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On a Generalization of Fubini's Theorem and Its Application to Green's Formula

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The object of the present paper is to establish a generalization of Fubini's theorem in the theory of integrals and to apply it to the proof of Green's formula under considerably general conditions.

The usual form of Fubini's theorem is concerned with the transformation of the integral of a summable function over the Euclidean space R_{p+q}(p and q natural numbers) into a repeated integral taken over R_p and R_q successively, the space R_{p+q} being the cartesian product of the spaces R_p and R_q. But this last circumstance is not essential for the validity of the theorem. In fact, we may take, roughly speaking, any two spaces Φ and Ψ with measures μ and ν respectively, define a mapping φ of Φ onto $\Psi=\varphi(\Phi)$, and denoting by Φ_y the inverse image $\varphi^{-1}(y)$ of $y \in \Psi$ under the mapping φ and by μ_y a measure on Φ_y, we have the formula (see Theorem 4)

$$\int_{\Phi} f(x)\, d\mu(x) = \int_{\Psi}\left[\int_{\Phi_y} f(x)\, d\mu_y(x)\right] d\nu(y)$$

for every $f(x)$ non-negative and measurable on Φ, provided that certain conditions involving the three measures μ, ν and μ_y are satisfied.

There is a research by P.R.Halmos [7] along similar lines of idea, but it seems to us that there is little point of contact between his paper and ours, since Halmos's interest lies chiefly in other directions.

Utilizing the generalized Fubini theorem thus established, we shall be able to prove our main theorem (Theorem 7) on the transformation of a Stieltjes integral into an ordinary one. In case the function $G(x)$ with respect to which we integrate is monotone, this is a well-known theorem and in fact is taken by Hobson (see Hobson [8], p. 605) as the very definition of the Stieltjes integral; but our theorem is concerned with a general function $G(x)$ of bounded variation and our result seems to be new.

We shall give here an account of the various conditions under which Green's formula has been proved by several writers. We denote dy C a rectifiable closed Jordan curve in the plane and by D the inner domain bounded by C, and we take the formula under consideration in the form

$$\int_C M(x, y)dy=\iint_D \frac{\partial M}{\partial x}dxdy, \tag{1}$$

where the integral is taken round C in the positive sense.

W.Gross [2] proved (1) under the condition that $M(x,y)$ is continuous on $\bar{D}=D+C$ and that $\frac{\partial M}{\partial x}$ is continuous and summable on D. Then W. T. Reid [4] proved the validity of (1) under the following three conditions:

(i) $M(x, y)$ is continuous on $\bar{D}$;

(ii) $M(x,y_0)$ is absolutely continuous in x on the intersection $D(y_0)$ of the domain D and the line $y=y_0$, for almost all values of y_0.

(iii) $\frac{\partial M}{\partial x}$ is summable over D.

The result of Reid was still extended by Tsuji [5]. He gave three conditions for the validity of (1), i. e. (ii) and (iii) of Reid, together with the condition (iv) : $M(x, y)$ is continuous and bounded on D, and $\lim M(x, y)$ exists almost everywhere on C, when the point (x,y) tends to C nontangentially.

Our result will show that Green's formula holds under conditions (ii) and (iii) of Reid, together with a new condition (see Theorem 8) which is weaker than the condition (iv) of Tsuji, and which is satisfied automatically if $M(x, y)$ is bounded in D. Our chief concern is, of course, the definition of the boundary values of $M(x, y)$, which is secured by the absolute continuity of $M(x, y)$ in x.

Reid [4] has given also a proof of the "strong form" of Cauchy's Fundamental Theorem, but our present result adds nothing new to Cauchy's theorem since we assume the summability of $\frac{\partial M}{\partial x}$ over D. Further research is necessary to justify the validity of this theorem under more general conditions. We shallt treat this problem on another opportunity.

It may be mentioned in passing that our main lemma (Theorem 8) may be deduced in a simpler way from Lusin's Theorem (see Saks [9], p. 72) on the approximation of a measurable function by a continuons one. But we retain our present proof on account of the methodological interest.

Finally the author wishes to express his heaıty thanks to Prof. Tsuji and Prof. Y. Kawada for their valuable helps and criticisms on the snbject of this paper.

We begin our subject with the following

Definition. *Let Ω be an abstract space. A class $\mathfrak{M}$ of subsets of Ω is called a primitive class, if*

(i) $\Omega \in \mathfrak{M}$;

(ii) *if $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$, then $AB \in \mathfrak{M}$;*

(iii) *if $A \in \mathfrak{M}$, then there is a disjoint sequence of sets $A_n \in \mathfrak{M}(n=1, 2, 3, \ldots\ldots)$ such that*

$$\Omega - A = \sum_{n=1}^{\infty} A_n.$$

Remark. Every primitive class $\mathfrak{M}$ contains the empty set, for taking $A=\Omega$, we have $A_n=0$ $(n=1, 2, 3, \ldots\ldots)$.

Theorem 1. *The smallest additive class $\mathfrak{N}$ containg a primitive class $\mathfrak{M}$ in a space Ω coincides with the smallest normal class $\mathfrak{N}$ containing $\mathfrak{M}$.*

Proof. The proof is the same as that for the Lemma of Saks [9], p. 83.

Theorem 2. *Let Φ and Ψ be non-empty abstract spaces, $\mathfrak{X}$ and $\mathfrak{Y}$ additive classes of sets in Φ and Ψ respectively, μ and ν measures defined for sets $(\mathfrak{X})$ and sets $(\mathfrak{Y})$ respectively, with $\mu(\Phi) < +\infty$. Let further φ be a mapping of Φ on Ψ, which is the image of Φ. For each $y \in \Psi$ we denote by Φ_y the inverse image of y under φ. Let $\mathfrak{X}_y$ be an additive class of sets in Φ_y for each $y \in \Psi$, and lei μ_y be a measure defined for the sets $(\mathfrak{X}_y)$. We denote by $\mathfrak{M}$ a primitive class af sets in Φ and by $\mathfrak{N}$ the smallest additive class in Φ containing $\mathfrak{M}$, so that we have $\mathfrak{N} \subset \mathfrak{X}$.*

Now we suppose that the following three conditions hold for every $A \in \mathfrak{M}$:

(i) *$A\Phi_y \in \mathfrak{X}$ for any $y \in \Psi$.*

(ii) *$\mu_y(A\Phi_y)$ is, as a function of y, measurable $(\mathfrak{Y})$ on Ψ.*

(iii) *We have*

$$\mu(A) = \int_{\Psi} \mu_y(A\,\Phi_y)\,d\nu(y).$$

Then these three statements hold also for every set $A \in \mathfrak{N}$.

Proof. Since $\mathfrak{N}$ coincides with the smallest normal class $\mathfrak{N}_0$ containing $\mathfrak{M}$ by Theorem 1, it is sufficient to prove that the class $\mathfrak{N}_1$ of all sets $(\mathfrak{X})$ satisfying the three conditions of Theorem 2 is normal.

Now let $X_n (n=1, 2, 3, \ldots\ldots)$ be a disjoint sequence of sets $(\mathfrak{N}_1)$ and put $X=\sum_{n=1}^{\infty} X_n$. Then for any $y \in \Psi$

$$X\Phi_y = \sum_{n=1}^{\infty} X_n \Phi_y \in \mathfrak{X}_y,$$

and $\mu_y(X\Phi_y) = \sum_{n=1}^{\infty} \mu_y(X_n\Phi_y)$ is measurable $(\mathfrak{Y})$ on Ψ; and since $\mu_y(X_n\Phi_y) \geqq 0$ for all n, we have

$$\mu(X) = \sum_{n=1}^{\infty} \mu(X_n) = \sum_{n=1}^{\infty} \int_{\Psi} \mu_y(X_n\Phi_y)\,d\nu(y)$$
$$= \int_{\Psi} \sum_{n=1}^{\infty} \mu_y(X_n\Phi_y)\,d\nu(y) = \int_{\Psi} \mu_y(X\Phi_y)\,d\nu(y).$$

Thus we find that $X \in \mathfrak{N}_1$.

Next let $X_n (n=1, 2, 3, \ldots\ldots)$ be a descending sequence of sets $(\mathfrak{N}_1)$ and put $X=\prod_{n=1}^{\infty} X_n$. Then for any $y \in \Psi$

$$X\Phi_y = \prod_{n=1}^{\infty} X_n \Phi_y \in \mathfrak{X}_y,$$

and $\mu_y(X\Phi_y) = \lim_{n\to\infty} \mu_y(X_n\Phi_y)$ almost everywhere $(\mathfrak{Y}, \nu)$ on Ψ, since $\mu_y(X_n\Phi_y) \leqq \mu_y(\Phi_y) = \mu_y(\Omega\Phi_y)$ $(n=1, 2, 3, \ldots\ldots)$ and

$$+\infty > \mu(\Omega) = \int_{\Psi} \mu_y(\Omega\Phi_y)\,d\nu(y).$$

Hence $\mu_y(X\Phi_y)$ is, as a function of y, measurable $(\mathfrak{Y})$ on Ψ and we have further, by Lebesgue's theorem,

$$\mu(X) = \lim_{n\to\infty} \mu(X_n) = \lim_{n\to\infty} \int_{\Psi} \mu_y(X_n\Phi_y)\,d\nu(y)$$
$$= \int_{\Psi} \lim_{n\to\infty} \mu_y(X_n\Phi_y)\,d\nu(y) = \int_{\Psi} \mu_y(X\Phi_y)\,d\nu(y).$$

Thus X is found to be a set $(\mathfrak{N}_1)$, completing the proof of our theorem.

Theorem 3. *Let us assume, in the hypothesis of Theorem 2, that the class* $\mathfrak{X}_y$ *is complete with respect to the measure* μ_y *for every* $y \in \Psi$, *and that for every* $A \in \mathfrak{X}$ *there is a set* $B \in \mathfrak{R}$ *such that* $A \subset B$ *and* $\mu(B-A)=0$. *Then for every* $A \in X$

(i) $A\Phi_y \in X$ *almost everywhere* $(\mathfrak{Y}, \nu)$ *on* Ψ;

(ii) $\mu_y(A\Phi_y)$ *as function of* y *is measurable* $(\mathfrak{Y})$ *on* Ψ *if we neglect a set of measure zero* $(\mathfrak{Y}, \nu)$;

(iii) *we have*

$$\mu(A)=\int_{\Psi}\mu_y(A\Phi_y)\,d\nu(y).$$

Proof. First suppose $\mu(A)=0$. Then $\mu(B)=0$ and

$$\int_{\Psi}\mu_y(\Phi_y)\,d\nu(y)=0$$

by Theorem 2, hence $\mu_y(B\Phi_y)=0$ almost everywhere $(\mathfrak{Y}, \nu)$ on Ψ. But $\mathfrak{X}_y$ is complete with respect to μ_y by hypothesis, and so $A\Phi_y \in \mathfrak{X}_y$ and $\mu_y(A\Phi_y)=0$ almost everywhere $(\mathfrak{Y}, \nu)$ on Ψ. Hence the result.

In the general case our theorem follows from the identity $A=B-(B-A)$ and Theorem 2 on account of $\mu(B-A)=0$.

Theorem 4. *Let us assume, in the hypothesis of Theorem 3, that* $f(x)$ *is a non-negative function measurable* $(\mathfrak{X})$ *on* Φ. *Then*

(i) $f(x)$ *is measurable* $(\mathfrak{X}_y)$ *on* Φ_y *for almost all* $(\mathfrak{Y}, \nu)$ *values of* $y \in \Psi$.

(ii) $\int_{\Phi_y} f(x)\,d\mu_y(x)$ *is, as a function of* y, *measurable* $(\mathfrak{Y})$ *on* Ψ, *if we neglect a set of measure zero* $(\mathfrak{Y}, \nu)$;

(iii) $$\int_{\Phi} f(x)\,d\mu(x)=\int_{\Psi}\left[\int_{\Phi_y} f(x)\,d\mu_y(x)\right]d\nu(y).$$

Proof. In case $f(x)$ is a finite step-function, our theorem is an immediate consequence of Theorem 3. In the general case $f(x)$ is the limit of an ascending sequence of finite non-negative step-functions $f_n(x)$ $(n=1, 2, 3, \ldots\ldots)$ measurable $(\mathfrak{X})$ on Φ, and the result follows from repeated applications of Lebesgue's theorem on integration of monotone sequences of functions.

Examples. Now we shall give some examples of primitive classes and Theorem 4.

(I) Let R denote the set of all finite real numbers and let $\mathfrak{M}$ denote the class of all closed intervals $[a, b]$, open intervals (a, b), and half-open intervals $[a, b)$ or $(a, b]$ (where in all cases we suppcse $-\infty < a < b < +\infty$), together with all one-pointic sets and the empty set. Then $\mathfrak{M}$ is clearly a primitive class.

(II) Let S and T be abstract spaces and let $\mathfrak{X}$ and $\mathfrak{Y}$ be primitive classes in S and T respectively. We denote by $\mathfrak{Z} = \mathfrak{X}\mathfrak{Y}$ the class of all sets of the form $X \times Y (X \in \mathfrak{X},\ Y \in \mathfrak{Y})$ in the product space $U = S \times T$. Then the class $\mathfrak{Z}$ is primitive in U. For clearly $U \in \mathfrak{Z}$, and if $A_n = X_n \times Y_n \in \mathfrak{Z}$ for $n = 1$ and 2, then

$$A_1 A_2 = (X_1 X_2) \times (Y_1 Y_2) \in \mathfrak{X}\mathfrak{Y},$$

since $X_1 X_2 \in \mathfrak{X}$ and $Y_1 Y_2 \in \mathfrak{Y}$; finally if $A = X \times Y \in \mathfrak{Z}$, then there are two disjoint sequences $\{X_n\}$ and $\{Y_n\}$ of sets $(\mathfrak{X})$ and sets $(\mathfrak{Y})$ respectively $(n = 1, 2, 3, \ldots\ldots)$, such that

$$S - X = \sum_{n=1}^{\infty} X_n, \quad T = Y = \sum_{n=1}^{\infty} Y_n.$$

Hence we have

$$U - A = X \times (T - Y) + (S - X) \times T = \sum_{n=1}^{\infty} X Y_n + \sum_{n=1}^{\infty} X_n T,$$

and this is a decomposition of U—A into a disjoint sequence of sets $(\mathfrak{Z})$.

(III) Let $\Phi = [0, 1] \times [0, 1]$ be the unit square in the plane, $\mathfrak{X}$ the class of all measurable sets in Φ, μ the Lebegue measure for the sets $(\mathfrak{X})$. Similarly let $\Psi = [0, 1]$ be the unit linear interval, $\mathfrak{Y}$ and ν the Lebesgue class and measure in Ψ. Let further φ be defined by $\varphi(x, y) = y$ for every point $(x, y) \in \Phi$. For every $y \in \Psi$ we put $\Phi_y = \Psi$, $\mathfrak{X}_y = \mathfrak{Y}$, $\mu_y = \nu$. We denote by $\mathfrak{M}$ the primitive class $\mathfrak{M}_0 \mathfrak{M}_0$ (see Example II), where $\mathfrak{M}_0$ is the class of all intervals, closed, open and half-open, together with all one-pointic sets and the empty set, in the unit linear interval. Then the smallest additive class $\mathfrak{N}$ in Φ containing $\mathfrak{M}$ is clearly the class of all Borel sets in Φ.

Now let $f(x)$ be a non-negative function measurable $(\mathfrak{X})$ on Φ. Then all the requirements of Theorem 4 are easily seen to be satisfied, and we derive the following form of Fubini's theorem:

A non-negative measurable function $f(x, y)$ on the unit square is, as a fnnction of x, measurable on $[0, 1]$ for every $y \in [0, 1]$ except at most a *linear set of measure zero, and we hcve*

$$\int_0^1\int_0^1 f(x, y)dxdy=\int_0^1\left[\int_0^1 f(x, y)dx\right]dy.$$

(IV) We can also deduce from our theory the following known theorem: *a function $f(x, y)$ measurable* ($\mathfrak{B}$) *on the unit square is measurable* ($\mathfrak{B}$) *in x for every $y \in [0, 1]$.* This is an immediate consequence of a theorem analogous to Theorem 4, to the following effect:

Let us assume, in the hypothesis of Theorem 2, that $f(x)$ is a non-negative function measurable ($\mathfrak{M}$) *on Φ. Then*

(i) *$f(x)$ is measurable* ($\mathfrak{X}_y$) *on Φ_y for every $y \in \Psi$;*

(ii) $\int_{\Phi_y} f(x)d\mu_y(x)$ *is, as a function of y, measurable* ($\mathfrak{Y}$) *on Ψ;*

(iii) *we have* $\int_{\Phi} f(x)d\mu(x)=\int_{\Psi}\left[\int_{\Phi_y} f(x)d\mu_y(x)\right]d\nu(y).$

Theorem 5. *Let F be a continuous function on an interval $I_0=[a, b]$ and let $s(y)$ denote for each y the number (finite or infinite) of the points of I_0 at which F assumes the ualue y. Then the function $s(y)$ is measurable* ($\mathfrak{B}$) *and we have*

$$\int_{-\infty}^{+\infty} s(y)dy=\mathrm{W}(F;\ I_0),$$

$\mathrm{W}(F;\ I_0)$ *denoting the total variation of F on I_0.*

Proof. This theorem, due to S. Banach, is proved on p. 280 of Saks [9].

Theorem 6. *Given a finite function F of a real variable, the set of the points at which the function F assumes a strict maximum or minimum is at most countable.*

Proof. This is proved on p. 261 of Saks [9].

Theorem 7. *Let F be a continuous fvnction of bounded variation on an interval $I_0=[a, b]$, and let G be a function integrable in the Lebesgue-Stieltjes sense with respect to F. We construct a new function $\tilde{G}(y)$ for all finite values of y by the following rule: if the function F assumes the value y an infinity of times on the open interval (a, b) or if there is a point x of (a, b) at which F assumes a strict extremum, we put $\tilde{G}(y)=0$. Otherwise the set of the points of (a, b) at which F assumes the value y is a finite set M, and at each point x of M the function F is strictly increasing or* *decreasing. Now we define a function* $\lambda(x)$ *equal to* 1 *or* -1 *according as* F *is strictly increasing or decreasing at* $x \in M$ *respectively, and using this* $\lambda(x)$ *we define* $\tilde{G}(y)$ *by*

$$\tilde{G}(y) = \sum_{x \in M} \lambda(x) G(x).$$

Then $\tilde{G}(y)$ *is summable and we have*

$$\int_a^b G(x) dF(x) = \int_{-\infty}^{+\infty} \tilde{G}(y) dy.$$

Proof. Clearly we may assume $G(x)$ non-negative. Let us denote by $F_1(x)$ and $F_2(x)$ the positive and negative variations of $F(x)$ on $[a, x]$ for $a \leqq x \leqq b$ respectively. We construct two functions $G_1(y)$ and $G_2(y)$ as follows: they are the functions constructed from $G(x)$ in a similar way as $\tilde{G}(y)$ was constructed from $G(x)$, with the modifications that now we take instead of $\lambda(x)$ the positive part $\lambda_1(x)$ of $\lambda(x)$ in the case of $G_1(y)$ and the negative part $\lambda_2(x)$ of $\lambda(x)$ in the case of $G_2(y)$. Then it is sufficient to prove

$$\int_a^b G(x) dF_m(x) = \int_{-\infty}^{+\infty} G_m(y) dy \quad (m=1, 2).$$

Since the proofs are the same for both cases, we carry out the proof for $m=1$ only, and this is done by application of Theorem 4.

Let us put, in the notations of Theorem 4,

$$\Phi = (a, b), \quad \Psi = F(I_0), \quad \varphi = F,$$

$\mathfrak{X}$ and μ the Lebesgue-Stieltjes additive class and measure with respect to F_1, and $\mathfrak{Y}$ and ν the Lebesgue additive class and measure. Further let $\mathfrak{X}_y$ be the class of all subsets of Φ_y for each $y \in \Psi$, and let us define the measure μ_y by

$$\mu_y(X) = \sum_{x \in X} \lambda_1(x) \quad (X \subset \Phi_y),$$

if Φ_y is finite and if $F(x)$ is strictly increasing or decreasing at every point of Φ_y; otherwise we put $\mu_y(X) = 0$ identically.

Now let $\mathfrak{M}$ be the class of all closed, open, or half-open intervals in Φ, together with the empty set and one-pointic sets in Φ. Then clearly $\mathfrak{M}$ is primitive, and the class $\mathfrak{N}$ coincides with the class of all

Borel sets in Φ.

This being so, we shall show that the three conditions of Theorem 2 are also satisfied. Let us take, as a representative case, that of a closed interval $A=[u, v]$ in Φ. Denoting by $W(x)$ the total variation of F on $[a, x]$, we have, almost everywhere on Ψ, on account of Theorems 5 and 6,

$$\mu(A)=F_1(v)-F_1(u)=\frac{1}{2}\,[W(v)-W(u)+F(v)-F(u)]$$

$$=\int_{-\infty}^{\infty}\frac{1}{2}\Big[\sum_{\substack{F(x)=y\\ x\in A}}|\lambda(x)|+\sum_{\substack{F(x)=y\\ x\in A}}\lambda(x)\Big]\,dy$$

$$=\int_{-\infty}^{\infty}\sum_{\substack{F(x)=y\\ x\in A}}\lambda_1(x)dy=\int_{\Psi}\mu_y(A\Phi_y)d\nu(y).$$

Further the class $\mathfrak{X}_y$ is complete with respect to the measure μ_y for every y, and it is well known that for every set A measurable in the Lebesgue-Stieltjes sense in (a, b) there is a Borel set $B\supset A$ with $\mu(B-A)=0$. Thus we can apply Theorem 4 and find immediately

$$\int_b^b G(x)dF_1(x)=\int_{-\infty}^{\infty}G_1(y)dy.$$

Theorem 8. *Let C be a rectifiable closed Jordan curve in the (x,y)-plane and let D denote the inner domain of C. Further let $M(x, y)$ be a continuous function in D with the following properties:*

(i) *$M(x,y)$ is absolutely continuous in x on the intersection of D and the line $y=y_0$ for almost all values of y_0.*

(ii) *$\dfrac{\partial M}{\partial x}$ is summable over D.*

(iii) *$M(x. y)$ is integrable in the Lebesgue-Stieltjes sense with respect to $y=y(s)$ (s is the arc length measured in the positive sense along C) around C, if we define $M(x, y)$ on C as follows: if the intersection $D(y_0)$ of D and the line $y=y_0$ consists of a finite number of linear open intervals, evry two of which have a positive distance, and if the point (x_0, y_0) is an end-point of one of these intervals, then we define $M(x_0, y_0)$ to be the limit of $M(x, y)$ as (x, y) approaches (x_0, y_0) through $D(y_0)$, if such a limit exists and is finite; in all other cases (that is to say, if we cannot define the value of $M(x_0, y_0)$ in this way) we put simply $M(x_0, y_0)=0$ for $(x_0, y_0)\in C$.*

Then we have

$$\int_C M(x, y)dy = \iint_D \frac{\partial M}{\partial x} dxdy,$$

the Stieltjes integral being taken in the positive sense around C.

Remark. (I) The measurablity of $M(x, y)$ in the Lebesgue-Stieltjes sense with respect to $y(s)$ on C is an easy consequence of the continuity and proyerty (i) of the function M and so we leave its verification to the reader.

(II) The derivative $\dfrac{\partial M}{\partial x}$ exists almost everywhere in D and is a measurable function. On this cf. Tsuji [5].

Proof. Applying Theorem 7 to $M(x, y)$ we have

$$\int_C M(x, y)dy = \int_{-\infty}^{\infty} \tilde{M}(y)dy.$$

where $\tilde{M}(y)$ is constructed from $M(s) = M(x(s), y(s))$ with respect to $y = y(s)$ in the way indicated in the proof of that theorem. But we find easily

$$\tilde{M}(y) = \int_{D(y)} \frac{\partial M}{\partial x} dx,$$

for almost all values of y, hence

$$\int_{-\infty}^{\infty} \tilde{M}(y)dy = \int_{-\infty}^{\infty} dy \int_{D(y)} \frac{\partial M}{\partial x} dx = \int_D \frac{\partial M}{\partial x} dxdy,$$

the last step being effected by the usual Fubini's theorem.

Corollary. *Let $M(x, y)$ be a bounded continuous function in D such that*

(i) *$M(x, y)$ is absolutely continuous in x on the intersection $D(y_0)$ of D and the line $y = y_0$ for almost all y_0;*

(ii) $\dfrac{\partial M}{\partial x}$ *is summable over D.*

Then Green's formula holds, if we define the value of M on C as in Theorem 8.

Proof. Since $M(x, y)$ is bounded, the condition (iii) of Theorem 8 is automatically satisfied.

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