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On the Differentability of the Unitary Representation of the Lie Group.

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J. von Neumann ([2]) has introduced the notion of the differentiablity of the matric group, and given a method of forming Lie algebras of matric Lie groups. This notion was extended further by K. Yosida ([4]) to the group embedded in tne normed ring.

In this paper, we shall utilize this idea to form the Lie algebra for the Lie group G embdeded in the unitary group with the weak topology in the Hilbert space $\mathfrak{H}$. Namely we shall show that the set of all operators

$$\tilde{A}_{\sigma(t)}=\lim_{t\to 0}\frac{U_{\sigma(t)}-E}{t}$$

for each one-parameter subgroup $\sigma(t)$ of G, forms the Lie algebra of G in a sense to be specified in Theorem 2, 3 below. There is diffculty on the domains of these operators. We shall show that they have a meet everywhere dense in $\mathfrak{H}$. By the way we obtain a new proof of M. H. Stone's theorem on the one-parameter group of unitary operators.

In § 1 we give a résumé of the theory of simple unitary structures, which we shall need in the proof of the fact that the domains of $\tilde{A}_{\sigma(t)}$ have an everywhere dense meet. § 2 contains a lemma that every element of $L^1(G)$ is approximable by C^2 functions. Our main results are Theorem 2, 3 in § 3.

§ 1.

Let G be a Lie group. We denote elements of G with σ, τ,…. On the other hand let $\mathfrak{H}$ be a Hilbert space, x, y… elements of $\mathfrak{H}$. A continuous unitary representation of G is a continuous homomorphic mapping $\sigma\to U_\sigma$ into the group of all unitary operators defined on $\mathfrak{H}$ and provided with the weak topology. The pair $\{U_\sigma,\mathfrak{H}\}$ is then called a *unitary structure* of G. If $\{U_\sigma,\mathfrak{H}\}$ is a unitary structure of G and if, moreover, there is such an element x of $\mathfrak{H}$ that closed linear manifold $\{U_\sigma x\,;\sigma\in G\}^{cl}$ coincides with $\mathfrak{H}$, then the triple $\{U_\sigma, \mathfrak{H}, x\}$ is said to be a *simple unitary structure* of G. Two such structure $\{U_\sigma, \mathfrak{H}, x\}$, $\{U_\sigma', \mathfrak{H}', x'\}$ are said to be *unitary-equivalent*, if there is a unitary mapping T of $\mathfrak{H}$ on $\mathfrak{H}'$ such that $T^{-1}U_\sigma' T = U_\sigma$ and $Tx=x'$.[1)]

Let $\{U_\sigma, \mathfrak{H}, x\}$ be a simple unitary structure of G. The function $\varphi(\sigma)=(U_\sigma x, x)$, (where the round brackets mean the inner product on $\mathfrak{H}$), is called the *characteristic function* of $\{U_\sigma, \mathfrak{H}, x\}$ It is a positive definite function on G, i. e. for any finite number of elements $\sigma_i(i=1,2,\cdots,n)$ of G and arbitrary complex numbers $\alpha_i(i=1,2,\cdots,n)$, the inequality

$$\sum_{i,j=1}^{n}\varphi(\sigma_i\sigma_j^{-1})\alpha_i\bar{\alpha}_j\geq 0$$

always holds. Conversely if any positive definite function $\varphi(\sigma)$ on G is given, there is a simple unitary structure, determined up to the unitary-equivalence, whose characteristic function is $\varphi(\sigma)$. According to [7], this simple unitary structure may be obtained as follows.

Let μ be a left-invariant Haar measure of G, $L^1(G)$ the Banach space consisting of all μ-integrable complex-valued functions $x(\sigma)$, $y(\sigma),\cdots$ on G, where the norm of $x(\sigma)$ is defined by $\|x\|_1=\int_G|x(\sigma)|\,d\mu(\sigma)$. $L^1(G)$ becomes a group algebras of G, if we define the convolution $x\times y(\sigma)=\int_G x(\tau)y(\tau^{-1}\sigma)d\mu(\tau)$ for any $x(\sigma)$, $y(\sigma)\in L^1(G)$. Next we define a $*$-operation for any element $x(\sigma)\in L^1(G)$ with $x^*(\sigma)=\overline{x(\sigma^{-1})}\Delta(\sigma)$, where $\overline{x(\sigma)}$ is the conjugate complex of $x(\sigma)$ and $\Delta(\sigma)$ is the density of the right-invariant Haar measure of G. Then we have clearly $\|x^*\|_1=\|x\|_1$, $(x^*)^*=x$ and $(x\times y)^*=y^*\times x^*$.

Now put

$$I_\varphi=\{x\,;\ x\in L^1(G),\int_G\varphi(\sigma)x^*\times x(\sigma)d\mu(\sigma)=0\}. \qquad (1\cdot 1)$$

I_φ is a closed left-ideal of $L^1(G)$ and left-G-invariant, so that we can consider the factor space $L^1(G)/I_\varphi$ of $L^1(G)$ by I_φ. We denote the point of this factor space containing $x(\sigma)$ with $[x]$, and introduce the inner product in this factor space by

$$([x],[y])_\varphi=\int_G\varphi(\sigma)y^*\times x(\sigma)d\mu(\sigma)$$
$$\text{for any } [x],[y]\in L^1(G)/I_\varphi.$$

We obtain the Hilbert space $\mathfrak{H}_\varphi$ by completing $L^1(G)/I_\varphi$ with respect to the norm defined from this inner product. For each element τ of G we define the mapping

$$U_\tau[x(\sigma)]=[x(\tau^{-1}\cdot\sigma)] \qquad \text{for any } [x]\in L^1(G)/I_\varphi.$$

As I_φ is left-G-invariant, this mapping is determined independently of the choice of a representative of the class $[x]$. On the other hand, $L^1(G)/I_\varphi$ is everywhere dense in $\mathfrak{H}_\varphi$, so this mapping can be uniquely extended to a unitary operator on $\mathfrak{H}_\varphi$, which we denote again with U_τ. Thus we obtain a unitary structure $\{U_\tau, \mathfrak{H}_\varphi\}$ of G.

Let $\{V_\alpha\}$ be a complete system of the neighbourhoods of the identity of G, and $C_{V\alpha}(\sigma)$ the characteristic function of the set V_α. Put $d_\alpha(\sigma)=C_{V_\alpha}(\sigma)/\mu(V_\alpha)$, $e_\alpha=d_\alpha^*\times d_\alpha$, then $\{[e_\alpha(\sigma)]\}$ is strongly convergent to an element x_φ in $\mathfrak{H}_\varphi$. It is proved that $\{U_\tau, \mathfrak{H}_\varphi, x_\varphi\}$ is then a simple unitary structure whose characteristic function is $\varphi(\sigma)$.

§ 2.

Let G be a Lie group of the dimension n, L the Lie algebra of G, and $\{A_1, A_2, \cdots A_n\}$ a basis of L. Using this basis we can introduce a cubic neighbourhood $V_\alpha=\{\tau\,;\ \tau=\exp(\sum_{i=1}^n x_i A_i),\ |x^i|<a,\ i=1,2,\cdots,n\}$ of the identity e of G, and a canonical system of coordinates C_e on V_α such that $\tau=\exp(\sum_{i=1}^n x_i A_i)$ has x_i as its i-th coordinate by C_e, where $\exp(\sum_{i=1}^n x_i A_i)$ is the element $\sigma(1)$ of the one-parameter subgroup $\sigma(t)$ having $\sum_{i=1}^n x_i A_i$ as its tangent vector. By translation we introduce the system of coordinates C_σ for the neighbourhood $V_\alpha\cdot\sigma$ of each point σ of G, i. e. the i-th coordinate of $\tau=\exp(\sum_{i=1}^n x_i A_i)\cdot\sigma$ in $V_\alpha\cdot\sigma$ is $x_i (i=1,2,\cdots,n)$.

Definition 1. *A complex-valued continuous function $x(\sigma)$ defined on an open set W contained in some cubic set $V_\alpha\cdot\sigma_0$ is said C^r function on W, if the expression $x(\sigma)=X(x_1, x_2, \cdots, x_n)$ by the system of cordinates C_{σ_0} is C^r function on W.*

Definition 2. *A curve $\sigma(t)$ in G, defined continuously with respect to real parameter t such that $\sigma(0)=e$, is colled a C^1 curve, when each coordinates of the elements of $\sigma(t)$ by C_e is continuously differentiable for t at $t=0$.*

Following properties of C^1 functions are earily verified from these definitions.

(1) Let $x(\sigma)$ be a C^1 function on W. For any C^1 curve $\sigma(t)$

$$\lim_{t\to 0}\frac{1}{t}(x(\sigma^{-1}(t)\cdot\sigma)-x(\sigma))=x_{\sigma(t)}(\sigma)$$

exists and is a continuous function of σ on W. Let the i-th coordinates of σ and $\sigma(t)$ by C_{σ_0} be x_i and $y_i(\sigma(t))$ respectively, then the i-th coordinate of $\sigma^{-1}(t)\cdot\sigma$ is $\varphi_i(x_1,x_2,\cdots,x_n\,;\,y_1(\sigma(t)),y_2(\sigma(t)),\cdots,y_n(\sigma(t)))$, where $\varphi_i(x_1,x_2,\cdots,x_n\,;\,y_1,y_2,\cdots,y_n)$ is analytic for $x_1,x_2,\cdots,x_n$ and $y_1,y_2,\cdots,y_n$. Then $x_{\sigma(\alpha)}(\sigma)$ is given by

$$x_{\sigma(\alpha)}(\sigma)=\sum_{i,j=1}^{n}\left[\frac{\partial x}{\partial x_i}\right]_{xk=xk}\cdot\left[\frac{\partial\varphi_i}{\partial y_i}\right]_{xk=xk,ye=0}\cdot\left[\frac{dy_j}{dt}\right]_{t=0,} \tag{2·1}$$

(2) Let $\sigma_k(t)$ be the one-parameter subgroup of G befined by $\sigma_k(t)=\exp(tA_k)$. If $x_{\sigma k}(\sigma)$ exists and is contiunous when σ is in W for each $\sigma_k(t)\,(k=1,2,\ldots,n)$, $x(\sigma)$ is a C^1 function. And for any C^1 curve $\sigma(t)$, $x_{\sigma(t)}(\sigma)$ is given by a linear combination of $x_{\sigma k(t)}(\sigma)\,(k=1,2,\cdots,n)$ with constant coefficients.

(3) To each C^1 curve $\sigma(t)$ corresponds one and only one one-parameter subgroup $\sigma'(t)$ such that $x_{\sigma(t)}(\sigma)=x_{\sigma'(t)}(\sigma)$ for every C^1 function $x(\sigma)$ on W.

(4) Let $\sigma(t)$ be a C^1 curve and $x(\sigma)$ a C^1 function on W with the expression $X(x_1,x_2,\cdots,x_n)$ by C_{σ_0}, and let $\dfrac{\partial X}{\partial x_i}$ be bounded on W for each i. Then there are a positive unmber t_0 and a neighbourhood of the identity V_1 both independent of σ_0, such that $\dfrac{d}{dt}x(\sigma^{-1}(t)\cdot\sigma)$ is continuous and bounded with respect to σ and t when $\sigma\in V_1\sigma_0$, and $|t|<t_0$.

Lemma 1. *Any function contained in $L^1(G)$ can be approximated as closely as we wish with respect to the topology in $L^1(G)$ by a C^2 function on G*[2] *contained in $L^1(G)$.*

Proof. Following Chevalley, we shall say a function $x(\sigma)$ defined on G to have the property P on $W\cdot\sigma$, when it is continuous and zero out of some cubic set $W\cdot\sigma$ of some point σ. A function in $L^1(G)$ can be sufficiently closely approximated by functions each of which is continuous and and zero out of some compact set. Such a function can be expressed as a finite sum of functions with the property P. Moreover, from Dieudonné's lemma ([1] p. 163) the cubic set W can be taken sufficiently small. Let V_β be a cubic neighbourhood of e, of breadth β, i. e. the absolute value of the i-th coordinate $x_i(\sigma)$ by C_e of an element σ contained in V_β is smaller than β for $i=1,2,\cdots,n$, and take β such as $V_\beta^2 C V_\alpha$. Then we have only to prove that for a function $x(\sigma)$ with the property P on $V_\beta\cdot\sigma_0$ of some point σ_0, there is a series of C^2 functions converging to $x(\sigma)$.

Let $X(x_1, x_2,\cdots, x_n)$ be the expression of $x(\sigma)$ by the system of coordinates C_{σ_0}, and put

$$x_m(\sigma)=X_m(x_1, x_2,\cdots x_n)$$
$$=\left(\frac{m}{2}\right)^n\int_{x_1-\frac{1}{m}}^{x_1+\frac{1}{m}}\int_{x_2-\frac{1}{m}}^{x_2+\frac{1}{m}}\cdots\cdots\int_{x_n-\frac{1}{m}}^{x_n+\frac{1}{m}}X(x_1,x_2,\cdots,x_n)dx_1\cdot dx_2\cdots dx_n \qquad \textit{if } \sigma\in V_{\alpha-\frac{1}{m}}\cdot\sigma_0,$$
$$x_m(\sigma)=0 \qquad \textit{if } \sigma\notin V_{\alpha-\frac{1}{m}}\cdot\sigma_0$$

for $m\geq m_0=\mathrm{Min}\left\{m\,;\,\alpha-\beta>\frac{4}{m}\right\}$, where $V_{\alpha-\frac{1}{m}}$ is a cubic set of e, of breadth $\alpha-\frac{1}{m}$. Then $x_m(\sigma)$ has clearly the property P on $V_{\beta+\frac{1}{m}}\cdot\sigma_0=\{\sigma\cdot\sigma_0\,;\, \sigma\in G,\ |\ x_i(\sigma)|<\beta+\frac{1}{m}$ by C_e $i=1,2,\cdots,n\}$. Moreover, $\frac{\partial X_m}{\partial x_i}$ exists in $V_{\alpha-\frac{1}{m}}\cdot\sigma_0$ and has the property P on $V_{\beta+\frac{1}{m}}\cdot\sigma_0$ for each i. If $\sigma\notin V_{\alpha-\frac{1}{m}}\cdot\sigma$, we have $x_{m\sigma(t)}(\sigma)=0$ for any C^1 curve $\sigma(t)$. Thus $x_m(\sigma)$ is a C^1 function on G and has the property P on $V_{\beta+\frac{1}{m}}\cdot\sigma_0$. As $X(x_1, x_2,\cdots, x_n)$ is uniformly continuous on the closure $V_{\beta+\frac{1}{m}}\cdot\sigma_0$, there is a positive number $\delta(\varepsilon)$ for any given positive unmber ε such that from $|x_i-x'_i|<\delta(\varepsilon)$, $i=1,2,\cdots,n$, follows

$$|X(x_1, x_2,\cdots x_n,)-X(x_1', x_2'\cdots, x_n')|<\varepsilon.$$

Take m larger than $m_1=\mathrm{Max}\left\{m_0, \frac{1}{\delta(\varepsilon)}\right\}$, then there is for each σ in $\bar{V}_{\beta+\frac{1}{m}}\cdot\sigma_0$ a point with the i-th coordinate $\xi_i(i=1,2,\cdots,n)$ such that $|\ x_i(\sigma)-\xi_i|<\frac{1}{m}<\delta(\varepsilon)$ and $X_m(x_1, x_2,\cdots, x_n)=X(\xi_1, \xi_2,\cdots,\xi_n)$ from the mean-value theorem of the integral. Therefore

$$|X_m(x_1, x_2,\cdots, x_n)-X(x_1, x_2,\cdots,x_n)|=|X(\xi_1, \xi_2,\cdots,\xi_n)-X(x_1, x_2,\cdots, x_n)|<\varepsilon \qquad \text{if } \sigma\in V_{\beta+\frac{1}{m}}\cdot\sigma_0,$$

and if $\sigma\notin\bar{V}_{\beta+\frac{1}{m}}\cdot\sigma_0$, we have $x_m(\sigma)=x(\sigma)=0$. So the series $\{x_m(\sigma)\}$ converges uniformly to $x(\sigma)$ and

$$\int_G |x(\sigma)-x_m(\sigma)|d\mu(\sigma)=\int_{\bar{V}_{\beta+\frac{1}{m}}\cdot\sigma_0} |x(\sigma)-x_m(\sigma)|d\mu(\sigma)\leq\varepsilon\mu(V_\alpha)$$

Thus the sequence $\{x(\sigma),\ m\geq m_1\}$ converges to $x(\sigma)$ with respect to the topology in $L^1(G)$. As it is already known that $x_m(\sigma)$ has the property P on $V_{\alpha+\frac{1}{m_1}}\cdot\sigma_0$, we can apply this method to each $x_m(\sigma)$ and obtain a series $\{x_{m,m'}(\sigma)\}$ of C^2 functions converging to $x_m(\sigma)$. $x_{m,m'}(\sigma)$ is indeed a C^2 function as it is easily seen from $(2\cdot 1)$ above. In taking a suitable partial sequence $\{x_{m,m'(m)}(\sigma)\}$ of $\{x_{m,m'}(\sigma)\}$. we obtain finally a series converging to $x(\sigma)$, q. e. d.

Definition 3. *Let V_β be a cubic set defined as above, $x(\sigma)$, $y(\sigma)$,$\cdots$ functions in $L^1(G)$ with the property P on $V_\beta\cdot\sigma_0$, $V_\beta\cdot\tau_0$,$\cdots$for some σ_0, τ_0,$\cdots$in G, $\{x_{m,m'(m)}(\sigma)\}$, $\{y_{n,n'(n)}(\sigma)\}$,$\cdots$the series of C^2 functions in$L^1(G)$ converging to $x(\sigma)$, $y(\sigma)$,$\cdots$respectively. We form linear combinations with complex coefficients of a finite number of such functions $x_{m,m'(m)}(\sigma)$, $y_{n,n'(n)}(\sigma)$,$\cdots$. The set of all these linear combinations forms a G-invariant linear manifold in $L^1(G)$ which is everywhere dense in $L^1(G)$. We denote this linear manifold with $D(G)$.*

§ 3.

The following lemma is important to deduce our main results.

Lemma 2. *Let $\{U_\sigma, \mathfrak{H}, x_0\}$ be a simple unitary structure of a Lie group G. The set of all elements x of $\mathfrak{H}$ for which*

$$\lim_{t\to 0}\frac{1}{t}(U_{\sigma(t)}-E)x \tag{3·1}$$

exists for any C^1 curve, is a linear manifold everywhere dense in $L^1(G)$.

Proof. As it was remarked in § 1, the given simple unitary structure $\{U_\sigma, \mathfrak{H}, x_0\}$ may be considered as $\{U_\sigma, \mathfrak{H}_\varphi, x_\varphi\}$, φ being a positive definite function on G. This remark will be often used in the sequel.

We use the same notation as in § 1 and § 2. Thus I_φ is the left-ideal defined by $(1\cdot 1)$, $\mathfrak{H}_\varphi$ the Hilbert space obtained by completion of $L^1(G)/I_\varphi$. Let $\mathfrak{D}_\varphi$ be the image of $D(G)$ by the natural mapping of

$L^2(G)$ onto $L^1(G)/I_\varphi$. As $D(G)$ is dense in $L^1(G)$, $\mathfrak{D}_\varphi$ is everywhere dense in $\mathfrak{H}_\varphi$. An element $[x]$ of $\mathfrak{D}_\varphi$ has the form $\sum_{i=1}^k a_i[x_i(\sigma)]$, where $x_i(\sigma)$ is a C^2 function with the property P on some cubic set $V_{\beta'}\cdot\sigma_0$, of a breadth β' such as $\beta<\beta'<\alpha$, constructed as in Definition 3. So our lemma will be proved, if we show the existence of (3·1) for such $[x_i(\sigma)]$ and an arbitrary C^1 curve $\sigma(t)$. Now we have

$$\lim_{t\to 0}\frac{1}{t}\{(U_{\sigma(t)}[x_i],[y])_\varphi-([x_i],[y])_\varphi\}$$

$$=\lim_{t\to 0}\int_G\int_G\varphi(\sigma^{-1}\cdot\tau)\,\bar{y}(\sigma)\frac{x_i(\sigma^{-1}(t)\cdot\tau)-x_i(\tau)}{t}d\mu(\sigma)d\mu(\tau)$$

$$\text{for any } [y]\in L^1(G)/I_\varphi. \tag{3·2}$$

Take t_1 so small that if $|t|<t_1$ and $\tau\notin\bar{V}_\alpha\cdot\sigma_0$, we have $\sigma^{-1}(t)\cdot\tau\notin V_\beta\cdot\sigma_0$ and $x_i(\sigma^{-1}(t)\cdot\tau)-x_i(\tau)=0$. Accordingly, the domain G of the integral with respect to τ in the right hand side of (3·2) can be replaced by $\bar{V}_\alpha\cdot\sigma$. Next, from the property (4) in § 2, $\frac{d}{dt}x_i(\sigma^{-1}(t)\cdot\tau)$ is continuous for τ and t and is bounded when τ is in some cubic set $V_1\cdot\tau_0$ and t in an interval $|t|<t_0$, where τ_0 is an arbitrary element of G and t_0 and V_1 may be both taken independently of τ_0. As $\bar{V}_\alpha\cdot\sigma_0$ is compact, it can be covered by a finite number of cubic sets $V_1\cdot\tau_i(i=1,2,\cdots,k)$. Therefore $\frac{d}{dt}x_i(\sigma^{-1}(t)\cdot\sigma)$ is bounded when τ is in $\bar{V}_\alpha\cdot\sigma_0$ and t in the interval $|t|<t_0$. On the other hand, if $|t|<\mathrm{Min}\{t_0,t_1\}$ we have

$$\frac{x_i(\sigma^{-1}(t)\cdot\tau)-x_i(\tau)}{t}=\left[\frac{d}{dt}x_i(\sigma^{-1}(t)\cdot\tau)\right]_{t=\xi} \tag{3·3}$$

where $0\leq\xi\leq t$ or $t\leq\xi\leq 0$. Then the left hand side of (3·3) is bounded when τ is in $\bar{V}_\alpha\cdot\sigma_0$ and t in the interval $|t|>\mathrm{Min}\{t_0,t_1\}$ and converges to $x_{ta(t)}(\tau)$ when t tends to zero. Consequently we can apply Lebegue's theorem, and we have

$$\lim_{t\to 0}\frac{1}{t}\{(U_{\sigma(t)}[x_i],y)_\varphi-([x_i],[y]_\varphi\}$$

$$=\int_{\bar{V}_{\alpha\cdot\sigma_0}}\int_G \varphi(\sigma^{-1}\cdot\tau)\bar{y}(\sigma)\cdot x_{i\sigma(t)}(\tau)d\mu(\sigma)\cdot d\mu(\tau)$$
$$=\int_G\int_G \varphi(\sigma^{-1}\cdot\tau)\bar{y}(\sigma)\cdot x_{i\sigma(t)}(\tau)\,d\mu(\sigma)\,d\mu(\tau)$$
$$=([x_{i\sigma(t)}],[y])_\varphi \qquad \text{for any } [y]\in L^1(G)/I_\varphi$$

Then the strong convergence of $(3\cdot1)$ is concluded from

$$\lim_{t\to 0}\left\|\frac{U_{\sigma(t)}[x_i]-[x_i]}{t}\right\|_\varphi^2=\int_G\int_G\varphi(\sigma^{-1}\cdot\tau)\bar{x}_{i\sigma(t)}(\sigma)\cdot x_{i\sigma(t)}(\tau)d\mu(\sigma)d\mu(\tau)$$
$$=\|[x_{i\sigma(t)}]\|_\varphi^2$$

which is proved in the same way as sbove. On the other hand, it is almost evident that the set of all elements of $\mathfrak{H}_\varphi$, for which $(3\cdot)$ exist for any C^1 curve $\sigma(t)$, forms a linear manifold, q. e. d.

Definition 4. *Let* $\{U_\sigma, \mathfrak{H}_\varphi, x_\varphi\}$ *be a simple unitary structure of a Lie group* G, *and* $\sigma(t)$ *a* C^1 *curve on* G. *We define the operator* $A_{\sigma(t)}$ *with the domain* $\mathfrak{D}_\varphi$ *in* $\mathfrak{H}_\varphi$ *by*

$$\lim_{t\to 0}\frac{1}{t}(U_{\sigma(t)}[x]-[x])=A_{\sigma(t)}[x] \qquad \text{for any } [x]\in\mathfrak{D}_\varphi. \qquad (3\cdot4)$$

Clearly $\sqrt{-1}\,A_{\sigma(t)}$ is a Hermitian operator, and from the properties (2) and (3) in §2 follows that $A_{\sigma(t)}$ is a linear combination of $A_{\sigma i(t)}$ $(i=1,2,\cdots,n)$ with real constant coefficients where $\sigma_i(t)$ $(i=1,2,\cdots,n)$ are linearly independet n one-parameter subgroups of G defined in §2, and that there is one and only one one-parameter subgroup $\sigma(t)$ with $A_{\sigma(t)}=A_{\sigma'(t)}$.

Lemma 3. *Let* $\{U_\sigma, \mathfrak{H}_\varphi, x_\varphi\}$ *be a simple unitary structure of* G, *and* $\sigma(t)$ *a one-parameter subgroup of* G. *If* $\lim_{t\to 0}\frac{1}{t}(U_{\sigma(t)}x-x)$ *exists we put*

$$\lim_{t\to 0}\frac{1}{t}(U_{\sigma(t)}x-x)=\tilde{A}_{\sigma(t)}x.$$

Then $\sqrt{-1}A_{\sigma(t)}$ *is a self-adjoint operator, with a domain containing* $\mathfrak{D}_\varphi$ *in* $\mathfrak{H}_\varphi$, *and is the one and only one self-adjoint extension of* $\sqrt{-1}\,A_{\sigma(t)}$ *defined in Definition* 4. *Moreover, let* $E_\lambda(-\infty\leq\lambda\leq\infty)$ *be the resolution of the identity of* $\sqrt{-1}\,\tilde{A}_{\sigma(t)}$, *then the one-parameter group* $U_{\sigma(t)}$ *can be expressed* *by*

$$U_{\sigma(t)}=\int_{-\infty}^{\infty} e^{\sqrt{-1}\lambda t} dE_{\lambda}.$$

Proof. (1) When the weak limit $w\text{-}\lim_{t\to 0}(U_{\sigma(t)}x-x)$ exists, we denote it temporarily with $\tilde{A}_{\sigma(t)}x$. As $\sigma(t)$ is a one-parameter subgroup of G, we have

$$w\text{-}\lim_{t'\to t}\frac{1}{t'-t}(U_{\sigma(t')}-U_{\sigma(t)})x=\tilde{A}_{\sigma(t)}U_{\sigma(t)}x=U_{\sigma(t)}\tilde{A}_{\sigma(t)}x$$

and

$$(U_{\sigma(t)}x-x,y)_{\varphi}=\int_0^t(U_{\sigma(t)}\tilde{A}_{\sigma(t)}x,\ y)_{\varphi}dt$$
$$=(\int_0^t U_{\sigma(t)}\tilde{A}x\ dt, y)_{\varphi}$$
$$\text{for any } y\in\mathfrak{H}_{\varphi},$$

so

$$U_{\sigma(t)}x-x=\int_0^t U_{\sigma(t)}\tilde{A}_{\sigma(t)}x\ dt.$$

Therefore from the weak convergence of $\frac{1}{t}(U_{\sigma(t)}x-x)$ follows the strong convergence. and the both definitions of $\tilde{A}_{\sigma(t)}$ coincide.

(2) $\sqrt{-1}\ A_{\sigma(t)}$ and $\sqrt{-1}\ \tilde{A}_{\sigma(t)}$ are both Hermitian operators, and the latter is an extension of the former. Let V and $\tilde{V}$ be the Cayley transforms of $\sqrt{-1}\ A_{\sigma(t)}$ and $\sqrt{-1}\ \tilde{A}_{\sigma(t)}$ respectively. These are both partially isometric operators and the latter is an extension of the former. We shall show that V has an everywhere dense linear manifold as the domain. For the purpose we have only to show that the set $\{(\sqrt{-1}(A_{\sigma(t)}+E)x\ ;\ x\in\mathfrak{D}_{\varphi}\}$ is dense in $\mathfrak{H}_{\varphi}$, because the graph of V is $\{\{\sqrt{-1}(A_{\sigma(t)}+E)x,\ \sqrt{-1}(A_{\sigma(t)}-E)x\}\ ;\ x\in\mathfrak{D}_{\varphi}\}$. Assume it were not true, then there would be a non-zero element y of $\mathfrak{H}_{\varphi}$ such that

$$(\sqrt{-1}(A_{\sigma(t)}+E)x,y)_{\varphi}=0 \qquad \text{for any } x\in\mathfrak{D}_{\varphi}.$$

Then from $U_{\sigma(t)}A_{\sigma(t)}x=A_{\sigma(t)}U_{\sigma(t)}x$, we obtain the differential equation

$$\frac{d}{dt}(U_{\sigma(t)}x,y)_\varphi = -(U_{\sigma(t)}x,y)_\varphi.$$

The solution of this differential equation under the initial condition $(U_{\sigma(0)} x,y)_\varphi=(x,y)_\varphi$ is given by

$$(U_{\sigma(t)}x,y)_\varphi = e^{-t}(x,y)_\varphi$$

Now, the absolute value of $(U_{\sigma(t)}x,y)_\varphi$ is bounded by $\|x\|_\varphi\cdot\|y\|_\varphi$, and on the other hand the absolute value $e^{-t}(x,y)_\varphi$ is not bounded, which is a contradiction. Thus V has an everywhere dense domain, and has the unique unitary extension $\tilde{\tilde{V}}$, which is also the unique unitary extension of $\tilde{V}$.

(3) Let $\tilde{\tilde{A}}$ be the self-adjoint operator having $\tilde{\tilde{V}}$ as its Cayley transform, then $\sqrt{-1}A_{\sigma(t)}$ and $\sqrt{-1}\tilde{A}_{\sigma(t)}$ both have the unique self-adjoint extension $\tilde{\tilde{A}}$. Let $E_\lambda(-\infty\leq\lambda\leq\infty)$ be the resolution of the identity of $\tilde{\tilde{A}}$ and consider the one-parameter group of unitary operators defined by

$$V_t=\int_{-\infty}^{\infty} e^{\sqrt{-1}\lambda t}\, dE_\lambda.$$

Then $U_{\sigma(t)}V_s x = V_s U_{\sigma(t)}x$ for any t and s, when x is contained in the domain of $\tilde{A}_{\sigma(t)}$, and for such x we have

$$\begin{aligned}\|U_{\sigma(t)}x - V_t x\|_\varphi &= \| \textstyle\sum_{i=1}^n (U_{\sigma(\frac{n+1-i}{n}t)} V_{\frac{i-1}{n}t} - U_{\sigma(\frac{n-i}{n}t)} V_{\frac{i}{n}t})x\|_\varphi \\ &= \| \textstyle\sum_{i=1}^n U_{\sigma(\frac{n-i}{n}t)} V_{\frac{i-1}{n}t}(U_{\sigma(\frac{t}{n})} - V_{\frac{t}{n}})x\|_\varphi \\ &\leq \textstyle\sum_{u=1}^n \|(U_{\sigma(\frac{t}{n})} - V_{\frac{t}{n}})x\|_\varphi \\ &= t\left\|\left(\frac{U_{\sigma(\frac{t}{n})} - V_{\frac{t}{n}}}{\frac{t}{n}}\right)x\right\|_\varphi. \end{aligned} \tag{3.5}$$

On the other hand, we have

$$\lim_{t\to 0}\frac{1}{t}(V_t-E)x = \lim_{t\to 0}\int_{-\infty}^{\infty}\frac{e^{\sqrt{-1}\lambda t}-1}{t}dE_\lambda x = \tilde{\tilde{A}}x,$$

and

$$\lim_{t\to 0}\frac{1}{t}(U_{\sigma(t)}-E)x=\tilde{A}_{\sigma(t)}x=\tilde{\tilde{A}}x,$$

whence we can conclude $U_{\sigma(t)}x=V_t x$ for x in the domain of $A_{\sigma(t)}x$, as the right hand side of $(3\cdot 5)$ tends to zero as n tends to ∞.[4] Finally, since the domain of $\tilde{A}_{\sigma(t)}$ is dense in $\mathfrak{H}$, $U_{\sigma(t)}$ coincides with V_t, and $\sqrt{-1}\tilde{A}_{\sigma(t)}$ with $\tilde{\tilde{A}}$. q. e. d.

From this lemma, we obtain a new proof of a theorem of M. H. Stone.

Theorem 1. (*M. H. Stone*) *For a unitary structure* $\{U_t, \mathfrak{H}\}$ *of the additive group* R^1 *of all real numbers provided with the usual topology, there is a resolution of the identity* $E_\lambda(-\infty\leq\lambda\leq\infty)$ *so that* U_t *is expressed by*

$$U_t=\int_{-\infty}^{\infty} e^{\sqrt{-1}\lambda t}\, dE_\lambda .$$

Proof. We define a closed linear manifold $\mathfrak{H}^{(\eta)}$ of $\mathfrak{H}$ for any transfinite number η by the method of transfinite induction as follows. For $\eta=1$, we take an arbitrary non-zero element $x^{(1)}$ of $\mathfrak{H}$, and define $\mathfrak{H}^{(1)}=\{U_t x^{(1)};-\infty<+<\infty\}^{cl}$. Let a closed linear manifold $\mathfrak{H}^{(\eta')}$ be defined for every $\eta'<\eta$. If the orthogonal complement of $\sum_{\eta'<\eta}\oplus\mathfrak{H}^{(\eta')}$ is not zero, we take and arbitrary non-zero element $x^{(\eta)}$ of this complement and define $\mathfrak{H}^{(\eta)}=\{U_t x^{(\eta)};\ -\infty<t<\infty\}^{cl}$. Otherwise we put $\mathfrak{H}^{(\eta)}=0$. Then we have $\sum_\eta\oplus\mathfrak{H}^{(\eta)}=\mathfrak{H}$, and if $\eta\neq\eta'$ $\mathfrak{H}^{(\eta)}$ and $\mathfrak{H}^{(\eta')}$ are always mutually orthogonal.

Let $E^{(\eta)}$ be the projection defined by the closed linear manifold $\mathfrak{H}^{(\eta)}$. The contraction of U_t on $\mathfrak{H}^{(\eta)}$ is then $U_t E^{(\eta)}$ and $\{U_t E^{(\eta)}, \mathfrak{H}^{(\eta)}, x^{(\eta)}\}$ is a simple unitary structure of R^1 for each η. Therefore by Lemma 3 there is a resolution of the identity $E_\lambda^{(\eta)}$ in the Hilbert space $\mathfrak{H}^{(\eta)}$ so that $U_t E^{(\eta)}$ is expressed by

$$U_t E^{(\eta)}=\int_{-\infty}^{\infty} e^{\sqrt{-1}\lambda t}\, dE_\lambda^{(\eta)},\quad \text{for each } \eta,$$

Clearly $\sum_\eta E_\lambda^{(\eta)}=E_\lambda(-\infty<\lambda<\infty)$ is a resolution of the identity in $\mathfrak{H}$ and

$$U_t=\sum_\eta U_t E=\int_{-\infty}^{\infty} e^{\sqrt{-1}\lambda t}\, dE_\lambda,$$

thus the theorem is proved.

Now, let L be the Lie algebra of G. We take the same basis $\{A_1, A_2,\cdots, A_n\}$ as used in § 2. Then an element A of L is expressed uniquely as the linear combination $a_1A_1+a_2A_2+\cdots+a_nA_n$, and there are C^1 curves $\sigma(t)$ satisfying

$$\left[\frac{d}{dt}x_i(\sigma(t))\right]_{t=0}=a_i,\quad i=1,2,\cdots n, \tag{3·6}$$

among which there is the unique one-parameter subgroup, i. e. the one-parameter subgroup defined by $\sigma(t)=\exp(t\sum_{i=1}^n a_iA_i)$.

Let $\{U_\sigma, \mathfrak{H}_\varphi, x_\varphi\}$ be a simple unitary structure of G, L_φ the set of all operators $A_{\sigma(t)}$ defined in Definition 3. To every element A of L take a C^1 curve $\sigma(t)$ satisfying (3·6) and let $A_{\sigma(t)}$ be the operator defined by (3·4). Then according to $A_{\sigma(t)}[x]=[x_{\sigma(t)}]$ and the property (3) in §2, the operator $A_{\sigma(t)}$ is determined uniquely by A alone. We shall write $\Phi(A)=A_{\sigma(t)}$. Φ is a mapping of L into L_φ. This mapping Φ is clearly linear, i. e., if $A, B\in L$ and a is a real number, we have $\Phi(aA)=a\Phi(A)$ and $\Phi(A+B)=\Phi(A)+\Phi(B)$. We shall now prove that $\Phi([A, B])=[\Phi(A), \Phi(B)]$. Let $\sigma(t)$ and $\tau(t)$ be one-parameter subgroups defined by $\Phi(A)=A_{\sigma(t)}$ and $\Phi(B)=A_{\tau(t)}$ respectively. Put $\rho(t)=\sigma^{-1}(\sqrt{t})\cdot\tau^{-1}(\sqrt{t})\cdot\sigma(\sqrt{t})\cdot\tau(\sqrt{t})$ when $t\geq 0$ and $\rho(t)=\sigma(\sqrt{-t})\cdot\tau(\sqrt{-t})\cdot\sigma^{-1}(\sqrt{-t})\cdot\tau^{-1}(\sqrt{-t})$ when $t<0$, then $\rho(t)$ is a C^1 curve and $[A, B]=\sum_{i=1}^n \left[\frac{d}{dt}x_i(\rho(t))\right]_{t=0} A_i$. On the other hand we have

$$A_{\rho(t)}[x]=\lim_{t\to+0}\frac{1}{t}U^{-1}_{\sigma(\sqrt{t})}U^{-1}_{\tau(\sqrt{t})}\{(U_{\sigma(\sqrt{t})}-E)(U_{\tau(\sqrt{t})}-E)$$
$$-(U_{\tau(\sqrt{t})}-E)(U_{\sigma(\sqrt{t})}-E)\}[x] \qquad \text{for any } [x]\in\mathfrak{D}_\varphi.$$

and, as $x(\sigma)\in D(G)$ is a C^2 function, we have

$$\frac{1}{t}(U_{\sigma(\sqrt{t})}-E)(U_{\tau(\sqrt{t})}-E)[x]=\frac{1}{\sqrt{t}}(U_{\sigma(\sqrt{t})}-E)U_{\tau(\xi_2)}A_{\tau(t)}[x]$$
$$=U_{\sigma(\xi_1)}A_{\sigma(t)}U_{\tau(\xi_2)}A_{\sigma(t)}[x]=[x_{\tau(t)\sigma(t)}(\sigma^{-1}(\xi_1)\cdot\tau^{-1}(\xi_2)\cdot\sigma)],$$

where $\sqrt{t}\geq\xi_1\geq 0$ and $\sqrt{t}\geq\xi_2\geq 0$, and ξ_2 is dependent of $[x]$, and ξ_1 of $[x]$ and ξ_2. As $x_{\tau(t)\sigma(t)}(\sigma)$ is continuous and zero out of some compact set K, for $|t|<t_0$ with sufficiently small t_0, $x_{\tau(t)\sigma(t)}(\sigma^{-1}(\xi_1)\cdot\tau^{-1}(\xi_2)\cdot\sigma)$ is always zero out of some compact set cotaining K. Therefore, it is bounded for $|t|<t_0$ and, when t tends to zero, converges to $x_{\tau(t)\sigma(t)}(\sigma)$. Hence we obtain

$$\lim_{t\to+0}\frac{1}{t}(U_{\sigma(t)}-E)(U_{\tau(t)}-E)[x]=A_{\sigma(t)}A_{\tau(t)}[x],$$

in the same way as in the proof of Lemma 2. Thus we have

$$A_{\rho(t)}[x]=A_{\sigma(t)}A_{\tau(t)}[x]-A_{\tau(t)}A_{\sigma(t)}[x] \quad \text{for any } [x]\in\mathfrak{D}_\varphi,$$

and

$$\Phi([A,B])=[\Phi(A),\Phi(B)].$$

Let N be the kernel of the representation given by the simple unitary structure $\{U_\sigma, \mathfrak{H}_\varphi, x_\varphi\}$ of G. It is now easily seen that L_φ is isomorphic to the Lie algebra of the factor group G/N.

Definition 5. *Let A be an operator of a Hilbert space $\mathfrak{H}$ such that $\sqrt{-1}A$ is a Hermitian operator with the unique self-adjoint extension. Then we denote with $\tilde{A}$ the extension of A such that $\sqrt{-1}\tilde{A}$ is the self-adjoint extension of $\sqrt{-1}A$.*

We have already used this notation in defining $\tilde{A}_{\sigma(t)}$ in Lemma 3. In the sequel, the operation $\sim$ for operators has always this meaning.

The following Theorem 2 is a direct consequence from what we have explained above.

Theorem 2. *Let $\{U_\sigma, \mathfrak{H}_\varphi, x_\varphi\}$ be a simple unitary structure of a Lie group G. Then the set $\tilde{L}_\varphi$ of all operators defined by*

$$\lim_{t\to0}\frac{1}{t}(U_{\sigma(t)}-E)=\tilde{A}_{\sigma(t)}$$

for each one-parameter subgroup $\sigma(t)$ of G forms a Lie algebra, a homomorphic image of the Lie algebra L of G over the field of real numbers, the addition and the formation of the commutator being defined as follows,

$$\tilde{A}_{\sigma(t)}+\tilde{A}_{\tau(t)}=(A_{\sigma(t)}+A_{\tau(t)})^{\sim},$$
$$[\tilde{A}_{\sigma(t)},\tilde{A}_{\tau(t)}]=[A_{\sigma(t)},A_{\tau(t)}]^{\sim}.$$

Theorem 3. *Let $\{U_\sigma, \mathfrak{H}\}$ be any (possibly not simple) unitary structure of G. By the same definition as for $\tilde{L}_\varphi$ in Theorem 2, we obtain a homomorphic image $\tilde{L}$ of L and if the representation given by $\{U_\sigma,\mathfrak{H}\}$ is faithful, $\tilde{L}$ is isomorphic to L.*

Proof. Just as in the proof of Theorem 1, we express $\mathfrak{H}$ as a direct sum $\mathfrak{H}=\sum_\eta\oplus\mathfrak{H}^{(\eta)}$, $\mathfrak{H}^{(\eta)}=\{U_\sigma x^{(\eta)},\sigma\in G\}^{cl}$ and we denote with $E^{(\eta)}$ the projection defined by $\mathfrak{H}^{(\eta)}$. $\{U_\sigma E^{(\eta)}, \mathfrak{H}^{(\eta)}, x^{(\eta)}\}$ is then a simple unitary structure for each η. We have

$$\lim_{t\to 0}\frac{1}{t}(U_{\sigma(t)}-E)x=\lim_{t\to 0}\sum_\eta(U_{\sigma(t)}E^{(\eta)}-E^{(\eta)})x$$
$$=\sum_\eta \tilde{A}_{\sigma(t)}E^{(\eta)}x,$$

and, if η is fixed, the set all $\tilde{A}_{\sigma(t)}E^{(\eta)}$ for each one-parameter subgroup $\sigma(t)$ of G is a homomorphic image of the Lie algebra L of G. So the set $\tilde{L}$ of all operators $A_{\sigma(t)}=\sum_\eta A_{\sigma(t)}E^{(\eta)}$ for each one-parameter subgroup $\sigma(t)$ in G is also a homomorphic image of L. If the representation $\sigma\to U_\sigma$ is faithful, $\tilde{A}_{\sigma)t)}$ and $\tilde{A}_{\tau(t)}$ are clearly different for different one-parameter subgroups $\sigma(t)$ and $\tau(t)$, so $\tilde{L}$ is isomorphic to L, q. e. d.

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Notes

1) These definitions are due to [2].
2) A C^2 function on$G$ means a C^2 function on any cubic set in G.
3) This method is due to [6].
4) This method is due to [4].