. 2 to Prop. 5.) Consequently it holds

$$S'\Xi=\Xi \text{ or } S'\Xi\cap\Xi=\phi \text{ for every } S'\in\mathfrak{S}_1(\Delta). \tag{7.2}$$

We put $\mathfrak{S}_1(\Delta)_\Xi=\{S';\ S'\in\mathfrak{S}_1(\Delta),\ S'\Xi=\Xi\}$. If we replace the group $\mathfrak{S}_1$ by $\mathfrak{S}$, quite analogous formulae will be obtained.

When $\mathfrak{G}$ is compact,[23] it is known that $\mathfrak{G}^{(s)}$ is a closed set whose dimension is by 3 less than that of $\mathfrak{G}$ and that the Poincaré group of $\mathfrak{G}$ is identical with that of the connected subset $\mathfrak{G}^{(r)}$.[29]

After these preparations we shall prove

Weyl's Theorem. *Let $\mathfrak{G}$ be a connected semi-simple Lie group corresponding to the compact form L of $\tilde{L}$. Then $\mathfrak{G}$ is compact and we have*

(i) *$\Xi\times[\mathfrak{G}/\mathfrak{H}]$ is a simply connected covering space of $\mathfrak{G}^{(r)}$ with respect to the mapping*

$$\Xi\times[\mathfrak{G}/\mathfrak{H}]\in(\theta,\bar{u})\longrightarrow u(\exp\theta)u^{-1}\in\mathfrak{G}^{(r)}. \tag{7.3}$$

(From the fact that $Z(\mathfrak{H})=\mathfrak{H}$, $\mathfrak{G}/\mathfrak{H}$ has a topological structure which depends only on the Lie algebra L, but not on the Lie group $\mathfrak{G}$.)

(ii) *Every element of the Poincaré group of $\Xi\times[\mathfrak{G}/\mathfrak{H}]$ with respect to the covering mapping* (7.3) *may be identified with a certain transformation S' in $\mathfrak{S}(\Delta)_\Xi$ by the relation*

$$(\theta_1, \bar{u}_1) \xrightarrow{S'} (\theta_2, \bar{u}_2) \rightleftarrows \theta_2 = S'\theta_1,\ u_2 = u_1 u^{-1},\text{[30]} \tag{7.4}$$

where $S' = T(\delta) S(u)$ $(\delta \in \Delta, u \in N(\mathfrak{H}))$. (In particular, $\mathfrak{S}(\Delta)_\Xi$ being a finite group, the Poincaré group is also finite!)

We limit ourselves only to sketch the proof, adding some considerations lacking in the original proof of Weyl. We devide our proof in several steps; the last one will prove Proposition 6 in the previous section. Replacing $\mathfrak{H}$ and $\mathfrak{S}$ by $Z(\mathfrak{H})$ and $\mathfrak{S}_1$ respectively, we first prove the theorem in a modified form.

1) We assume first $\mathfrak{G}$ is compact and is a compact form of $\tilde{\mathfrak{G}}$.[23] Weyl's proof starts from the verification of the fact that every conjugate class of $\mathfrak{G}$ contains at least an element in $\mathfrak{H}$; in other words, the image of the mapping

$$\mathfrak{H} \times [\mathfrak{G}/Z(\mathfrak{H})] \ni (w, \bar{u}) \xrightarrow{f} uwu^{-1} \in \mathfrak{G} \tag{7.5}$$

coincides with $\mathfrak{G}$. Moreover it is shown that this mapping is locally homeomorphic at the regular points of $\mathfrak{G}$.[29] If we put

$$\tilde{\mathfrak{B}} = \mathfrak{H} \times [\mathfrak{G}/Z(\mathfrak{H})],\quad \tilde{\mathfrak{B}}_1 = \mathfrak{H}^{(r)} \times [\mathfrak{G}/Z(\mathfrak{H})],$$
$$\mathfrak{B} = \mathfrak{G},\qquad \mathfrak{B}_1 = \mathfrak{G}^{(r)},$$

the following conditions are fulfiled:

(i) $\tilde{\mathfrak{B}}$ is compact,

(ii) $\tilde{\mathfrak{B}}_1$ is a locally connected open subset of $\tilde{\mathfrak{B}}$,

(iii) f is a continuous mapping of $\tilde{\mathfrak{B}}$ onto $\mathfrak{B}$, and especially it is a local homeomorphism on $\tilde{\mathfrak{B}}_1$,

(iv) $\tilde{\mathfrak{B}}_1 = f^{-1}(\mathfrak{B}_1)$.

From these facts we can conclude that every point of $\mathfrak{B}_1 (= \mathfrak{G}^{(r)})$ has an "evenly covered" neighbourhood.[26] Since $\mathfrak{G}^{(r)}$ is connected, any component of $\mathfrak{H}^{(r)} \times [\mathfrak{G}/Z(\mathfrak{H})]$ becomes a covering space of $\mathfrak{G}^{(r)}$ with respect to the mapping (7.5).

2) By (7.2), it halds $T(\delta)\Xi \cap \Xi = \phi$ for any $\delta \in \Delta$, $\delta \neq 0$. Therefore the exponential mapping induces a homomeorphism of Ξ onto one of the components of $\mathfrak{H}^{(r)}$. Thus $\Xi \times [\mathfrak{G}/Z(\mathfrak{H})]$ is a covering space of $\mathfrak{G}^{(r)}$ with repect to the mapping which is defined by replacing $\mathfrak{H}$ by $Z(\mathfrak{H})$ in (7.3). It follows from the particular form of this covering mapping and from the fact the fact that Ξ is a polyhedron, that a closed curve in $\mathfrak{G}^{(r)}$ is homotopic to zero if and only if its continuous inverse-image in $\Xi \times [\mathfrak{G}/$

$Z(\mathfrak{H})]$ becomes a closed curve. This proves the simply-connectedness of $\Xi\times[\mathfrak{G}/Z(\mathfrak{H})]$.

3) As $\mathfrak{H}$ is clearly a maximal connected abelian subgroup of $\mathfrak{G}$, it follows readily that $\mathfrak{H}$ coincides wth its closure and that it is a maximal torus subgroup of $\mathfrak{G}$. If it were not identical with its centralizer $Z(\mathfrak{H})$, the homogeneous space $\mathfrak{G}/\mathfrak{H}$ would cover $\mathfrak{G}/Z(\mathfrak{H})$ essentially, which contradicts to the above statement. The first half of the theorem is thereby proved.

4) Now we proceed to the proof of the latter half. It can be readily seen that the homeomorphism of $\Xi\times[\mathfrak{G}/\mathfrak{H}]$ defined by replacing $\mathfrak{S}$ by $\mathfrak{S}_1$ in (7.4), gives an element of the Poincaré group. Conversely, let us suppose that a transformation $(\theta_1,\bar{u}_1)\rightarrow(\theta_2,\bar{u}_2)$ of the Poincaré group is given, namely $u_1(\exp\theta_1)u_1^{-1}=u_2(\exp\theta_2)u_2^{-1}$. It is sufficient to prove (ii) at one special point (θ_1,u_1) of $\Xi\times[\mathfrak{G}/\mathfrak{H}]$ such that $\exp\theta_1$ generates an everywhere dense subgroup of $\mathfrak{H}$. We have then $u=u_2^{-1}u_1\epsilon N(\mathfrak{H})$ and $\exp\theta_2=\exp(S(u)\theta_1)$. Putting

$$\theta_2=S(u)\theta_1+\delta,\ \delta\epsilon\Delta,$$
$$S'=T(\delta)S(u)\epsilon\mathfrak{S}_1(\Delta),$$

we have $S'\Xi\cap\Xi\neq\phi$, which means by (7.2), $S'\Xi=\Xi$, $S'\epsilon\mathfrak{S}(\Delta)_\Xi$. (ii) is thus proved in a modified form.

5) Since Ξ is a (finite) polyhedron, the Poincaré group identified with $\mathfrak{S}_1(\Delta)_\Xi$ is finite. It follows, as the adjoint group $\mathfrak{G}_0$ of L is surely compact, that all connected Lie groups corresponding to L are compact, for they are covering groups of $\mathfrak{G}_0$. It follows that $\tilde{\mathfrak{G}}$ is the "algebraic group" associated with the compact group $\mathfrak{G}$ and thus $\tilde{\mathfrak{G}}$ is topologically the direct product of $\mathfrak{G}$ and an n-dimensional Euclidean space.[24] In particular, $\mathfrak{G}^0$ is compact and, as $\tilde{\mathfrak{G}}^0$ is simply connected, $\mathfrak{G}_0$ is also simply connected. We may now, therefore, legitimately consider the simply connected compact group $\mathfrak{G}^0$.

6) If we take up the simply connected group $\mathfrak{G}^0$, the Poincaré group $\mathfrak{S}_1(\Delta^0)_\Xi$ contains only the identical transformation; this means that

$$S'\Xi\quad(S'\epsilon\mathfrak{S}_1(\Delta^0))$$

cover $H^{(r)}$ at most once. On the other hand, as the group $\mathfrak{S}(\{\{\mathfrak{r}^*\}\})$ contains all reflections of θ with respect to the hyperplanes of the form $\pi_\alpha(k)$ $(\alpha\epsilon\mathfrak{r},\ k=0,\pm1,\cdots)$,

$$S'\Xi \quad (S \in S(\{\{\mathfrak{r}^*\}\})$$

cover $H^{(r)}$ at least once. But from the formula (6.1), which was justified just above, we have

$$\mathfrak{S}(\{\{\mathfrak{r}^*\}\}) \subseteq \mathfrak{S}_1(\Delta^0)$$

From these facts we can conclude $\mathfrak{S}(\{\{\mathfrak{r}^*\}\}) = \mathfrak{S}_1(\Delta^0)$ and consequently $\mathfrak{S} = \mathfrak{S}_1$, $\mathfrak{T}(\{\{\mathfrak{r}^*\}\}) = \mathfrak{T}(\Delta^0)$.

Weyl's theorem is thus proved completely.

As a consequence of above considerations we add here

Proof of Lemma 3. If we take Ξ to be $\{\theta;\ \theta \in \Theta,\ 0 < (\alpha, \theta) < 1$ for all $\alpha \in \mathfrak{r}^*\}$, Ξ is the only component of $\Pi - H^{(s)}$ such that the origin of Θ is contained in the boundary of Ξ. Hence if $S\Pi = \Pi$ for $S \in \mathfrak{S}$, $S \neq 1$, we would have $S\Xi = \Xi$, namely $S \in \mathfrak{S}(\Delta^0)_{\Xi}$. Then the group $\mathfrak{G}^0$ would be covered essentially by $\Xi \times [\mathfrak{G}^0/\mathfrak{H}^0]$, contrary to the simply-connectedness of $\mathfrak{G}^0$, q. e. d.

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1) The numbers in brackets refer to the bibliography at the end of the paper.

2) See Weyl [10], K. IV, §§ 3, 4.

3) See Weyl [11], Ch. III, §§ 4, 5.

4) See Weyl [10], K. IV, §§ 1, 2. Recently H. Chandra [4] gave a new proof of Weyl's theorem as a consequence of Cartan's theorem.

5) *Cf.* van der Waerden [9]. He added to the definition of root systems, a supplementary condition that under the same assumption as our Lemma 1, all the following vectors

$$\beta,\ \beta-\varepsilon\alpha,\ \beta-2\varepsilon\alpha,\cdots,\ \beta-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha(=S_\alpha\beta)$$

also belong to $\mathfrak{r}$ (*cf.* (1.5)). This condition, however, is unnecessary. For we can prove it by the successive applications of Lemma 1 starting from β and $S_\alpha\beta$.

6) $\mathrm{sign}(\alpha,\lambda)$ is $+1,\ 0,-1$ according as $(\alpha,\lambda)>,=,<0$.

7) The proof is analytical in spite of the algebraic nature of Proposition 1 and a more direct (and algebraical!) proof seems to be desirable. *Cf.* Remark at the end of § 6.

8) *Cf.* Cartan [1], or van der Waerden [9]. See also § 3 of the present paper.

9) Conversely, if two irreducible root systems have one and the same transformation group in the above sense, they are congruent or inverse to each other. From this point of view, H. Weyl ([11], Ch. III, § 6) gave a new classification-theory.

10) δ_{ij} is Kronecker's delta.

11) The symbol $-$ denotes the set-theoretical difference.

12) Weyl proved Prop. 2 as a consequence of this Corollary ([11], Ch. III, §§ 4, 5).

13) The symbol ϕ denotes the empty set.

14) 1 denotes the identical transformation of E^n.

15) A direct proof is given in Weyl [11], Ch. III, § 4.

16) *Cf.* Gantmacher [7], Ch. III.

17) Our schemata are quite analogous to those of Coxeter ([6], or Weyl [11], Ch. III, §6).

18) It seems to me that the algebraic source of the analogies between the groups of type B_n and C_n, such as the coincidence of the Poincaré groups of their adjoint groups or that of the Poincaré polynomials of their compact forms, lies in this very point (see Cor. to Prop. 6 in § 6).

19) $SL(n+1,C)$ is the Lie algebra of all complex matrices of degree $n+1$ with vanishing traces.

20) As is seen from (4.7), the 1-st and n-th fundamental representations are realized by $SL(n+1,C)$, where the matrices representing the same element of A_n are contragredient to each other. (This is another proof of the fact that the outer automorphism group of A_n consists of two elements.) The Cartan composite of these two representations is the adjoint representation. The i-th and $(n+1-i)$-th fundamental representations are the irreducible representations of $SL(n+1,C)$ indicated by the so-called Young's diagram $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline \vdots\\ \hline i\\ \hline\end{array}$ ....These results are simple applications of Cartan's Theorem I.

21) For fundamental concepts on Lie groups, see, for example, Chevalley [5].

22) As $\tilde{L}$ is semi-simple, the adjoint group $\tilde{\mathfrak{G}}_0$ is one of the Lie groups corresponding to $\tilde{L}$. Remark also that $\mathfrak{G}_0$ is the adjoint group of L.

23) This will be proved afterwards to be always true.

24) See Chevalley [5], Ch. VI.

25) That $\mathfrak{S} \subseteq \mathfrak{S}_1$ can be proved directly. *Cf.* Weyl [11], Ch. III, § 5, or Gantmacher [7], Ch. III.

26) Concerning the notion of covering space we also refer to Chevalley [5], Ch. II.

27) *Cf.* Weyl [11], Ch. III, § 5.

28) Ξ can be proved to be a simplex, when $\tilde{L}$ is a simple Lie algebra. *Cf.* Cartan [3], or Weyl [11], Ch. III, § 5.

29) See Weyl [10], K. IV, § 1.

30) $\bar{u}_1$ denotes the left coset of u_1 in the homogeneous space of $\mathfrak{G}$ modulo $\mathfrak{H}$(or $Z(\mathfrak{H})$). When $u \in N(\mathfrak{H})$, $\overline{u_1 u^{-1}}$ is uniquely determined by $\bar{u}_1$.