Journal of the Mathematical Society of Japan Vol. 2, Nos. 1–2, Sept., 1950.

Change of variables in the multiple Lebesgue integrals.

Masatsugu TSUJI.

(Received April 4, 1948)

Rademacher's theorem[1] on the change of variables in the multiple Lebesgue integrals, though very important, is not found in any book on the theory of functions of real variables, so that I will give a simple proof of it in the following lines.

Let D be a domain in the $(x_1,\ldots\ldots,x_n)$-space and $\varDelta$ be one in the $(u_1,\ldots\ldots,u_n)$-space and D be mapped on $\varDelta$ topologically by

$$T: \qquad u_i=f_i(x_1,\ldots\ldots,x_n), \tag{1}$$

$$T^{-1}: \qquad x_i=\varphi_i(u_1,\ldots\ldots, u_n), \quad (i=1,2,\ldots\ldots,n),$$

where f_i and φ_i are continuous in D and $\varDelta$ respectively.

If any measurable set in D is mapped on a measurable set in $\varDelta$, then T is called a measurable mapping. It is well known[2] that the necessary and sufficient condition that T is a measurable mapping is that any null set[3] in D is mapped on a null set in $\varDelta$.

Theorem 1.[4] *If at every point* $(x_1,\ldots\ldots,x_n) \in D$,

$$\overline{\lim_{h_1^2+\ldots+h_n^2\to 0}} \frac{|f_i(x_1+h_1,\ldots,x_n+h_n)-f_i(x_1,\ldots,x_n)|}{\sqrt{h_1^2+\ldots+h_n^2}}$$

$$=L_i(x_1,\ldots,x_n)<\infty \qquad (i=1,2,\ldots\ldots,n), \tag{2}$$

then T is a measurable mapping.

Proof. We define a set A_k $(k=1,2,\ldots)$ of points $(x_1,\ldots,x_n)$ by the condition

---

(1) Rademacher: Über die partielle und totale Differentierbarkeit von Funktionen mehererer Veränderlichen und über die Transformation der Doppelintegrale. Math. Ann. **79**.

(2) Rademacher: Eineindeutige Abbildung und Messbarkeit. Monathefte f. Math. u. Phys. **27** (1916). Carathéodory: Vorlesungen über reelle Funktionen. p. 354.

(3) A set is called a null set, if its Lebesgue measure is zero.

(4) Rademacher. 1. c. (1), (2).

that for any $h_1,\dots,h_n$, such that $h_1^2+\dots+h_n^2<\frac{1}{k^2}$,

$$|f_i(x_1+h_1,\dots,x_n+h_n)-f(x_1,\dots,x_n)|\leqq k\sqrt{h_1^2+\dots+h_n^2}, \qquad (3)$$
$$(i=1,\ 2,\dots\dots,n).$$

From the continuity of $f_i(x_1,\dots,x_n)$, it follows that A_k are closed sets and from (2)

$$D=\sum_{k=1}^{\infty}A_k. \qquad (4)$$

Let $(x_1^0,\dots,x_n^0)\in A_k$ and

$$u_i=f_i(x_1^0+h_1,\dots,x_n^0+h_n),\quad u_n^0=f_i(x_1^0,\dots\dots,x_n^0),\quad (h_1^2+\ +h_n^2<\frac{1}{k^2}), \qquad (5)$$

then by (3),

$$\sum_{i=1}^{n}(u_i-u_i^0)^2\leqq nk^2(h_1^2+\dots+h_n^2). \qquad (6)$$

Let $K(r)$ be the inside of a sphere of radius r:

$$K(r)\ :\ \sum_{i=1}^{n}(x_i-x_k^0)^2\leqq r^2 \qquad (7)$$

and $K'(r)$ be its image in $\varDelta$ by (1), then by (6),

$$\sum_{i=1}^{n}(u_i-u_i^0)^2\leqq(\sqrt{n}\ k\ r)^2,$$

so that $K'(r)$ is contained in a sphere of radius $\sqrt{n}\ k\ r$.

Since

$$mK(r)=Cr^n,\ \left(C=\frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}\right), \qquad (8)$$

we have

$$mK'(r)\leqq M\ mK(r),\quad \text{if}\quad r\leqq\frac{1}{k},\ (M=\sqrt{n}\,k)^n. \qquad (9)$$

To prove that T is a measurable mapping, it suffices to prove that any null set e in D is mapped on a null set e' in $\varDelta$.
Let $e\subset D$, $me=0$ and put $e_k=eA_k$, then $me_k=0$ and

$$e=\sum_{k=1}^{\infty} e_k, \qquad e'=\sum_{k=1}^{\infty} e_k', \tag{10}$$

where e', e_k' are images of e, e_k in $\varDelta$ respectively. We will prove that $me_k'=0$. Since $me_k=0$, there exists for any $\varepsilon > 0$ an open set O, such that $e_k \subset O$, $mO < \varepsilon$. We express O as a sum of enumerably infinite number of non-overlapping cubes $\{\varDelta_i\}$,

$$O=\sum_{i=1}^{\infty} \varDelta_i, \qquad mO=\sum_{i=1}^{\infty} m\varDelta_i < \varepsilon. \tag{11}$$

Let K_i be the inside of a sphere which is concentric with $\varDelta_i$ and passes through the vertices of $\varDelta_i$, then

$$mK_i \leqq M' \ m\varDelta_i, \ (M'=\text{const.}). \tag{12}$$

By taking $\varDelta_i$ sufficiently small, we may assume that all K_i are contained in O and their radii are less than $\dfrac{1}{k}$. Let K_i' be the image of K_i in $\varDelta$. If K_i has common points with e_k, then, since its radius is less than $\dfrac{1}{k}$, we have from (9), $mK'_i \leqq M \ mK_i$. Hence if we denote the sum of K_i, which have comnon points with e_k by $\sum_i' K_i$, then

$$e_k \subset \sum_i' K_i, \qquad e_k' \subset \sum_i' K_i',$$

hence from (9), (12), (11),

$$m^* e_k' \leqq \sum_i{}' mK_i' \leqq M \sum_i mK_i \leqq MM' \ \sum_i{}' m\varDelta_i < MM'\varepsilon,$$

where m^*e_k' is the outer measure of e_k'. Since ε is arbitrary, we have $me'_k=0$ and hence $me'=0$ from (10).
Hence T is a measurable mapping.

2.

$f(x_1,\ldots\ldots,x_n)$ is said totally differentiable in Stolz's sense at $(x_1^0,\ldots\ldots,x_n^0)$, if $f(x_1,\ldots\ldots,x_n)$ is expressed in the neighbourhood of $(x_1^0,\ldots\ldots,x_n^0)$ in the form:

$$f(x_1^0+h_1,\ldots\ldots, x_n^0+h_n)=f(x^0,\ldots\ldots,x^0_n)+\sum_{k=1}^{n}\frac{\partial f(x_1^0,\ldots,x_n^0)}{\partial x_k^0} \ h_k \\ +D(h_1,\ldots\ldots,h_n), \tag{13}$$

where

$$\lim_{h_1^2+\cdots+h^2_n \to o} \frac{|D(h_1,\ldots,h_n)|}{\sqrt{h_1^2+\ \ +h^2_n}}=0.$$

If in (1), $f_i(x_1,\ldots,x_n)$ $(i=1,2,\ldots,n)$ satisfy the condition (2) at every point of D, then $f_i(x_1,\ldots,x_n)$ are totally differentiable almost-everywhere in D[5]. Hence $\frac{\partial f_i}{\partial x^k}$ and so

$$J(x_1,\ldots,x_n)=\begin{vmatrix} \frac{\partial f_1}{\partial x_1},\ldots,\frac{\partial f_1}{\partial x_n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \frac{\partial f_n}{\partial x_1},\ldots,\frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

exists almost everywhere in D and since f_i are continuous, $\frac{\partial f_i}{\partial x_k}$ and hence $J(x_1,\ldots,x_n)$ is measurable in D.

Theorem 2. *Let $f_i(x_1,\ldots,x_n)$ $(i=1,2,\ldots,n)$ satisfy the condition* (2) *at every point of D and $F(u_1,\ldots,u_n)$ be integrable in Δ, then $F(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n))\,|J(x_1,\ldots x_n)|$ is measurable in D and*

$$\int_\Delta\!\!\cdots\!\int F(u_1,\ldots,u_n)\,du_1\ldots du_n=\int_D\!\!\cdots\!\int F(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n))\,|J(x_1,\ldots,x_n)|\,dx_1\ldots dx_n.$$

Proof. Since $f_i(x_1,\ldots,x_n)$ are totally differentiable almost everywhere in D, we may assume that $(o,\ldots,o)\in D$, $f_i(o,\ldots\ldots,o)=o$ and $f_i(x_1,\ldots,x_n)$ are totally differentiable at $(o,\ldots,o)$, so that in the neighbourhood of $(o,\ldots o)$, (1) can be expressed in the form:

$$\left.\begin{array}{l} u_1=f_1(x_1,\ldots,x_n)=a_{11}x_1+\ldots+a_{1n}x_n+D_1(x_1,\ldots,x_n), \\ \cdots\cdots\cdots\cdots \\ u_n=f_n(x_1,\ldots x_n)=a_{n1}x_1+\ldots+a_{nn}x_n+D_n(x_1,\ldots,x_n) \end{array}\right\}, \tag{14}$$

where $$a_{ik}=\left(\frac{\partial f_i(x_1,\ldots,x_n)}{\partial x_k}\right)_{x_1=\ldots=x_n=0}$$

$$\lim_{x_1^2+\cdots+x^2_n\to 0}\frac{|D_i(x_1,\ldots\ldots,x_n)|}{\sqrt{x_1^2+\ldots+x^2_n}}=o,\quad (i=1,2,\ldots,n), \tag{15}$$

(5) Rademacher: l.c. (1). Saks: Theory of the integral. p. 311.
(6) Rademcher: l.c. (1).

so that

$$J(o,\dots,o)=\begin{vmatrix} a_{11},\dots\dots,a_{1n} \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ a_{n1},\dots\dots,a_{nn} \end{vmatrix}. \tag{16}$$

Let $K(r)$ be the inside of a sphere of radius r:

$$K(r):\quad x_1^2+\dots+x_n^2 \leqq r^2 \tag{17}$$

and $K'(r)$ be its image in $\varDelta$ by (14). Then we will prove that

$$\lim_{r\to 0}\frac{mK'(r)}{mK(r)}=|J(o,\dots,o)|. \tag{18}$$

To prove this, we associate to (14) an affine transformation:

$$\left.\begin{aligned} v_1&=a_{11}\,x_1+\dots+a_{1n}x_n, \\ &\dots\dots\dots\dots\dots \\ v_n&=a_{n1}x_1+\dots+a_{nn}x_n. \end{aligned}\right\}. \tag{19}$$

Then by (15),

$$\sum_{i=1}^{n}(u_i-v_i)^2 \leqq \varepsilon^2(x_1^2+\dots+x_n^2), \tag{20}$$

where $\varepsilon\to o$ with $x_1^2+\dots+x_n^2\to o$.

We have two cases to consider according as $J(o,\dots.o)=o$ or $J(o,\dots,o)\neq o$.

(i) $J(o,\dots,o)=o$.

In this case, $(x_1,\dots,x_n)$-space is mapped on a linear sub-space of at most $(n-1)$-dimensions in $(v_1,\dots,v_n)$-space by (19), hence from (20), we have easily $mK'(r)\leqq \varepsilon r^n$, where $\varepsilon\to o$ with $r\to o$, hence from (8), we have $mK'(r)\leqq \delta mK(r)$, where $\delta\to o$ with $r\to o$, so that

$$\lim_{r\to 0}\frac{mK(r)}{mK(r)}=o=|J(o,\dots,o)|. \tag{21}$$

(ii) $J(o,\dots,o)\neq o$.

In this case, we can solve (19) with respect to x_i, such that

$$\left.\begin{aligned} x_1&=b_{11}v_1+\dots.+b_{1n}v_n, \\ &\dots\dots\dots\dots\dots\dots\dots \\ x_n&=b_{n1}v_1+\dots+b_{nn}v_n \end{aligned}\right\}, \tag{22}$$

where

$$\begin{vmatrix} b_{11},\dots, b_{1n} \\ \dots\dots\dots \\ b_{n1}\dots, b_{nn} \end{vmatrix} = \begin{vmatrix} a_{11},\dots,a_{1n} \\ \\ a_{n1},\dots,a_{nn} \end{vmatrix}^{-1} = \frac{1}{J(o,\dots\dots,o)}. \tag{23}$$

Let $K(r)$ be mapped on $\Delta(r)$ by (19), then by (22),

$$\Delta(r):\ (b_{11}v_1+\dots+b_{1n}v_n)^2+\dots+(b_{n1}v_1+\ +\dots+b_{nn}\ v_n)^2=\sum_{i,k}^{1,\dots n} B_{ik}v_i\ v_k \leqq r^2,$$

where

$$B=\begin{vmatrix} B_{11},\dots\dots,B_{1n} \\ \dots\dots\dots\dots \\ B_{n1},\dots\dots,B_{nn} \end{vmatrix} = \begin{vmatrix} b_{11},\dots\dots,b_{1n} \\ \dots\dots\dots \\ b_{n1},\dots\dots,b_{nn} \end{vmatrix}^2 = \frac{1}{J^2(o\dots\dots,o)}. \tag{25}$$

By a suitable orthogonal transformation, $\Delta(r)$ can be brought into the form:

$$\Delta(r):\ \frac{\xi_1^2}{a_1^2}+\dots+\frac{\xi_n^2}{a_n^2} \leqq r^2, \quad (a_i>0), \tag{26}$$

where $\lambda_1=\frac{1}{a_1^2},\dots,\lambda_n=\frac{1}{a_n^2}$ are the roots of the characteristic equation:

$$B(\lambda)=\begin{vmatrix} B_{11}-\lambda,\dots\dots,B_{1n} \\ \dots\dots\dots\dots\dots\dots \\ B_{n1},\ \dots\dots B_{nn}-\lambda \end{vmatrix}=0,$$

so that by (25),

$$\frac{1}{(a_1\dots a_n)^2}=\lambda_1\dots\lambda_n=B(o)=\frac{1}{J^2(o\dots,o)}, \text{ or}$$

$$a_1\dots a_n=|J(o,\dots,o)|. \tag{27}$$

Since by a transformation: $\xi_i=a_iX_i$, (26) can be transformed into the form: $X_1^2+\dots+X_n^2 \leqq r^2$, we have from (8),

$$m\Delta(r)=a_1\dots a_n\ mK(r)=C.\ a_1\dots a_n\ r^n. \tag{28}$$

If $(x_1,\dots,x_n)$ lies on a sphere: $x^2+\dots+x_n^2=r^2$, then $(v_1,\dots,v_n)$ lies on an ellipsoid: $\frac{\xi_1^2}{a_1^2}+\dots+\frac{\xi_n^2}{a_n^2}=r^2$, so that by (20), $(u_1,\dots,u_n)$ lies between two ellipsoids:

$$\frac{\xi_1^2}{a_1^2}+\dots+\frac{\xi_n^2}{a_n^2}=r^2(1-\delta)^2, \quad \frac{\xi_1^2}{a_1^2}+\dots+\frac{\xi_n^2}{a_n^2}=r^2(1+\delta)^2, \tag{29}$$

where $\delta\to o$ with $r\to o$. Hence we have from (27), (28),

$C\ a_1\dots a_n\ r^n(1-\delta)^n \leqq mK'(r) \leqq Ca_1\dots a_n\ r^n\ (1+\delta)^n$, or

$|J(o,\dots,o)|\,(1-\delta)^n\ mK(r) \leqq mK'(r) \leqq |J(o,\dots,o)|\,(1+\delta)^n mK(r)$, so that

$$\lim_{r\to 0}\frac{mK'(r)}{mK(r)}=|J(o,\dots,o)|. \tag{29}$$

From (21), (29), we have (18).

By Theorem 1, the mapping $u_i=f_i(x_1,\dots x_n)$ is a measurable mapping, so that any measurable set $e\subset D$ is mapped on a measurable set $e'\subset \varDelta$ and a null set in D is mapped on a null set in $\varDelta$. Hence if we put

$$me'=\varPhi(e),$$

then $\varPhi(e)$ is an absolutely continuous additive set function defined on measurable sets in D, so that by Lebesgue's theorem,

$$me'=\varPhi(e)=\int\dots\int_e D\varPhi(x_1,\dots,x_n)dx_1,\dots dx_n,$$

where $D\varPhi(x_1,\dots,x_n)$ is the derivative of $\varPhi(e)$ at $(x_1,\dots x_n)$, which exists almost everywhere in D. From (18), we have $D\varPhi(x_1,\dots x_n)=|J(x_1,\dots x_n)|$ almost everywhere in D, so that

$$me'=\int\dots\int_e |J(x_1,\dots,x_n)|dx_1\dots\dots dx_n. \tag{30}$$

Let e_0 be the set of $(x_1\dots,x_n)$, such that $J(x_1,\dots,x_n)=o$, then e_0 is measurable, so that from (30),

$$me'_0=o, \tag{31}$$

where e'_0 is the image of e_0 in $\varDelta$.

Now $\varDelta-e_0'$ is mapped on $D-e_0$ by $x_i=\varphi_i(u_1,\dots,u_n)$. We will prove that this is a measurable mapping. To prove this, it suffices to prove that any null set $e'\subset\varDelta-e_0'$ is mapped on a null set $e\subset D-e_0$.

Let $e'\subset\varDelta-e'_0$ be a null set and e be its image in $D-e_0$. Since $me'=o$, there exists a G_δ-set H' in $\varDelta$, such that

$$e'\subset H',\quad mH'=o. \tag{31}$$

Let H be the image of H' in D, then H is a G_δ-set, so that by (30),

$$o=mH'=\int\dots\int_H|J(x_1,\dots,x_n)|dx_1\dots\dots dx_n.$$

Hence $J(x_1,\dots,x_n)=o$ almost everywhere in H. From this we conclude that $m(H-He_0)=o$. Since $e\in H-He_0$, we have $me=o$, q.e.d.

Hence any measurable set in $\Delta - e_0'$ is mapped on a measurable set in $D - e_0$.

Let $F(u_1,\ldots,u_n)$ be a measurable function in Δ and for any real number α, let E_α' be the set of $(u_1,\ldots,u_n) \in \Delta - e'_0$, such that

$$F(u_1,\ldots,u_n) > \alpha. \tag{32}$$

Then E_α' is measurable, so that its image $E_\alpha \subset D - e_0$ by $x_i = \varphi_i(u_1,\ldots,u_n)$ is measurable. Evidently E_α is the set of $(x_1,\ldots,x_n) \in D - e_0$, such that

$$F(f_1(x_1,\ldots,x_n),\ldots,f_n(x,\ldots,x_n)) > \alpha. \tag{33}$$

Hence $F(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n))$ and so

$$F(f_1(x_1,\ldots,x_n),\ldots f_n(x_1,\ldots,x_n))\,|J(x_1,\ldots,x_n)| \tag{34}$$

is measurable in $D - e_0$. Since $J(x_1,\ldots,x_n) = o$ in e_0, (34) is measurable in D.

First we suppose that $F(u_1,\ldots,u_n)$ is bounded and $A < F(u_1,\ldots u_n) < B$ in Δ. Let

$$A = l_0 < l_1 < \ldots < l_{p+1} = B \quad (l_{i+1} - l_i < \varepsilon), \tag{35}$$

$$E_i' = E(l_i \leqq F(u_1,\ldots,u_n) < l_{i+1}), \text{ where } E = \Delta - e'_0,$$

and E_i be the image of E_i' in $D - e_0$, then E_i', E_i are measurable and

$$\sum_{i=0}^{p} E_i' = \Delta - e_0', \quad \sum_{i=0}^{p} E_i = D - e_0. \tag{36}$$

By (30),

$$mE_i' = \int \ldots \int_{Ei} |J(x_1,\ldots,x_n)|\,dx_1\ldots dx_n\,.$$

We put

$$S = \sum_{i=0}^{p} l_i \; mE_i' = \sum_{i=0}^{p} l_i \int \ldots \int_{Ei} |J(x_1,\ldots,x_n)|\,dx_1\ldots dx_n, \tag{87}$$

$$I = \int_D \ldots \int F(f_1,\ldots,f_n)\,|J(x_1,\ldots,x_n)|\,dx_1\ldots dx_n = \int \ldots \int_{D-e_0} F(f_1,\ldots,f_n)$$

$$|J(x_1,\ldots,x_n)|\,dx_1\ldots dx_n$$

$$= \sum_{i=0}^{p} \int \ldots \int_{Ei} F(f_1,\ldots,f_n)\,|J(x_1,\ldots,x_n)|\,dx_1\ldots dx_n, \tag{38}$$

then

$$|I-S| \leqq \sum_{i=0}^{p}\int_{Ei}\cdots\int |F(f_1,\dots f_n)-l_i|\,|J(x_1,\dots,x_n)|\,dx_1\dots dx_n \leqq$$

$$\sum_{i=0}^{p}\int_{Ei}\cdots\int (l_{i+1}-l_i)\,|J(x_1,\dots,x_n)|\,dx_1\dots dx_n \leqq \varepsilon\int_{D-e_0}\cdots\int |J(x_1,\dots x_n)|\,dx_1,\dots dx_n$$

$$=\varepsilon\int_{D}\cdots\int |J(x_1,\dots x_n)|\,dx_1\dots dx_n=\varepsilon m\Delta. \tag{39}$$

Since for $\varepsilon \to o$.

$$S\to\int_{\Delta-e'o}\cdots\int F(u_1,\dots u_n)\,du_1\dots du_n=\int_{\Delta}\cdots\int F(u_1,\dots u_n)\,du_1,\dots du_n,$$

we have from (39),

$$\int_{\Delta}\cdots\int F(u_1,\dots,u_n)\ du_1,\dots du_n=\int\cdots\int F(f_1,\dots,f_n)\,|J(x_1,\dots,x_n)|\,dx_1,\dots dx_n. \tag{40}$$

If $F(u_1,\dots,u_n) \geqq o$ in Δ, then put

$$F_N(u_1,\dots,u_n)=[F(u_1,\dots,u_n)]_0^N,$$

where $F_N=F$, if $F\leqq N$ and $F=N$, if $F\geqq N$. Then from (40)

$$\int_{\Delta}\cdots\int F_N(u_1,\dots,u_n)\ du_1\dots du_n=\int_{D}\cdots\int F_N(f_1,\dots\dots,f_n)\,|J(x_1,\dots,x_n)|\,dx_1\dots dx_n.$$

If we make $N\to\infty$, then we have (40). In the general case we put $F=F_1-F_2$, where $F_1=\dfrac{|F|+F}{2}\geqq o$, $F_2=\dfrac{|F|-F}{2}\geqq o$ and apply (40) on F_1 and F_2 and we have (40).

Hence the theorem is completely proved.

Mathematical Institute,
Tokyo University.