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On Baire's Theorem concerning a Function f(x, y), which is Continuous with respect to Each Variable x and y.

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The purpose of this paper is to give a simple proof of the following Baire's theorem[1].

Theorem. *Let $f(x, y)$ be defined in a square Δ: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and be continuous with respect to each variable x and y. Then there exists a set X on the x-axis, which is dense on $[0,1]$, such that for any $x_0 \in X$, $f(x, y)$, considered as a function of two varibles (x, y), is continuous at every point of the segment $x=x_0$, $0 \leqq y \leqq 1$. Similarly there exists a set Y on the y-axis, which is dense on $[0, 1]$, such that for any $y_0 \in Y$, $f(x, y)$ is continuous on the segment $y=y_0$, $0 \leqq x \leqq 1$*

Proof. We will prove the existence of the set X, which satisfies the conditions of the theorem. The existence of the set Y can be proved similarly.

We define $f_n(x, y)$ $(n=1, 2\cdots)$ in Δ as follows:

$$f_n(x,y)=f(x,\frac{\nu}{2^n})+\frac{f(x,\frac{\nu+1}{2^n})-f\left(x,\frac{\nu}{2^n}\right)}{\frac{1}{2^n}}\left(y-\frac{\nu}{2^n}\right) \tag{1}$$

for $0 \leqq x \leqq 1, \frac{\nu}{2^n} \leqq y \leqq \frac{\nu+1}{2^n}$, $\quad (\nu=0,1,2,\cdots\cdots,2^n-1)$.

Then $f_n(x, y)$ is continuous in Δ and

$$f(x, y)=\lim_{n\to\infty} f_n(x, y), \tag{2}$$

$$\lim_{n\to\infty} [\operatorname*{Max.}_{0\leqq y\leqq 1} |f(x, y)-f_n(x, y)|]=0, \text{ for a fixed } x. \tag{3}$$

From (2), it follows that $f(x, y)$ is of the first class of Baire.

For a fixed $\varepsilon>0$, we define a set $E_n(\varepsilon)$ on the x-axis by

$$E_n(\varepsilon)=\underset{x}{E}[\operatorname*{Max.}_{0\leqq y\leqq 1} |f(x, y)-f_n(x, y)| \leqq \varepsilon], \tag{4}$$

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1) R. Baire: Sur les fonctions de variables réelles. Annali di Matematica. (3) 3 (1899). K. Bögel: Über die Stetigkeit und die Schwankung von Funktionen zweier Veränderlichen. Math Ann. 81 (1920).

then $E_n(\varepsilon)$ is a closed set and from (3) follows

$$I_0=\sum_{n=1}^{\infty} E_n(\varepsilon), \text{ where } I_0=[0, 1]. \tag{5}$$

Hence by Baire's theorem, in any interval $(\alpha, \beta)\subset I_0$, there exists a certain interval $U(x_0; r_0)$: $|x_0-x_0|<r_0$, such that for a suitable n_0,

$$I_0\cdot U(x; r_0)=E_{n_0}(\varepsilon)\cdot U(x_0; r_0). \tag{6}$$

If we take $\varepsilon=\frac{1}{\nu}$ $(\nu=1,2,\cdots.)$ then there exists a neighbourhood $U(x_\nu; r_\nu)$ and n_ν such that

$$I_0\cdot U(x_\nu; r_\nu)=E_{n_\nu}\left(\frac{1}{\nu}\right)\cdot U(x_\nu; r_\nu), \tag{7}$$

where we may assume that $U(x_\nu; r_\nu)\subset U(x_{\nu+1}; r_{\nu+1})$, $r_\nu\to 0$, so that $U(x_\nu; r_\nu)$ converges to a point x_0, such that

$$x_0 \in U(x_\nu; r_\nu) \quad (\nu=1,2,\cdots\cdots). \tag{8}$$

We will prove that $f(x, y)$ is contnuous at every point (x_0, φ_0) on the segment: $x=x_0$, $0\leqq y\leqq 1$.

Since $f_{n_\nu}(x, y)$ is continuous at (x_0, y_0), there exists a neighbourhood U_0: $|x-x_0|<\delta$, $|y-y_0|<\delta$, such that for any $(x, y)\in U_0$,

$$\left|f_{n_\nu}(x, y)-f_{n_\nu}(x_0, y_0)\right|<\frac{1}{\nu}. \tag{9}$$

Since $x_0 \in U(x_\nu, r_\nu)$, we can take δ so small that the interval $|x-x_0|<\delta$ is contained in $U(x_\nu; r_\nu)$.

Hence if $(x, y) \in U_0$, then $x \in U(x_\nu; r_\nu)$, so that by (7), $x \in E_{n_\nu}\left(\frac{1}{\nu}\right)$, hence by (4),

$$|f(x, y)-f_{n_\nu}(x, y)|\leqq\frac{1}{\nu}, \qquad 0\leqq y\leqq 1, \tag{10}$$

especially

$$|f(x_0, y_0)-f_{n_\nu}(x_0, y_0)|\leqq\frac{1}{\nu}. \tag{11}$$

Hence for $(x, y) \in U_0$,

$$|f(x_0, y_0)-f(x, y)| \leqq |f(x, y)-f_{n_\nu}(x, y)|+|f_{n_\nu}(x, y)-f_{n_\nu}(x_0, y_0)|$$

$$+ |f_{n_\nu}(x_0, y_0) - f(x_0, y_0)| < \frac{3}{\nu}.$$

Hence $f(x, y)$ is continuous at (x_0, y_0). Since $x_0 \in (\alpha, \beta)$ and α, β are arbitrary, the set X of x_0 is dense on $[0, 1]$, which proves theorem.

Remark. Let K be a continuum contained in Δ and K_x, K_y be its projection on the x-and y-axis respectively, then at least one of K_x, K_y consists of a closed interval, so that $K_x \cdot X \neq 0$, or $K_y \cdot Y \neq 0$, hence there exists a point on K, where $f(x, y)$ is continuous. From this it follows that *any continuum in Δ contains infinitely many points, where $f(x, y)$ is continuous.*

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