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Note on the Cluster Sets of Analytic Functions.

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1. Let D be an arbitrary connected domain and C be its boundary. Let E be a closed set of capacity[1] zero, included in C and z_0 be a point in E. Suppose that $W=f(z)$ is a single-valued function meromorphic in D. We associate with z_0 three cluster sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$ and $S_{z_0}^{*(C)}$ as follows: $S_{z_0}^{(D)}$ is the set of all values a such that $\lim_{\nu\to\infty} f(z_\nu)=a$ with a sequence $\{z_\nu\}$ of points tending to z_0 inside D. $S_{z_0}^{*(C)}$ is the intersection $\bigcap_r M_r$, where M_r denotes the closure of the union $\bigcup_{z'} S_{z'}^{(D)}$ for all z' belonging to the common part of $C-E$ and $U(z_0, r)$: $|z-z_0|<r$. In the particular case when E consists of a single point z_0, we denote $S_{z_0}^{*(C)}$ by $S_{z_0}^{(C)}$ for the sake of simplicity. Obviously $S_{z_0}^{(D)}$ and $S_{z_0}^{*(C)}$ are closed sets such that $S_{z_0}^{*(C)}\subset S_{z_0}^{(D)}$, and $S_{z_0}^{(D)}$ is always non-empty while $S_{z_0}^{*(C)}$ becomes empty if and only if there exists a positive number r such that $C-E$ and $U(z_0, r)$ have no point in common.

Concerning the cluster sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$ and $S_{z_0}^{*(C)}$ the following theorems are known:

Theorem I. (Iversen-Beurling-Kunugi)[2] *$B(S_{z_0}^{(D)})\subset S_{z_0}^{(C)}$, where $B(S_{z_0}^{(D)})$ denotes the boundary of $S_{z_0}^{(D)}$, or, what is the same, $\Omega=S_{z_0}^{(D)}-S_{z_0}^{(C)}$ is an open set.*

Theorem II. (Beurling-Kunugi)[3] *Suppose that $\Omega=S_{z_0}^{(D)}-S_{z_0}^{(C)}$ is not empty and denote by Ω_n any connected component of Ω. Then $w=f(z)$ takes every value, with two possible exceptions, belonging to Ω_n infinitely often in any neighbourhood of z_0.*

Theorem. I* (Tsuji)[4] *$B(S_{z_0}^{(D)})\subset S_{z_0}^{*(C)}$, that is, $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is an open set.*

Theorem II*. (Kametani-Tsuji)[5] *Suppose that $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is not empty. Then $w=f(z)$ takes every value, except a possible set of w-values of capacity zero, belonging to Ω infinitely often in any neighbourhood of z_0.*

Evidently Theorem I* is a complete extension of Theorem I. It seems however that there exists a large gap between Theorem II and Theorem II*. The object of the present note is to show that under the assumption that *D is simply connected,* Theorem II* can be written in the form of Theorem II.

Namely, the writer proposes to prove the following

Theorem 1. *Suppose that D is simply connected and $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is not empty. Let Ω_n be any connected component of Ω. Then, $w=f(z)$ takes every value, with two possible exceptions, belonging to Ω_n infinitely often in any neighbourhood of z_0.*

2. *Proof of Theorem* 1. Without loss of generality we may suppose that Ω_n does not contain $w=\infty$. Suppose, contrary to the assertion, that there are three exceptional values w_0, w_1 and w_2 in Ω_n. Then, there exists a positive number r_1 such that

$$f(z)\neq w_0,\ w_1,\ w_2$$

in the common part of D and $U(z_0, r_1)$: $|z-z_0|<r_1$. Inside Ω_n we draw a simple closed regular analytic curve Γ which surrounds w_0, w_1 and passes through w_2, and whose interior consists only of interior points of Ω_n. By hypothesis, we can select a positive number r $(<r_1)$, arbitrarily small, such that, K denoting the circle $|z-z_0|=r$, $K\cap(C-E)\neq 0$ and the closure M_r of the union $\bigcup_{z'} S_{z'}^{(D)}$ for all z' belonging to the common part of $C-E$ and $|z-z_0|\leq r$ lies outside Γ. Now, by an extension of Iversen's theorem[6], either w_0 is an asymptotic value of $w=f(z)$ at z_0 or there exists a sequence of points z'_n in E tending to z_0 such that w_0 is an asymptotic value at each z'_n. Consequently it is possible to find a point z'_0 (distinct from z_0 or not) belonging to $E\cap U(z_0, r)$ such that w_0 is an asymptotic value of $w=f(z)$ at z_0'. Let Λ be the asymptotic path with the asymptotic value w_0 at z_0'. We may assume that the image of Λ by $w=f(z)$ is a curve lying completely in the interior of Γ. Consider the set D_r of points z inside the intersection of D and $U(z_0, r)$ such that $w=f(z)$ lies in the interior of Γ. Then the open set D_r consists of at most an enumerable number of connected components. We shall denote by Δ the component which contains the asymptotic path Λ. It is easily seen that the boundary of Δ consists of a finite number of arcs of the circle K, a finite or an enumerable number of analytic contours inside D and a closed subset E_0 of E. Further it should be noticed that Δ is simply connected. For, any connected component of the intersection $D\cap U(z_0, r)$ is simply connected, as by hypothesis D is simply connected, and the frontier of Δ contains no closed analytic contour, since every analytic contour of Δ is transformed by $w=f(z)$ into a curve lying on the simple closed curve Γ passing through an exceptional value w_2.

Here we apply Evans' theorem[7] on the logarithmic potential, to find that there exists a distribution of positive mass $d\mu(a)$ entirely on E_0 such that

$$(1) \qquad u(z)=\int_{E_0}\log\left|\frac{1}{z-a}\right|d\mu(a), \quad \int_{E_0}d\mu(a)=1$$

is harmonic outside E_0, excluding $z=\infty$, and has boundary value $+\infty$ at any point of E_0. Let $v(z)$ be its conjugate harmonic function and put

$$(2) \qquad \zeta=\chi(z)=e^{u(z)+iv(z)}=\rho(z)e^{iv(z)};$$

for the sake of convenience, we shall call the function $\zeta=\chi(z)$ "Evans' function." Let C_λ be the niveau curve $\rho(z)=\text{const.}=\lambda$ $(0<\lambda<+\infty)$. Then C_λ consists of a finite number of simple closed curves surrounding E_0. Let us use the niveau curve C_λ: $\rho(z)=\lambda$ and v-line $v(z)=\text{const.}=\theta$ in the same manner as the circle $|z|=\lambda$ and the ray $\arg z=\theta$ in the theory of meromorphic functions for $|z|<+\infty$. Further, Evans' function has the important property

$$(3) \qquad \int_{C_\lambda}dv(z)=\int_{C_\lambda}\frac{\partial u}{\partial n}ds=2\pi,$$

where ds is the arc length of C_λ and n is the inner normal of C_λ. Let λ_0 be a fixed positive number such that for $\lambda_0\leqq\lambda$ all the niveau curves C_λ intersect the asymptotic path Λ. For $\lambda_0\leqq\lambda$, let Θ_λ denote the common part of the niveau curve C_λ and the domain Δ; Θ_λ consists only of a finite number of cross-cuts and does not contain any loop-cut, as Δ is simply connected. Denote $\Delta(\lambda)$ the common part of Δ and the domain exterior to C_λ. It is clear that the open set $\Delta(\lambda)$ consists of a finite number of simply connected components. Let $A(\lambda)$ denote the area of the Riemannian image of the open set $\Delta(\lambda)$ by the function $w=f(z)$ and let $L(\lambda)$ denote the total length of the image of the curve Θ_λ. Then,

$$A(\lambda)=\iint_{\Delta(\lambda)}|f'(z)|^2d\sigma \quad (d\sigma\text{: the area element on the z-plane}),$$

$$L(\lambda)=\iint_{\Theta_\lambda}|f'(z)||dz|.$$

Next we prove that

$$(4) \qquad \lim_{\lambda\to\infty}A(\lambda)=+\infty$$

and

(5) $$\lim_{\lambda\to\infty}\frac{L(\lambda)}{S(\lambda)}=0 \quad \text{where } S(\lambda)=\frac{A(\lambda)}{\textit{area of the interior of } \Gamma}.$$

To prove these, we use Evans' function

$$\zeta=\chi(z)=e^{u(z)+iv(z)}, \quad (0\leqq v(z)<2\pi).$$

By putting

$$W(\zeta)\equiv f[z(\zeta)],$$

we have

$$A(\lambda)-A(\lambda_0)=\int_{\lambda_0}^{\lambda}\int_{\tilde{\Theta}_\lambda}|W'(\zeta)|^2\lambda d\lambda\, d\theta, \quad (\zeta=\lambda e^{i\theta}),$$

where $\tilde{\Theta}_\lambda$ denotes the image of Θ_λ on the circle $|\zeta|=\lambda$ transformed by $\zeta=\chi(z)$ $(0\leqq v(z)<2\pi)$, and

$$L(\lambda)=\int_{\tilde{\Theta}_\lambda}|W'(\zeta)|\lambda d\theta.$$

Denote by $\eta>0$ the distance of Γ from the image of Λ. Then a geometrical consideration gives

(6) $$L(\lambda)\geqq 2\eta \quad \text{for} \quad \lambda_0\leqq\lambda<+\infty.$$

Applying Schwarz's inequality

$$[L(\lambda)]^2\leqq\int_{\tilde{\Theta}_\lambda}\lambda d\theta\int_{\tilde{\Theta}_\lambda}|W'(\zeta)|^2\lambda d\theta=\lambda\theta(\lambda)\int_{\tilde{\Theta}_\lambda}|W'(\zeta)|^2\lambda d\theta,$$

we have

(7) $$\frac{[L(\lambda)]^2}{\lambda\theta(\lambda)}\leqq\int_{\tilde{\Theta}_\lambda}|W'(\zeta)|^2\lambda d\theta.$$

Consequently

(8) $$\frac{2\eta^2}{\pi}\int_{\lambda_0}^{\lambda}\frac{d\lambda}{\lambda}\leqq\int_{\lambda_0}^{\lambda}\int_{\tilde{\Theta}_\lambda}|W'(\zeta)|^2\lambda d\lambda d\theta=A(\lambda)-A(\lambda_0),$$

since

(9) $$\theta(\lambda)=\int_{\Theta_\lambda}dv(z)\leqq\int_{C_\lambda}dv(z)=2\pi.$$

(8) gives (4) when λ tends to infinity. Next we obtain from (7)

$$\frac{d\lambda}{\lambda\theta(\lambda)}\leqq\frac{dA(\lambda)}{[L(\lambda)]^2}.$$

Hence, on denoting by M_λ the set of all λ such that

$$L(\lambda)\geqq A(\lambda)^{\frac{1}{2}+\varepsilon}, \quad (\varepsilon>0),$$

we see, by (9), that

$$\frac{1}{2\pi}\int_{M_\lambda} d\log\lambda \leq \int_{M_\lambda}\frac{d\lambda}{\lambda\theta(\lambda)} \leq \int_{M_\lambda}\frac{dA(\lambda)}{[A(\lambda)^{\frac{1}{2}+\varepsilon}]^2} \leq \int^{\infty}\frac{dt}{t^{1+2\varepsilon}} < +\infty,$$

whence $L(\lambda) < A(\lambda)^{\frac{1}{2}+\varepsilon}$ for all λ not belonging to a set M_λ where $\int_{M_\lambda} d\log \lambda < +\infty$. Thus (5) holds good.

If $\lambda_0 \leqq \lambda$, the open set $\Delta(\lambda)$ consists of a cert in numbei of simply connected components which we will denote by

$$\Delta^{(1)}(\lambda),\ \Delta^{(2)}(\lambda), \dots\dots\dots, \Delta^{(m)}(\lambda),$$

where $m=m(\lambda)$, $m\geqq 1$ depends on λ. Denote by $\Phi^{(i)}(\lambda)$ the Riemannian image of $\Delta^{(i)}(\lambda)$ transformed by $w=f(z)$ in a one-one manner, where $i=1,2,\dots\dots\dots,m$. If we denote by Φ_0 the domain obtained by excluding two points w_0 and w_1 from the interior of Γ, then, by hypothesis, $\Phi^{(i)}(\lambda)$ $(i=1,2,\dots\dots\dots, m)$ is a finite covering surface of the basic surface Φ_0. By Ahlfors' principal theorem on covering surfaces[8], we have

$$(10) \qquad S^{(i)} \leqq hL^{(i)} \quad (i=1,2,\dots\dots\dots,m),$$

where $S^{(i)}$ denotes the average number of sheets of $\Phi^{(i)}(\lambda)$, i. e., $S^{(i)}$ denotes the ratio between the area of $\Phi^{(i)}(\lambda)$ and the area of Φ_0 and $L^{(i)}$ the length of the boundary of $\Phi^{(i)}(\lambda)$ relative to Φ_0, h being a constant dependent only upon Φ_0. From (10)

$$\sum_{i=1}^{m} S^{(i)} \leqq h\sum_{i=1}^{m} L^{(i)},$$

that is

$$(11) \qquad S(\lambda) \leqq h(L(\lambda)+L_0),$$

where L_0 denotes the total length of the image of arcs of K included in the boundary of Δ. Accordingly

$$(12) \qquad \varliminf_{\lambda\to\infty} \frac{L(\lambda)}{S(\lambda)} \geqq \frac{1}{h} > 0.$$

It is clear that (12) contradicts (5), which proves our theorem.

Remark. In our proof of Theorem 1, the assumption that Δ is simply connected plays an important rôle.

3. Consider a particular case that $w=f(z)$ is regular in the common part of the simply connected domain D and a certain neighbourhood $U(z_0)$ of z_0; that is, $f(z) \neq \infty$ in $D \cap U(z_0)$. Under an additional condition we want to show that $w=f(z)$ takes every finite value, save one possible exceptional value, belonging to Ω_n in any neighbourhood of z_0. Suppose, namely, that there are two finite exceptional values w_0 and w_1 within Ω_n, and let Γ be any closed simple regular analytic curve, in Ω_n, which surrounds w_0 and w_1 and whose interior consists only of interior points of Ω_n. Let Δ be the domain defined in the same way as in the proof of Theorem 1. Then, we easily see that Δ is also simply connected. If Δ were not simply connected, the boundary of Δ would contain at least one closed analytic contour q such that q be a loop-cut of D. Accordingly, $w=f(z)$ would take inside q a value lying outside the simple closed curve Γ, while $w=f(z)$ be regular both inside q and on q and the image of q by $w=f(z)$ would lie on Γ. Repeating the same argument as in the proof of Theorem 1, we would arrive at a contradiction. Thus we have

Theorem 2. *Suppose that D is simply connected, $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is not empty, and further $f(z)$ is regular in the common part of D and a certain neighbourhood $U(z_0)$ of z_0. Let Ω_n be any connected component of Ω. Then, $w=f(z)$ takes every finite value, with one possible exception, belonging to Ω_n infinitely often in any neighbourhood of z_0.*

As an immediate consequence, we see that under the same condition as in Theorem 2, for any connected component Ω_n which does not contain $w=\infty$, $w=f(z)$ takes every value, with one possible exception, belonging to Ω_n infinitely often near z_0. Thus we obtain the following

Theorem 3. *Suppose that D is simply connected, $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is not empty, and further that $f(z)$ is regular and bounded in the common part of D and a certain neighbourhood $U(z_0)$ (or that $S_{z_0}^{(D)}$ does not coincide with the whole w-plane). Let Ω_n be any connected component of Ω. Then $w=f(z)$ takes every value, with one possible exception, belonging to Ω_n infinitely often in any neighbourhood of z_0.*

As another immediate consequence of Theorem 2, we get, by using a linear transformation,

Theorem 4. *Under the same condition as in Theorem 1, if there are two exceptional values w_0, w_1 ($w_0 \neq w_1$) belonging to the same component Ω_n, $w=f(z)$ takes every w-value other than w_0 and w_1 infinitely often in any neighbourhood of z_0 and so $S_{z_0}^{(D)}$ coincides with the whole w-plane.*

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