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Riemann Spaces of Class two and their Algebraic Characterization

Part III

Makoto MATSUMOTO.

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In the two preceding papers (part I and II)[1] we have defined the type number of $n(\geqq 4)$-dimensional Riemann space R_n and, making use of it, have got a necessary and sufficient condition that there be a set of functions H^{I}_{ij} and H^{II}_{ij} satisfying the Gauss equation

$$R_{ijkl}=H^P_{ik}H^P_{jl}-H^P_{il}H^P_{jk} \quad (P=I,\ II;\ i,\ j,\ k,\ l=1,\ldots\ldots,\ n),$$

when $R_n(n\geqq 6)$ is of type $\geqq 3$. And then we have had the theorem 4. 4 of the part II, i. e. a necessary and sufficient condition that $R_n(n\geqq 8)$ of type $\geqq 4$ be of class two, making use of the theorem 1. 5 of the part I.

In this part III, we conider the Codazzi and Ricci equations when $R_n(n\geqq 6)$ is of type $\geqq 3$, and get a necessary and sufficient condition that $R_n(n\geqq 6)$ of type *three* be of class two.

In the writting of those three papers I have received many invaluable advices and criticism by Prof. J. Kanitani in Kyoto University. Those papers also could never have been written had it not been for the works of T. Y. Thomas and C. B. Allendoerfer.

§ 1. Introduction

In § 1 of Part I we put

$$L_{ijkl}=H^{I}_{ij}H^{II}_{kl}-H^{I}_{il}H^{II}_{jk}-H^{II}_{ij}H^{I}_{kl}+H^{II}_{il}H^{I}_{jk}, \tag{1·1}$$

and

$$K^{\ \ P}_{Q\cdot ij}=g^{cd}(H^{Q}_{ci}H^{P}_{dj}-H^{Q}_{cj}H^{P}_{di}). \tag{1·2}$$

If we put $K_{ij}=K^{I}_{II,ij}$, we have

$$K_{ij}=\frac{1}{2}g^{ab}L_{ajbi}. \tag{1·3}$$

We have had the intrinsic expressions of K_{ij} and L_{ijkl} in the part II, and the theorem 2.2 of the part I, i. e. K_{ij} satisfy the equation

$$K_{ij,k}+K_{jk,i}+K_{ki,j}=0, \tag{1·4}$$

if R_n is of type $\geqq 4$. From (1·1) we have

$$L_{a(i|b|j,k)}=0, \tag{1·5}$$

according to the Codazzi equation

$$H^{P}_{ai,j}-H^{P}_{aj,i}=H^{Q}_{ai}H^{Q}_{Pj}-H^{Q}_{aj}H^{Q}_{Pi}. \tag{1·6}$$

Now, if there be a set of functions $H^{P}_{ij}(P=I,\ II)$ satisfying the Gauss equation in $R_n(n\geqq 6)$ of type $\geqq 3$, the equation (1·2) are satisfied (Cf. the end of the part II) and also the equation (1·1), accodring to (1·7) of the part I. Differentiating (1·7) of the part I covariantly with respect to x^m and subtracting from that equation four equations obtained by interchanging m with i, j, k and l, we have from the Bianchi's idantity and (1·4)

$$K_{i(j}L_{|ab|kl)m}+K_{(kl}L_{|ab|m)ij}+K_{j(k}L_{|ab|lm)i}+K_{im}L_{ablkj}=0 \tag{1·7}$$

where $L_{abijk}=L_{a(i|b|j,k)}$. If R_n $(n\geqq 8)$ be of type $\geqq 4$, we have (1·5) from (1·7) with the similar way of the case when we have proved (1·4) (Cf. § 2 of the part I).

§ 2. The Codazzi equation

If $R_n(n\geqq 6)$ be of type $\geqq 3$, L_{aibj} is expressed intrinsically and then from § 1 we have the

Lemma:...... *If $R_n(n\geqq 6)$ be of class two and of type 3, it is necessary that the metric tensor g_{ij} satisfy the equation* (1·5).

It is to be noted that contracting (1·5) by g^{ab} we have the equation (1·4).

Substituting (1·1) in (1·5) we have

$$H^{Q}_{(i|b|}D^{P}_{|a|jk)}+H^{Q}_{(i|a|}D^{P}_{|b|jk)}-H^{P}_{(i|b|}D^{Q}_{|a|jk)}-H^{P}_{(i|a|}D^{Q}_{|b|jk)}=0\,; \tag{2·1}$$

where $D^{P}_{aij}=H^{P}_{ai,j}-H^{P}_{aj,i}$. Differentiating the Gauss equation

$$R_{abij}=H^P_{ai}H^P_{bj}-H^P_{aj}H^P_{bi}$$

covariantly with respect to x^k and making use of the Bianchi's identity we have

$$H^P_{(i|a|}D^P_{|b|jk)}-H^P_{(i|b|}D^P_{|a|jk)}=0 \qquad (2\cdot 2)$$

Now, if R_n is of type 3, transforming the coordinate system, we have $D_3\neq 0$[4], and then

$$H^P_{ia}H^{ja}_Q=\delta^P_Q\delta^j_i,\ H^P_{ia}H^{ia}_Q=3\delta^P_Q,\ H^P_{ia}H^{ib}_P=\delta^b_a\ (i,j=1,2,3\ ;\ a,b=k_1,\ldots\ldots,k_6) \qquad (2\cdot 3)$$

Taking $i, j, k=1, 2, 3$, and $a, b=k_1,\ldots\ldots, k_6$ in (2·2) and contracting by H^{kb}_Q we have

$$D^Q_{aij}+H^P_{ia}(H^{kb}_Q D^P_{bkj})-H^P_{ja}(H^{kb}_Q D^P_{bki})+H^{kb}_Q H^P_{ka}D^P_{bji}=0, \qquad (2\cdot 4)$$

and contracting (2·4) by H^{ia}_R we have

$$H^{ia}_R D^Q_{aij}+H^{kb}_Q D^R_{bkj}=0. \qquad (2\cdot 5)$$

Now we define

$$H^P_{Qi}=-\frac{1}{4}H^{kb}_Q D^P_{bkj}\ (j,\ k=1,2,3\ ;\ b=k_1,\ldots\ldots,k_6), \qquad (2\cdot 6)$$

and those H^P_{Qj} are skew-symmetric in P and Q from (2·5). Substituting (2·6) in (2·4) we have

$$D^Q_{aij}-4(H^P_{ia}H^P_{Qj}-H^P_{ja}H^P_{Qi})+H^{kb}_Q H^P_{ka}D^P_{bji}=0. \qquad (2\cdot 7)$$

Next taking $i, j, k=1, 2, 3$ and $a, b=k_1,\ldots\ldots\ldots, k_6$ in (2·1) and contracting by H^{kb}_P we have, making use of (2·6),

$$2D^Q_{aij}-4(H^P_{ia}H^P_{Qj}-H^{ja}_P H^P_{Qi})+H^Q_{ka}H^{kb}_P D^P_{bji}=0 \qquad (2\cdot 8)$$

Subtracting (2·7) from (2·8) we have

$$D^Q_{aij}+H^Q_{ka}H^{kl}_P D^P_{bji}-H^{kb}_Q H^P_{ka}D^P_{aji}=0$$

and contracting by H^{la}_R and then H^Q_{lb} $(l=1, 2, 3)$ we have

$$3D^R_{bij}=H^{la}_R H^Q_{lb}D^Q_{aji}.$$

Accordingly (2·7) becomes

$$D^Q_{aij}=H^P_{ia}H^P_{Qj}-H^P_{ja}H^P_{Qi}. \tag{2·9}$$

Next taking $i, j=1, 2, 3$; $k>3$ and $a, b=k_1, \ldots\ldots k_6$ in (2·2) and substituting (2·9) in (2·2) we have

$$H^P_{ia}\Delta^P_{bkj}-H^P_{ja}\Delta^P_{bki}-H^P_{ib}\Delta^P_{akj}+H^P_{jb}\Delta^P_{aki}=0\ ; \tag{2·10}$$

where

$$\Delta^P_{bkj}=D^P_{bkj}-H^Q_{kb}H^Q_{Pj}. \tag{2·11}$$

Contracting (2·10) by H^{ia}_Q we have

$$2\Delta^Q_{bkj}-H^{ia}_QH^P_{ib}\Delta^P_{akj}+H^{ia}_QH^P_{jb}\Delta^P_{aki}=0, \tag{2·12}$$

and contracting (2·12) by H^{jb}_S we have

$$H^{ib}_S\Delta^Q_{bkj}+H^{ia}_Q\Delta^S_{aki}=0. \tag{2·13}$$

Now we define

$$H^P_{Qk}=\frac{1}{3}H^{ia}_Q\Delta^P_{aki}\ \ (k>3\ :\ j=1,\ 2,\ 3\ ;\ a=k_1, \ldots\ldots k_6), \tag{2·14}$$

and those H^P_{Qk} are skew-symmetric in P and Q from (2·13). Substituting (2·14) in (2·12) we have

$$2\Delta^Q_{bkj}-H^{ia}_QH^P_{ib}\Delta^P_{akj}+3H^P_{jb}H^P_{Qk}=0. \tag{2·15}$$

Contracting (2·15) by H^{lb}_R and then H^Q_{bc} ($l=1, 2, 3$; $c=k_1, \ldots\ldots k_6$) we have

$$-\Delta^Q_{bkj}+2H^{la}_QH^R_{lb}\Delta^R_{akj}-3H^R_{jb}H^R_{Qk}=0 \tag{2·16}$$

Summing (2·16) to (2·15) we have $\Delta^Q_{bkj}+H^{ia}_QH^P_{ib}\Delta^P_{akj}=0$ and substituting this equation in (2·15) we have $\Delta^Q_{bkj}+H^P_{jb}H^P_{Qk}=0$, from which we deduce $D^P_{bkj}=H^Q_{Pb}H^Q_{Pj}-H^P_{jb}H^Q_{Pk}$, ($j=1, 2, 3$; $k>2$; $b=k_1, \ldots\ldots, k_6$) from (2·11).

Similarly we have the Codazzi equation (1·6) ($a, i, j=1, \ldots\ldots, n$). Consequently from the theorem 1.4 of the part I we have the

Theorem 2. 1: ...... *If a real Riemann space $R_n(n\geqq 6)$ is of type three, then there will be two sets of real functions $H^P_{ij}(=H^P_{ji})$ and $H^P_{Qi}(=-H^Q_{Pi})$ $(P, Q=I, II;\ i, j=1, \ldots\ldots, n)$ satisfying the Gauss, Codazzi and Ricci equations, i. e. R_n will be of class two if, and only if, the conditions of theorem 4. 2 of the part II and the equation (1·5) are satisfied.*

If the Gauss equation be satisfied and there be two sets of functions H^P_{Qi} and $\bar{H}^P_{Qi}$ satisfying the Codazzi equation (1·6) in R_n, contracting (1·6) by H^P_{bk} and summing those equations obtained by cyclic permutation of i, j and k; and contracting by g^{ab} we have

$$g^{ab}H^P_{b(i}D^P_{|a|jk)}=H^Q_{P(i}K^Q_{|P|jk)}. \tag{2·17}$$

As we have similar equation for $\bar{H}^Q_{Pi}$, we have $H^Q_{P(i}K^Q_{|P|jk)}=\bar{H}^Q_{P(i}K^Q_{|P|jk)}$ and, if we put $H^I_{IIi}=H_i$, we have

$$\varDelta_{(i}K_{jk)}=0\,; \tag{2·18}$$

where $\varDelta_i=H_i-\bar{H}_i$. By the similar way as in the case when we proved (1·4) we have from (2·18) $\varDelta_i=0\,(i=1,\ldots\ldots, n)$, if type $\geqq 2$.

Consequently we have the

Theorem 2. 2: ...... *If $R_n(n\geqq 4)$ is of class two and type $\geqq 2$, a set of functions H^P_{Qi} satisfying the Codazzi equation is unique.*

§ 3. The Ricci equation

We shall prove the theorem 1. 4 of the part I, not making use of Allendoerfer's method.[6]

Differentiating (1·6) covariantly with respect to x^k and summing three equations obtained by cyclic permutation of i, j and k we have $H^P_{b(i}R^b_{|a|\cdot jk)}=H^Q_{a(i}H^Q_{|P|jk)}$; where $H^Q_{Pjk}=H^Q_{Pj,k}-H^Q_{Pk,j}$. Making use of (1·6) of the part I we have

$$H^Q_{a(i}K^P_{|Q|jk)}=H^Q_{a(i}H^Q_{|P|jk)}. \tag{3·1}$$

Contracting (3·1) by H^P_{bl} and subtracting from the equation three equations obtained by interchanging l with i, j and k, and contracting by g^{ab} we have

$$K_{i(j}\varDelta_{kl)}+\varDelta_{i(j}K_{kl)}=0\,; \tag{3·2}$$

where $\varDelta_{kl}=K_{kl}-H^I_{IIkl}$. We have immediately $\varDelta_{ij}=0$ $(i, j=1,\ldots\ldots, n)$ from (3·2) and consequently the theorem 1. 4 of the part I is proved.

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References

1) M. Matsumoto, Riemann spaces of class two and......... J. Math. Soc. Japan,

2) T. Y. Thomas, Riemann spaces of class one and.......... Acta Math. 67 (1936)

3) C. B. Allendoerfer, Rigidity for spaces of class greater than one. Amer. J. Math. 61 (1939)

4) 1. c. § 2

5) 1. c. Lemma IV

6) 1. c. Lemma V

Mathematical Institute,
Doshisha University.