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A Generalization of Laguerre Geometry 1.

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§ 1. *Introduction*. In this paper we shall try to generalize the classical Laguerre differential geometry[1] in making use of the tensor calculus. Let V_n be an n-dimensional Riemannian space with the fundamental metric tensor $g_{ij}(x^k)$[2]. In each tangent Euclidean space referred to a cartesian coordinate system (X^i), a hypersphere is determined by the coordinates V^i of the center and its radius V^0, and is represented by an equation of the form

$$g_{jk}(X^j-V^j)(X^k-V^k)=(V^0)^2. \tag{1.1}$$

The V^i are components of a covariant vector and V^0 is that of a scalar of V_n. A hypersphere will be hereafter denoted by the symbol V^λ[3] Thus each tangent space of V_n contains ∞^{n+1} hyperspheres. When it is regarded as the space whose elements are hyperspheres, we shall call it the *tangent space of hyperspheres*. Now, the tangential distance D between two hyperspheres V^λ and W^λ is given by

$$D^2=g_{jk}(V^j-W^j)(V^k-W^k)-(V^0-W^0)^2 \tag{1.2}$$

or by

$$D^2=g_{\mu\nu}(V^\mu-W^\mu)(V^\nu-W^\nu), \tag{1.3}$$

where we have put

$$g_{00}=-1, \qquad g_{0k}=g_{k0}=0.$$

Now, a hyperplane in tangent space is represented by an equation of the form

$$t_i X^i=p. \tag{1.4}$$

The necessary and sufficient condition that a hypersphere V^λ touches the hyperplane (1.4) is given by

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1) T. Takasu. Differentialgeometrieen in den Kugelräumen, Bd. 2, Laguerresche Differentialkugelgeometrie.

2) The indices i, j, k,······take the values 1, 2,······, n.

3) The indices λ, μ, ν,······, take the values 0, 1,···, n.

$$\frac{t_i V^i - p}{\sqrt{g^{ij} t_i t_j}} = V^0,$$

or by

(1.5) $$t_\lambda V^\lambda = p,$$

where we have put

$$t_0 = -\sqrt{g^{ij} t_i t_j}$$

and consequently we have

(1.6) $$g^{\lambda\mu} t_\lambda t_\mu = 0 \quad (g^{\lambda\mu} g_{\mu\omega} = \delta^\lambda_\omega).$$

Therefore, any hypersphere satisfying (1.5) where the coefficients t_λ satisfy (1.6) touches always to a hyperplane whose normal is $g^{ij} t_j$ and whose distance from the origin (x^i) is $p/\sqrt{g^{ab} t_a t_b}$. In this sense, the equation (1.5) defines a hyperplane. We shall denote hereafter such a hyperplane by the symbol (t_λ, p) or t_λ.

Of course t_i is a covariant vector and t_0 and p are scalars.

§ 2. *Linear connection which leaves invariant the tangentital distance between two hyperspheres.*

Each tangent space being regarded as space of hyperspheres, we shall define a correspondence between the tangent space of hyperspheres at (x_i) and that at $(x^i + dx^i)$. We shall assume that the hypersphere $V^\lambda + dV^\lambda$ in the tangent space of hyperspheres at $x^i + dx^i$ corresponds to the hypersphere $V^\lambda + dV^\lambda$ in the tangent space of hyperspheres at (x^i) and δV^λ is given by the equation of the form

(2.1) $$\delta V^\lambda = dV^\lambda + \Gamma^{\lambda}_{\mu k} V^\mu dx^k.$$

If V^λ is a field of hypersphere, we can put

$$\delta V^\lambda = V;^\lambda_k dx^k,$$

where

(2.2) $$V;^\lambda_k = \frac{\partial V^\lambda}{\partial x^k} + \Gamma^{\lambda}_{\mu k} V^\mu.$$

The linear connection being thus defined, we assume that it leaves invariant the tangential distance between two hyperspheres, when these hyperspheres are displaced according to the above defined linear connection.

Thus we must have

(2.3) $$g_{\mu\nu,k} \equiv \frac{\partial g_{\mu\nu}}{\partial x^k} - g_{\alpha\nu} \Gamma^{\alpha}_{\mu k} - g_{\alpha\mu} \Gamma^{\alpha}_{\nu k} = 0$$

from which follow the equations;

$$(2.4)\qquad \begin{cases} \dfrac{\partial g_{ij}}{\partial x^k}-g_{aj}\,\Gamma^a_{ik}-g_{ai}\,\Gamma^a_{jk}-0, \\ \Gamma^o_{jk}-g_{ja}\Gamma^a_{ok}=0, \\ \Gamma^o_{ok}=0. \end{cases}$$

To the differential dx^i we can associate a hypersphere whose center is at dx^i and whose radius is zero; denoting it by dx^λ, we have

$$(2.5)\qquad \delta_2 dx_1{}^\lambda-\delta_1 dx_2{}^\lambda=(\Gamma^\lambda_{jk}-\Gamma^\lambda_{kj})\,\underset{1}{dx^j}\,\underset{2}{dx^k}$$

for any two infinitesimal displacements along $\underset{1}{dx^i}$ and $\underset{2}{dx^j}$ where dx^0 is assumed to be zero. The tensor

$$S^\lambda_{jk}=\Gamma^\lambda_{jk}-\Gamma^\lambda_{kj}$$

defines the torsion of the space. We shall assume that our space has no torsion, so that we have

$$(2.6)\qquad \Gamma^o_{jk}=\Gamma^o_{kj},\quad \Gamma^i_{jk}=\Gamma^i_{kj}.$$

Consequently, from the first of the equations (2.4), we find

$$(2.7)\qquad \Gamma^i_{jk}=\left\{ {i \atop jk} \right\}=\frac{1}{2}g^{ia}\left(\frac{\partial g_{aj}}{\partial x^k}+\frac{\partial g_{ak}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^a}\right).$$

On the other hand, we have

$$(2.8)\qquad \underset{2}{\delta}\,\underset{1}{\delta} V^\lambda-\underset{1}{\delta}\,\underset{2}{\delta} V^\lambda=B^\lambda{}_{\mu jk}\,V^\mu\underset{1}{dx^j}\,\underset{2}{dx^k}$$

for any two displacements $\underset{1}{dx^i}$ and $\underset{2}{dx^i}$, where $B^\lambda_{\mu jk}$ are components of the curvature tensor given by

$$(2.9)\qquad B^\lambda_{\mu jk}=\frac{\partial\Gamma^\lambda_{\mu j}}{\partial x^k}-\frac{\partial\Gamma^\lambda_{\mu k}}{\partial x^j}-\Gamma^a_{\mu j}\,\Gamma^\lambda_{ak}+\Gamma^a_{\mu k}\,\Gamma^\lambda_{aj}.$$

Writing down fully the components of the curvature tensor, we have

$$(2.10)\qquad \begin{cases} B^m{}_{ijk}=R^m{}_{ijk}+\Gamma^m_{ok}\,\Gamma^o_{ij}-\Gamma^m_{oj}\,\Gamma^o_{ik}, \\ B^o_{\cdot ijk}=\Gamma^o_{ij,k}-\Gamma^o_{ik,j}, \\ B^i_{\cdot ojk}=g^{im}B^o_{\cdot mjk}, \\ B^o_{\cdot ojk}=0, \end{cases}$$

where $R^m_{\cdot ijk}$ are components of ordinary curvature tensor formed with Γ^i_{jk}:

$$R^{m}_{\cdot ijk}=\frac{\partial\{^{m}_{ij}\}}{\partial x^{k}}-\frac{\partial\{^{m}_{ik}\}}{\partial x^{j}}-\{^{a}_{ij}\}\{^{m}_{ak}\}+\{^{a}_{ik}\}\{^{m}_{aj}\}$$

and the $\Gamma^{o}_{ij,k}$ denotes the covariant derivative with respect to the Christoffel symbols $\{^{i}_{jk}\}$.

§3. *One parameter family of hyperspheres along a curve.* Let

$$V^{\lambda}=V^{\lambda}(t) \tag{3.1}$$

be a family of ∞' hyperspheres defined along a curve

$$x^{i}=x^{i}(t).$$

The covariant derivative of the hypersphere V^{λ} being defined by

$$\delta V^{\lambda}=dV^{\lambda}+\Gamma^{\lambda}_{\mu k}\,V^{\mu}\,dx^{k},$$

the hypersphere $V^{\lambda}+dV^{\lambda}$ in the tangent space at the point $x^{i}+dx^{i}$ is mapped on the hypersphere in the tangent space at the point x^{i} whose center is at $dx^{i}+V^{i}+\delta V^{i}$ and whose radius is $V^{0}+\delta V^{0}$. This hypersphere may be represented by

$$V^{\lambda}+\delta V^{\lambda}+dx^{\lambda},$$

where we assume that

$$dx^{0}=0.$$

If we have

$$g_{\lambda\mu}(\delta V^{\lambda}+dx^{\lambda})(\delta V^{\mu}+dx^{\mu})=0,$$

the hypersphere V^{λ} and $V^{\lambda}+\delta V^{\lambda}+dx^{\lambda}$ are tangent to each other. If we have

$$g_{\lambda\mu}(\delta V^{\lambda}+dx^{\lambda})(\delta V^{\mu}+\delta x^{\mu})\neq 0$$

along the curve, we can introduce a parameter s such that

$$\left|g_{\lambda\mu}\left(\frac{\delta V^{\lambda}}{ds}+\frac{dx^{\lambda}}{ds}\right)\left(\frac{\delta V^{\mu}}{ds}+\frac{dx^{\mu}}{ds}\right)\right|=1.$$

Then putting

$$\overset{(1)}{V^{\lambda}}=\frac{\delta V^{\lambda}}{ds}+\frac{dx^{\lambda}}{ds}, \tag{3.5}$$

we have

$$g_{\lambda\mu}\overset{(1)}{V^{\lambda}}\,\overset{(1)}{V^{\mu}}=\overset{(1)}{e}=1 \text{ or } -1. \tag{3.6}$$

Now, the covariant derivative $\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}$ of $\overset{(1)}{V}{}^{\lambda}$ along the curve is a hypersphere satisfying

$$g_{\lambda\mu} \overset{(1)}{V}{}^{\lambda} \frac{\delta \overset{(1)}{V}{}^{\mu}}{ds} = 0.$$

Thus, if

$$\frac{d\overset{(1)}{V}{}^{\lambda}}{ds} \neq 0 \quad \text{and} \quad g_{\lambda\mu} \frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds} \frac{\delta \overset{(1)}{V}{}^{\mu}}{ds} \neq 0, \tag{3.7}$$

then we can define a hypersphere such that

$$\delta \overset{(1)}{V}{}^{\lambda}/ds = \overset{(1)}{k} \overset{(2)}{V}{}^{\lambda} \tag{3.8}$$

where

$$\overset{(1)}{k} = \sqrt{|g_{\lambda\mu}(\delta \overset{(1)}{V}{}^{\lambda}/ds)(\delta \overset{(1)}{V}{}^{\mu}/ds)|} \tag{3.9}$$

and

$$g_{\lambda\mu} \overset{(1)}{V}{}^{\lambda} \overset{(2)}{V}{}^{\mu} = 0, \qquad g_{\lambda\mu} \overset{(2)}{V}{}^{\lambda} \overset{(2)}{V}{}^{\mu} = \overset{(2)}{e} = 1 \text{ or } -1. \tag{3.10}$$

If

$$g_{\lambda\mu}\left(\frac{\delta \overset{(2)}{V}{}^{\lambda}}{ds} + \overset{(1)}{e}\overset{(2)}{e}\overset{(1)}{k}\overset{(1)}{V}{}^{\lambda}\right)\left(\frac{\delta \overset{(2)}{V}{}^{\mu}}{ds} + \overset{(1)}{e}\overset{(2)}{e}\overset{(1)}{k}\overset{(1)}{V}{}^{\mu}\right) \neq 0, \tag{3.11}$$

then we can define a hypersphere $\overset{(3)}{V}{}^{\lambda}$ such as

$$\delta \overset{(2)}{V}{}^{\lambda}/ds + \overset{(1)}{e}\overset{(2)}{e}\overset{(3)}{k}\overset{(1)}{V}{}^{\lambda} = \overset{(2)}{k}\overset{(3)}{V}{}^{\lambda} \tag{3.12}$$

where

$$\overset{(2)}{k} = \sqrt{|g_{\lambda\mu}(\delta \overset{(2)}{V}{}^{\lambda}/ds + \overset{(1)}{e}\overset{(2)}{e}\overset{(1)}{k}\overset{(1)}{V}{}^{\lambda})(\delta \overset{(2)}{V}{}^{\mu}/ds + \overset{(1)}{e}\overset{(2)}{e}\overset{(1)}{k}\overset{(1)}{V}{}^{\mu})|} \tag{3.13}$$

and

$$g_{\lambda\mu} \overset{(3)}{V}{}^{\lambda} \overset{(2)}{V}{}^{\mu} = g_{\lambda\mu} \overset{(2)}{V}{}^{\lambda} \overset{(1)}{V}{}^{\mu} = 0, \quad g_{\lambda\mu} \overset{(3)}{V}{}^{\lambda} \overset{(3)}{V}{}^{\mu} = \overset{(3)}{e} = 1 \text{ or } -1. \tag{3.14}$$

Proceeding in this way, we can arrive at a hypersphere $\overset{(n)}{V}{}^{\lambda}$ such that

$$0 = g_{\lambda\mu} \overset{(n)}{V}{}^{\lambda} \overset{(n-1)}{V}{}^{\mu} = \cdots\cdots = g_{\lambda\mu} \overset{(n)}{V}{}^{\lambda} \overset{(1)}{V}{}^{\mu}, \quad g_{\lambda\mu} \overset{(n)}{V}{}^{\lambda} \overset{(n)}{V}{}^{\mu} = \overset{(n)}{e} = 1 \text{ or } -1. \tag{3.15}$$

If

$$(3.16)\qquad g_{\lambda\mu}\left(\frac{\delta \overset{(n)}{V}{}^{\lambda}}{ds}+\overset{(n-1)}{e}\overset{(n)}{e}\overset{(n-1)}{k}\overset{(n-1)}{V}{}^{\lambda}\right)\left(\frac{\delta \overset{(n)}{V}{}^{\mu}}{ds}+\overset{(n-1)}{e}\overset{(n)}{e}\overset{(n-1)}{k}\overset{(n-1)}{V}{}^{\lambda}\right)\neq 0,$$

then we can define a hypersphere $\overset{(n+1)}{V}{}^{\lambda}$ such that

$$(3.17)\qquad \frac{\delta \overset{(n)}{V}{}^{\lambda}}{ds}+\overset{(n-1)}{e}\overset{(n)}{e}\overset{(n-1)}{k}\overset{(n-1)}{V}{}^{\lambda}=\overset{(n)}{k}\overset{(n+1)}{V}{}^{\lambda}$$

where

$$(3.18)\qquad \overset{(n)}{k}=\sqrt{\left|g_{\lambda\mu}(\delta \overset{(n)}{V}{}^{\lambda}/ds+\overset{(n-1)}{e}\overset{(n)}{e}\overset{(n-1)}{k}\overset{(n-1)}{V}{}^{\lambda})(\delta \overset{(n)}{V}{}^{\mu}/ds+\overset{(n-1)}{e}\overset{(n)}{e}\overset{(n-1)}{k}\overset{(n-1)}{V}{}^{\mu})\right|}$$

and

$$(3.19)\qquad g_{\lambda\mu}\overset{(n+1)}{V}{}^{\lambda}\overset{(n)}{V}{}^{\mu}=\cdots\cdots=g_{\lambda\mu}\overset{(n+1)}{V}{}^{\lambda}\overset{(1)}{V}{}^{\mu}=0,$$

$$g_{\lambda\mu}\overset{(n-1)}{V}{}^{\lambda}\overset{(n-1)}{V}{}^{\mu}=\overset{(n-1)}{e}=1 \text{ or } -1.$$

Putting

$$(3.20)\qquad W^{\lambda}=\frac{\delta \overset{(n+1)}{V}{}^{\lambda}}{ds}+\overset{(n)}{e}\overset{(n-1)}{e}\overset{(n)}{k}\overset{(n)}{V}{}^{\lambda},$$

we have from (3.17) and (3.39)

$$(3.21)\qquad g_{\lambda\mu}W^{\lambda}\overset{(n-1)}{V}{}^{\mu}=\cdots\cdots=g_{\lambda\mu}W^{\lambda}\overset{(1)}{V}{}^{\mu}=0.$$

Then W^{λ} must be zero, since the $n+1$ hyperspheres $\overset{(1)}{V}{}^{\lambda},\cdots\cdots,\overset{(n+1)}{V}{}^{\lambda}$ form a base of the tangent space of hyperspheres, and we have

$$(3.22)\qquad \frac{\delta \overset{(n+1)}{V}{}^{\lambda}}{ds}+\overset{(n)}{e}\overset{(n-1)}{e}\overset{(n)}{k}\overset{(n)}{V}{}^{\lambda}=0.$$

The epuations (3.8), (3.12), (3.17) and (3.22) constitute the so-called Frenet formulae in our space with Laguerre connection.

§ 4. *Some special one parameter families of hyperspheres.*

In this section, we prove some theorems on special one parameter family of hyperspheres.

Theorem 1. *If a hypersphere of the form* $V^{\lambda}+pV^{\lambda}$ *is fixed by our connection along the curve, then we have*

$$(4.1)\qquad \frac{\delta V^{\lambda}}{ds}=0.$$

The converse is also true.

Proof. By the assumption, the center and the radius of the hypersphere $V^\lambda + p\overset{(1)}{V^\lambda}$ being fixed, we have

$$\frac{\delta}{ds}(x^\lambda + V^\lambda + p\overset{(1)}{V^\lambda}) = 0, \tag{4.2}$$

from which we have

$$\overset{(1)}{V^\lambda} + \frac{dp}{ds}\overset{(1)}{V^\lambda} + p\frac{\delta V^\lambda}{ds} = 0. \tag{4.3}$$

Contracting this equation with $g_{\lambda\mu}V^\mu$, we find

$$1 + \frac{dp}{ds} = 0. \tag{4.4}$$

Hence we have from (4.3)

$$\delta\overset{(1)}{V^\lambda}/ds = 0. \tag{4.5}$$

Conversely, if the equation (4.1) is satisfied by some $\overset{(1)}{V^\lambda}(s)$, then we have

$$0 = \frac{\delta}{ds}(x^\lambda + V^\lambda + p\overset{(1)}{V^\lambda}) = \left(1 + \frac{dp}{ds}\right)\overset{(1)}{V^\lambda},$$

where

$$p = c - s.$$

Thus we know that the hypersphere

$$V^\lambda + (c-s)\overset{(1)}{V^\lambda}$$

is fixed by our connection along the curve. Thus the theorem is proved.

Theorem 2. *If a hypersphere of the form* $V^\lambda + p\overset{(2)}{V^\lambda}$ *is fixed by our connection along the curve, then we have*

$$\frac{\delta^2\overset{(1)}{V^\lambda}}{ds^2} + \overset{(1)}{e}g_{\alpha\beta}\frac{\delta\overset{(1)}{V^\alpha}}{ds}\frac{\delta\overset{(1)}{V^\beta}}{ds}\overset{(1)}{V^\lambda} = 0. \tag{4.6}$$

The converse is also true.

Proof. By the assumption, we have

$$\frac{\delta}{ds}\left(x^\lambda + V^\lambda + p\frac{\delta V^\lambda}{ds}\right) = 0, \tag{4.6}$$

from which

$$\overset{(1)}{V}{}^{\lambda}+\frac{dp}{ds}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}+p\frac{\delta^2 \overset{(1)}{V}{}^{\lambda}}{ds^2}=0. \tag{4.8}$$

Contracting this by $g_{\lambda\mu}\overset{(1)}{V}{}^{\mu}$ we get

$$\overset{(1)}{e}=pg_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta \overset{(1)}{V}{}^{\mu}}{ds}=1 \text{ or } -1. \tag{4.9}$$

If we differentiate (4.9) covariantly along the curve, we have

$$\frac{dp}{ds}g_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta \overset{(1)}{V}{}^{\mu}}{ds}+2pg_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta^2 \overset{(1)}{V}{}^{\mu}}{ds^2}=0. \tag{4.10}$$

On the other hand, we have from (4.8)

$$\frac{dp}{ds}g_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta \overset{(1)}{V}{}^{\mu}}{ds}+pg_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta^2 \overset{(1)}{V}{}^{\mu}}{ds}=0. \tag{4.11}$$

From (4.10) and (4.11) we get

$$\frac{dp}{ds}=0, \qquad p=\text{const.}=\overset{(1)}{e}\left(g_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta \overset{(1)}{V}{}^{\mu}}{ds}\right)^{-1}. \tag{4.12}$$

Therefore (4.8) becomes (4.6). Conversely, if (4.6) holds and

$$g_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta \overset{(1)}{V}{}^{\mu}}{ds}\neq 0,$$

we find from (4.6)

$$\frac{\delta}{ds}\left(g_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta \overset{(1)}{V}{}^{\mu}}{ds}\right)=0. \tag{4.13}$$

Putting

$$p=\overset{(1)}{e}\left(g_{\lambda\mu}\frac{\delta \overset{(1)}{V}{}^{\lambda}}{ds}\frac{\delta \overset{(1)}{V}{}^{\mu}}{ds}\right)^{-1} \tag{4.14}$$

we have from (4.13)

$$\frac{dp}{ds}=0. \tag{4.15}$$

Therefore (4.6) becomes (4.8). In this case the tangential distance between V^λ and the fixed hypersphere $V^\lambda+p\frac{\delta \overset{(1)}{V^\lambda}}{ds}$ becomes

$$D=\left(g_{\lambda\mu}\frac{\delta \overset{(1)}{V^\lambda}}{ds}\frac{\delta \overset{(1)}{V^\mu}}{ds}\right)^{-\frac{1}{2}}. \tag{4.16}$$

§ 5. *Family of* ∞^{n-1} *hyperspheres.*

Let

$$V^\lambda=V^\lambda(u^{\dot 1}, u^{\dot 2},\dots,u^{\dot{n}-\dot 1}) \tag{5.1}$$

be a family of ∞^{n-1} hyperspheres defined along a hypersurface $x^i=x^i(u)$. The difference between two consecutive hyperspheres belonging to our family is given by

$$\delta V^\lambda+dx^\lambda={}^*B^\lambda_A\, du^A, \tag{5.2}$$

where

$$B^\lambda_A=\frac{\delta V^\lambda}{\partial u^A}+\frac{\partial x^\lambda}{\partial u^A}=\frac{\partial V^\lambda}{\partial u^A}+\Gamma^\lambda_{\mu k}V^\mu\frac{\partial x^k}{\partial u^A}+\frac{\partial x^\lambda}{\partial u^A}. \tag{5.3}$$

We put

$$g_{\lambda\mu}\, B^\lambda_A\, B^\mu_B=G_{AB}, \tag{5.4}$$

and assume that

$$|G_{AB}|\neq 0. \tag{5.5}$$

Then we can define the conjugate tensor

$$G^{AB}=G^{BA}, \tag{5.6}$$

by means of the equations

$$G^{AC}\, G_{CB}=\delta^A_B. \tag{5.7}$$

There exist two hyperplanes t_λ and $\bar{t}_\lambda$ which touch V^λ and $V^\lambda+\delta V^\lambda$ for any displacement du^A, that is, which satisfy

$$\text{(A)}\begin{cases} t_\lambda\, B^\lambda_A=0,\\ t_\lambda\, t^\lambda=0,\\ g^{\lambda\mu}t_\lambda=t^\mu,\end{cases}\qquad \text{(B)}\begin{cases} \bar{t}_\lambda\, B^\lambda_A=0,\\ \bar{t}_\lambda\, \bar{t}^\lambda=0,\\ g^{\lambda\mu}\,\bar{t}_\mu=\bar{t}^\lambda,\end{cases} \tag{5.8}$$

where the factors for t_λ and $\bar{t}_\lambda$ are assumed to satisfy the following condition :

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* A, B, C,$\cdots\cdots=\dot{1}, \dot{2},\cdots\cdots, \dot{n}-\dot{1}$.

(5.9) $$t_\lambda \bar{t}^\lambda = 2.$$

Then, we can define two hyperspheres B_o^λ and B_∞^λ by

(5.10) $$B_o^\lambda = t^\lambda + \frac{\bar{t}^\lambda}{4}, \quad B_\infty^\lambda = t^\lambda - \frac{\bar{t}^\lambda}{4}.$$

Then, from (5.8) and (5.9) we have

(5.11) $$g_{\lambda\mu} B_o^\lambda B_A^\mu = g_{\lambda\mu} B_\infty^\lambda B_A^\mu = 0, \quad g_{\lambda\mu} B_o^\lambda B_\infty^\mu = 0, \quad B_{\lambda\mu} B_o^\lambda B_o^\mu = 1,$$
$$g_{\lambda\mu} B_\infty^\lambda B_\infty^\mu = -1.$$

Thus, the $n+1$ hyperspheres B_o^λ, B_∞^λ and B_A^λ are linearly independent. Next, we shall differentiate these hyperspheres covariantly along the hypersurface. Then we must have the equations of the form

(5.12) $$\begin{cases} \delta B_A^\lambda / \partial u^B = B_C^\lambda K_{AB}^C + B_o^\lambda K_{AB}^o + B_\infty^\lambda K_{AB}^\infty, \\ \delta B_o^\lambda / \partial u^B = B_C^\lambda K_{oB}^C + B_o^\lambda K_{oB}^o + B_\infty^\lambda K_{oB}^\infty, \\ \delta B_\infty^\lambda / \partial u^B = B_C^\lambda K_{\infty B}^C + B_o^\lambda K_{\infty B}^o + B_\infty^\lambda K_{\infty B}^\infty, \end{cases}$$

where

$$\delta B_A^\lambda / \partial u^B = \partial B_A^\lambda / \partial u^B + \Gamma_{\mu k}^\lambda B_A^\mu \, \partial x^k / \partial u^B \qquad \text{etc.}$$

But, from (5.11), we have

(5.14) $$K_{oB}^o = K_{\infty B}^\infty = 0, \qquad K_{\infty B}^o = K_{oB}^\infty,$$
$$K_{AB}^o + G_{AC} K_{oB}^C = 0, \qquad K_{AB}^\infty - G_{AC} K_{\infty B}^C = 0.$$

These quantities are not independent, but satisfy the following integrability conditions. First we have from (5.3)

(5.14) $$B_{\cdot\mu kl}^\lambda V^\mu \frac{\partial x^k}{\partial u^A} \frac{\partial x^l}{\partial u^B} = B_C^\lambda (K_{AB}^C - K_{BA}^C) + B_o^\lambda (K_{AB}^o - K_{BA}^o)$$
$$+ B_\infty^\lambda (K_{AB}^\infty - K_{BA}^\infty).$$

Next, we have from (5.12)

(5.15) $$B_{\cdot\mu kl}^\lambda B_X^\mu \frac{\partial x^k}{\partial u^C} \frac{\partial x^l}{\partial u^C} = B_Z^\lambda \left\{ \left(\frac{\partial K_{XB}^Z}{\partial u^C} + K_{YC}^Z K_{XB}^Y \right) - \left(\frac{\partial K_{XC}^Z}{\partial u^B} + K_{XB}^Z K_{YO}^Y \right) \right\}$$

where the indices X, Y, Z run over the range o, ∞, $\dot{1}, \cdots\cdots, \dot{n-1}$.

§ 6. *Several properties of a* ∞^{n-1} *family of hyperspheres.*

From (5.10) we have

(6.1) $$t^\lambda = \frac{1}{2}(B_o^\lambda + B_\infty^\lambda), \quad \bar{t}^\lambda = 2(B_o^\lambda - B_\infty^\lambda).$$

Therefore, the equation (5.12) becomes respectively

$$(6.2)\qquad \begin{cases} \delta B_A^\lambda/\partial u^B = B_C^\lambda K_{AB}^C + \dfrac{t^\lambda}{2} H_{AB} + \dfrac{\bar{t}^\lambda}{2}\bar{H}_{AB}, \\ \delta t^\lambda/\partial u^B = -B_C^\lambda \bar{H}_{.B}^C + t^\lambda K_{oB}^\infty, \\ \delta \bar{t}^\lambda/\partial u^B = -B_C^\lambda H_{.B}^C - \bar{t}^\lambda K_{oB}^\infty, \end{cases}$$

where

$$(6.3)\qquad \begin{cases} H_{AB} = 2(K_{AB}^o + K_{AB}^\infty), \quad \bar{H}_{AB} = \dfrac{1}{2}(K_{AB}^o - K_{AB}^\infty), \\ H_{.B}^C = G^{CA} H_{AB}, \qquad \bar{H}_{.B}^C = G^{CA}\bar{H}_{AB}. \end{cases}$$

Conjugate directions. For two displacements $\underset{1}{du^A}$ and $\underset{2}{du^A}$ we have from (6.2)

$$(6.4)\qquad g_{\lambda\mu}\,\underset{2}{\delta t^\lambda}\, B_A^\mu\, \underset{1}{du^A} = -\bar{H}_{AB}\,\underset{1}{du^A}\,\underset{2}{du^B}.$$

If the hyperplane $t^\lambda + \underset{1}{\delta t^\lambda}$ touches the pencil $(V^\lambda,\ V^\lambda + B_A^\lambda \underset{2}{du^A})$, we must have

$$(6.5)\qquad g_{\lambda\mu}(t^\lambda + \underset{2}{\delta t^\lambda})(V^\mu + B_A^\lambda \underset{1}{du^A}) = g_{\lambda\mu}(t^\lambda + \underset{2}{\delta t_\lambda})\, V^\mu,$$

$$(6.6)\qquad g_{\lambda\mu}\,\underset{2}{\delta t^\lambda}\cdot B_A^\mu\, \underset{1}{du^A} = -\bar{H}_{AB}\,\underset{1}{du^A}\,\underset{2}{du^B}.$$

The converse is also true. In this case, the two directions $\underset{1}{du^A}$ and du^A are said to be *conjugate* to each other.

Normal curvature. Directions of curvature.

For a ∞^1 family of hyperspheres belonging to our ∞^{n-1} family we have

$$(6.7)\qquad \overset{(1)}{V^\lambda} = \frac{\delta V^\lambda}{ds} + \frac{dx^\lambda}{ds} = B_A^\lambda \frac{du^A}{ds},$$

where we assume that

$$(6.8)\qquad g_{\lambda\mu}\overset{(1)}{V^\lambda}\,\overset{(1)}{V^\mu} = G_{AB}\frac{du^A}{ds}\frac{du^B}{ds} = \overset{(1)}{e} = 1 \text{ or } -1.$$

As we have

$$(6.9)\qquad g_{\lambda\mu}\overset{(1)}{V^\lambda}\, t^\mu = g_{\lambda\mu} B_A^\lambda\, t^\mu \frac{du^A}{ds} = 0,$$

differentiating (6.9) covariantly we find

$$(6.10) \qquad g_{\lambda\mu}\frac{\delta \overset{(1)}{V^{\lambda}}}{ds}t^{\mu}+g_{\lambda\mu}V^{\mu}\frac{\delta t^{\mu}}{ds}=0.$$

Then we have from (6.9) and (6.2)

$$(6.11) \qquad g_{\lambda\mu}\frac{\delta \overset{(1)}{V^{\lambda}}}{ds}t^{\mu}=\bar{H}_{AB}\frac{du^A}{ds}\frac{du^B}{ds},$$

that is

$$(6.12) \qquad g_{\lambda\mu}\frac{\delta \overset{(1)}{V^{\lambda}}}{ds}t^{\mu}=\overset{(1)}{k}\, g_{\lambda\mu}\overset{(2)}{V^{\mu}}\, t^{\mu}=\frac{\bar{H}_{AB}\,du^A\,du^B}{e\overset{(1)}{G}_{AB}\,du^A\,du^B}$$

Especially, if $g_{\lambda\mu}\overset{(2)}{V^{\lambda}}\,t^{\mu}=\overset{(1)}{e}$, then the equation (6.12) becomes

$$(6.13) \qquad \overset{(1)}{k}=\frac{1}{R}=\frac{\bar{H}_{AB}\,du^A\,du^B}{G_{AB}\,du^A\,du^B}$$

We shall call R^{-1} the *normal curvature.* When R^{-1} takes an extremal value, we call such a direction given by du^A the *direction of curvature.*

Theorem 3. *If a hypersphere of the form $V^{\lambda}+pt^{\lambda}$ is fixed by our connection in any direction touching to the hypersurface $x^{\lambda}=x^{\lambda}(u)$, then we must have*

$$(6.15) \qquad \bar{H}_{AB}=\rho^{-1}\,G_{AB},$$

$$(6.16) \qquad K^{\infty}_{oB}=\frac{\partial}{\partial u^B}\log\rho^{-1},$$

$$(6.17) \qquad \rho=(n-1)(\bar{H}_{.C}^{C})^{-1}$$

The converse is also true. In this case $V^{\lambda}+pt^{\lambda}$ is a hypersphere which touches the hyperplane t_{λ} and the hypersphere V^{λ}. We call such a ∞^{n-1} family the *totally umbilical family.*

Proof. By the assumption we have

$$(6.18) \qquad \delta(x^{\lambda}+V^{\lambda}+\rho t^{\lambda})=0,$$

that is

$$(6.19) \qquad B^{\lambda}_{A}\,du^A+(d\rho)t^{\lambda}+\rho(-B^{\lambda}_{C}\,\bar{H}^{C}_{.A}\,du^A+t^{\lambda}\,K^{\infty}_{oA}\,du^A)=0.$$

Rewriting (6.19) we have

$$\{B^{\lambda}_{C}\,(\delta^{C}_{A}-\rho\,\bar{H}^{\cdot C}_{\cdot A})+t^{\lambda}\Big(\frac{\partial\rho}{\partial u^{A}}+\rho K^{\infty}_{oA}\Big)\Big\}du^{A}=0. \tag{6.20}$$

Contracting this with $g_{\lambda\mu}\,B^{\mu}_{B}$ and $\bar{t}_{\lambda}$, we have respectively

$$(G_{AB}-\rho\bar{H}_{AB})\;du^{\kappa}=0, \tag{6.21}$$

$$\Big(\frac{\partial\rho}{\partial u^{A}}+\rho K^{\infty}_{oA}\Big)du^{A}=0. \tag{6.22}$$

As du^{A} can be taken arbitrarily this gives us (6.15), (6.16) and (6.17). The converse is evident. Q.E.D.

If the family is totally umbilical, then the directions of curvature will be indeterminate.

§ 7. *Normalizations of t_{λ} and $\bar{t}^{\lambda}$.*

To determine the factors for t_{λ} and $\bar{t}_{\lambda}$, let us change t_{λ} and $\bar{t}_{\lambda}$ as follows:

$$\overset{*}{t}_{\lambda}=\rho t_{\lambda},\ \overset{*}{\bar{t}}_{\lambda}=\rho^{-1}\bar{t}_{\lambda}.\quad (\overset{*}{t}_{\lambda}\ \overset{*}{\bar{t}}{}^{\lambda}=2) \tag{7.1}$$

Putting

$$\overset{*}{\delta}t^{\lambda}=-B^{\lambda}_{C}\ \overset{*}{\bar{H}}{}^{C}_{\cdot B}\ du^{B}+\overset{*}{t}{}^{\lambda}\ \overset{*}{K}{}^{o}_{\infty B}\ du^{B} \tag{7.2}$$

we have from (7.1), (7.2) and (5.12)

$$\overset{*}{\bar{H}}{}^{\cdot C}_{\cdot B}=\rho\ \bar{H}^{\cdot C}_{\cdot B}. \tag{7.3}$$

Contracting this with respect to the indices C and B, we get

$$\overset{*}{\bar{H}}{}^{\cdot C}_{\cdot C}=\rho\bar{H}^{\cdot C}_{\cdot C}. \tag{7.4}$$

Similarly, we have

$$\overset{*}{H}{}_{\cdot B}^{\cdot C}=\rho^{-1}\ H_{\cdot B}^{\cdot C} \tag{7.5}$$

from which follows

$$\overset{*}{H}{}_{\cdot C}^{\cdot C}=\rho^{-1}\ H_{\cdot C}^{\cdot C}. \tag{7.6}$$

If

$$H_{\cdot C}^{\cdot C}\neq 0.$$

or

$$\bar{H}_{\cdot C}^{\cdot C}\neq 0,$$

then we put

(7.7) $$\hat{t}_\lambda = H^{c}_{\cdot c}\, t_\lambda, \qquad \hat{\bar{t}}_\lambda = (H^{\cdot c}_{\cdot c})^{-1} \bar{t}_\lambda$$

or

(7.8) $$\hat{\hat{t}}_\lambda = (\bar{H}^{\cdot c}_{\cdot c})^{-1}\, t_\lambda, \qquad \hat{\hat{\bar{t}}}_\lambda = \bar{H}^{\cdot c}_{\cdot c}\, \bar{t}_\lambda.$$

Then $(\hat{t}_\lambda, \hat{\bar{t}}_\lambda)$ or $(\hat{\hat{t}}_\lambda, \hat{\hat{\bar{t}}}_\lambda)$ are invariant under the change of (7.1).