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Homotopy Properties of Fibre Bundles

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Introduction

Recently, Steenrod [6]* has laid the foundation for the construction of tensor functions by introducing the notion of the so-called characteristic cocycle, which plays an important rôle in the theory of fibre bundles. The characteristic cocycle was thereby defined for a special type of fibre bundle, so that the characteristic class obtained is a topological invariant.

The object of the present note is to show that the generalization of the ideas involved is likewise of some help in studying homotopy properties of fibre bundles. We shall restrict ourselves largely to the case that the base space has a simple topological nature. It turns out that the characteristic class is a topological invariant. The method is then applied to homogeneous spaces. In particular, we take the spheres as such and are led to some results concerning group manifolds.

§ 1. Properties of the ψ-cocycle

1. Let R be a fibre buudle, with the base space B and with fibres which are simple in every dimension. B is supposed to be a polyhedron, and we take a simplicial decomposition K of B which is so fine that the star of each simplex lies in a coordinate neighborhood of B. We then denote by π the projection of R onto B. Let K^r be the subcomplex of K, consisting all simplexes of dimension not greater than r.

Suppose it is defined a continuous map ψ of K^r in R such that each point is mapped into a point belonging to the fibre over it. Such a map we refer to as *slicing map*. Let T^{r+1} be a simplex of dimension $r+1$ of K and N a coordinate neighborhood containing T^{r+1}. We resolve $\pi^{-1}(N)$ into the topological product of N and a fixed fibre F and denote by λ the projection of $\pi^{-1}(N)$ onto F. Since the map $\lambda\psi$ is defined over the boundary $\dot{T}^{r+1}$, we get a map of a sphere of dimension r into a fibre and hence an element of the homotopy group of dimension r, which we denote by $c(\psi, T^{r+1})$. Following Steenrod, we shall define the *ψ-cocycle*, by assigning $c(\psi, T^{r+1})$ to each simplex T^{r+1}:

*) Numbers in brackets refer to the bibliography.

$$c^{r+1}(\psi)=\sum c(\psi, T_i^{r+1})\ \ T_i^{r+1}$$

which depends on the decomposition used and on the map ψ. In general, the cohomology group is based on local coefficient groups connected by local isomorphisms [6], [7].

We assume except for Theorem (1.3) throughout this paper that *the fibre is connected and simple in every dimension.*

Lemma 1.*) $c^{r+1}(\psi)=0$ *is a necessary and sufficient condition that* ψ *can be extended to a slicing map of* K^{r+1}.

Proof. Suppose ψ is defined on K^{r+1}. Then, we have $c(\psi, T^{r+1})=0$ for every T^{r+1}. Conversely, if $c(\psi, T^{r+1})=0$, $\lambda\psi$ can be extended to a map of T^{r+1} in F. We obtain the extension of ψ, by assignment $x\rightarrow(x, \lambda\psi(x))$ for $x \in T^{r+1}$.

Lemma 2. **) *If* $c^{r+1}(\psi)\sim 0$, *there exists a slicing map* ψ' *of* K^{r+1}, *such that* $\psi'=\psi$ *on* K^{r-1}.

Proof. By hypothesis, there exists an r-chain c^r with the coboundary $\delta c^r=c^{r+1}(\psi)$, namely, we have $c^{r+1}(\psi, T^{r+1})=\sum_j c^r(T_j^r)$. For each simplex T_j^r, we choose a homothetic simplex t_j^r. Let τ_j map $T_j^r-t_j^r$ linearly onto $T_j^r-p_j$, $p_j \in T_j^r$. Let further μ_j be a map of t_j^r into F carrying $\dot{t}_j^r$ into $\lambda\,\psi(p_j)$, which represents $-c^r(T_j^r)$ of $\pi^r(F)$. We define the map ψ' of T_j^r into $T_j^r\times F$:

$$\begin{aligned}\psi'(x)&=(x,\ \lambda\psi\tau_j(x)), \quad x\in T_j^r-t_j^r,\\ &=(x,\ \mu_j(x)), \quad x\in t_j^r.\end{aligned}$$

ψ' coincides with ψ on $\dot{T}^r$. It follows directly from the construction just given, that the map $\lambda\psi'$ of $\dot{T}^{r+1}$ in F determines 0 of $\pi^r(F)$. Hence we have $c^{r+1}(\psi', T^{r+1})=0$. This leads to a map ψ' of K^r with $c^{r+1}(\psi')=0$. Thus, by Lemma 1, ψ' can be extended to a slicing map of K^{r+1}.

With these preparations we can state the following theorem:

Theorem (1. 1) *Let* $H^r(B)$ *be the r-th cohomology group* (with local coefficient group $\pi^{r-1}(F)$) *of B. If* $H^2(B)=H^3(B)=\ldots=H^s(B),=0$ *it is possible to define a slicing map over* K^s.

Proof. For a vertex of K, we choose a point in the fibre over it. Since F is connected, we can easily define a slicing map ψ over all simplexes of K^1. Suppose inductively that ψ has been defined over K^{r-1}.

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*) This is identical with Theorem 4 (a) in the paper of Steenrod [6].

**) Cf. [6] Theorem 4 (c).

Then ψ is defined on $\mathring{T}^r$. Thus, we have, as before, an r-dimensional ψ-cocycle $c^r(\psi)$, by putting

$$c^r(\psi)=\sum_i c^r(\psi,\ T_i^r)\ T_i^r$$

By assumption, $c^r(\psi)\sim 0$, hence by Lemma 2, there can be defined a slicing map ψ' over K^r. Repeating this process, we can, finally construct a slicing map over K^s.

2. Let again R be a fibre bundle over a polyhedron B. If for any simplicial decomposition K of B, there is no slicing map of K^r in R, we say B has an *r-dimensional hindrance*. If there is no slicing map of B, we simply say, B has a *hindrance*.

In the following, B is taken to be a sphere S^n of dimension n. We suppose that K is a fixed simplicial decomposition of S^n. Let further p be a inner point of an n-dimensional simplex T of K. Since S^n-p is contractible, we can easily construct a slicing map ψ over S^n with exactly one singular point p. To the map ψ, we attach ψ-cocycle $c^n(\psi)$ as before. It is clear that $c^n(\psi)=\alpha T$, $\alpha\in\pi^{n-1}(F)$.

Let ψ_1 be another slicing map over S^n-p, and $c^n(\psi_1)=\alpha_1 T$ be a corresponding ψ-cocycle. Since $\overline{S^n-T}$ is an n-cell, it follows by Feldbau's theorem [1] that the part of R over $\overline{S^n-T}$ is fibre homeomorphic to the topological product $\overline{S^n-T}\times F$. Hence, the maps ψ, ψ_1 are given by

$$\psi(x)=(x,\ f_0(x))$$
$$\psi_1(x)=(x,\ f_1(x))$$

for $x\in\overline{S^n-T}$. Clearly, the maps f_0, f_1 of the boundary $(\overline{S^n-T})^\cdot$ into F are inessential and are therefore homotopic to each other. We denote such a homotopy by f_t $(0\leq t\leq 1)$ and put

$$\psi_t(x)=(x,\ f_t(x)),\quad x\in(\overline{S^n-T})^\cdot$$

Since $(\overline{S^n-T})^\cdot=\dot{T}$ and $\psi=\psi_0$, it follows that ψ and ψ_1 are homotopic in $\pi^{-1}(\mathring{T})$, which means, by definition, that $\alpha=\alpha_1$. Thus we find that the ψ-cocycle $c^n(\psi)$ does not depend on the choice of ψ.

It is easily verified that the cohomology class of $c^n(\psi)$ is independent of the simplicial decomposition, which is used to define ψ-cocycle. The cohomology class of $c^n(\psi)$ will be called the characteristic class of S^n. We summarize these results as follows:

Theorem (1.2) *The characteristic class of S^n is a topological invariant of R.*

This leads us naturally to the definition of the characteristic number.

The characteristic class of S^n is determined by the homotopy group $\pi^{n-1}(F)$. In fact, the cohomology class of $c^n(\psi)$ is determined by the sum of its coefficients, which we call the *characteristic number*. In particular, if R is the fibre bundle of non zero tangent vectors of S^n, the characteristic number is nothing else than the so-called Euler-Poincaré characteristic. This has the value 2, if n is even, and 0, if n is odd. The characteristic number is, a fortiori, a topological invariant.

For the further investigations we shall need the following

Theorem (1.3)*) *Let R be a fibre bundle over a normal space B and F the fibre. If there exists a slicing map ψ, we have*

$$\pi^r(R)\approx\pi^r(B)+\pi^r(F), \quad r>1.$$

Moreover, if $\pi^1(F)$ is abelian, the same formula is also valid for $r=1$**)

Proof. We first consider the homotopy sequence

$$\dots\to\pi^{r+1}(R, F)\to\pi^r(F)\to\pi^r(R)\to\pi^r(R,F)\to\dots$$

It follows, from the covering homotopy theorem [8], that the boundary homomorphisms $\pi^{r+1}(R, F)\to\pi^r(F)$ are all trivial. Hence we have

$$\pi^r(R)-\pi^r(F)\approx\pi^r(R, F)\approx\pi^r(B).$$

Moreover ψ is a homeomorphism $B\to B'=\psi(B)$.

It is easily seen that the injection map of B' in R defines in a natural way the imbedding of $\pi^r(B')$ in $\pi^r(R)$. Thus, given $\alpha\in\pi^r(R)$, there exists β and γ such that $\alpha=\beta+\gamma$, where $\beta\in\pi^r(B')$, $\gamma\in\pi^r(F)$. Since $\pi^r(B')\cap\pi^r(F)=0$, we have

$$\pi^r(R)\approx\pi^r(B')+\pi^r(F)\approx\pi^r(B)+\pi^r(F),$$

which is to be proved.

§ 2. Homogeneous spaces

3. In this §, we consider homogeneous space W with a compact, transitive, Lie group G of automorphisms; such a space can also be defined as the space G/U of the cosets determined by a closed subgroup U; the cosets may be considered as fibres in G, making G into a fibre bundle, with the base-space G/U. W is a compact, orientable, differentiable manifold. In particular, W can be triangulated. As is well known, U is simple

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*) As indicated above, we do not require here that the fibre is simple in every dimension.

**) If R is orientable relative to $\pi^1(F)$ in the sense explained by Steenrod, this relation still holds for $r=1$ [6].

dimension. Thus, if U is supposed to be connected, our method can be applied to fiberings of G.

Lemma 3. *If there exists a slicing map* ψ *over* W, $U \not\sim O$ *in* G (with rational coefficients) *and* W *is a* Γ*-manifold with unit.**)

Proof. U and $\psi(W)$ meet only in one point. Clearly, the index of intersection of U with $\psi(U)$ is equal to ± 1. Hence, by Poincaré-Veblen's duality theorem, $U \not\sim O$.

Now we shall show that W is a Γ-manifold with unit. In fact, the product in W is induced by group multiplication in G. Let π be the projection, which takes each point of G into the coset of U containg it. We set $f(p, q) = \pi(\psi(p) \cdot \psi(q))$. It is easily seen that the so defined multiplication is continuous in both p and q. Putting $\pi(e) = \bar{e}$, where e is the unit in G, we obtain $f(\bar{e}, q) = \pi(\psi\pi(e) \cdot \psi(q)) = \pi\psi(q) = q$ for every q in W. Similarly, $f(p, \bar{e}) = p$ for every p in W. Therefore, we have $c_r = c_l = 1$. Thus, W is a Γ-manifold with unit.

The Euler-Poincaré characteristic of a Γ-manifold is always 0 [3]. Hence,

Theorem (2.1) *Let* $\chi(W)$ *denote the Euler-Poincaré characteristic of* W. *If* $\chi(W) > 0$, W *has a hindrance.*

We may now prove the following

Theorem (2.2) *Let* W *be simply connected and acyclic in every dimensions* $\leq s$. *If* $U \sim O$ *in* G (with rational coefficients), *amongst the groups* $\pi^s(U), \ldots, \pi^{n-1}(U)$ $(n = dim\ W)$, *there exists at least a non-trivial one.*

Proof. We suppose a suitable simplicial decomposition K of W. Clearly, we have a slicing map ψ over the s-dimensional skeleton K^s, and hence a ψ-cocycle $c^{s+1}(\psi)$ with the coefficient group $\pi^s(U)$. Suppose $\pi^s(U)=0$, then $c^{s+1}(\psi)=0$, which implies, by Lemma 1, that ψ can be extended to a map of K^{s+1}. In just the same manner, if $\pi^{s+1}(U), \ldots, \pi^{n-1}(U)$ are all trivial, we can define a slicing map of the whole K. Since $U \sim 0$ in G, by Lemma 3, W has a hindrance. The assumption that $\pi^s(U), \ldots, \pi^{n-1}(U)$ are trivial, has now led to a contradiction.

4. As an application, we shall prove the following

Theorem (2.3) *Let* G *be a compact, semi-simple, connected Lie group*

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*) Let W be a closed manifold. Given a continuous map of $W \times W$ in W, for p fixed, $W \times p$ and $p \times W$ are mapped with degrees c_r and c_l respectively. W is said to be a Γ-manifold, if there exists a continuous map of $W \times W$ in W such that $c_r \neq 0 \neq c_l$. In particular, if it is the case with $c_r = c_l = 1$, W is called a Γ-manifold with unit [3].

and T be a toral subgroup of G. Then G/T can not be homemorphic to the r-dimensional sphere S^r $(r \neq 2)$.

Proof. We suppose $G/T = W$ to be sudivided into a suitable simplicial complex K. Since G is arcwise connected, it is always possible to construct a slicing map ψ over K^1. If $W = S^1$, by Theorem (1.3), we have $\pi^1(G) \approx \pi^1(S^1) + \pi^1(T)$. It is impossible, because $\pi^1(G)$ is a finite group [4], while $\pi^1(T)$ is infinite. Hence W can not be homeomorphic to S^1.

Since T is a toral group, $\pi^r(T) = 0 (r > 1)$. We denote by $c^2(\psi)$ a 2-dimensional ψ-cocycle. If $c^2(\psi) \sim 0$, there exists, by Lemma 2, a slicing map ψ' over K^2. Since $\pi^2(T) = 0$, we have $c^3(\psi') = 0$. Hence we can define a slicing map over K^3. Repeating the same argument with $c^4(\psi')$ etc., it is easily verified, that we can construct a slicing map over the whole W. Again, we obtain $\pi^1(G) \approx \pi^1(W) + \pi^1(T)$. This is impossible, since $\pi^1(T)$ is infinite. Thus, we have $c^2(\psi) \not\sim 0$. In other words, W has a 2-dimensional hindrance.

Suppose $W = S^r$ $(r > 2)$. Then there exists a slicing map at least over K^2 contrary to the existence of a 2-dimensional hindrance. Thus, the Theorem is proved.

Remark. Let R^2 denote the group of all rotations of S^2. R^1 be the subgroup consisting of those rotations, which leave invariant a fixed point. Clearly, R^1 is a toral group and $R^2/R^1 = S^2$.

In the rest of this §, we shall take spheres S^n as homogeneous spaces. In this case, we need only to consider the n-dimensional hindrance. This in turn is determined by the characteristic number. Hence

Theorem (2.4) *The necessary and sufficient condition that there exists a hindrance in $G/U = S^n$ is that the characteristic number is not the unit* 0 *of $\pi^{n-1}(U)$.*

By Theorem (2.1), there exists a hindrance for even n. Thus

Theorem (2.5) *If $G/U = S^{2n}$, $\pi^{2n-1}(U) \neq 0$.*

Let R^n be the group of all rotations of S^n. We may suppose R^m $(m < n)$ to be the subgroup of R^n. It is easily seen that $R^n/R^{n-1} = S^n$. Hence, as a corollary, we obtain

Corollary. $\pi^{2n-1}(R^{2n-1}) \neq 0$.

Finally, Theorem (2.4) combined with Theorem (1.3) and Lemma 3 imply the following

Theorem (2.6)*) *If $\pi^{n-1}(U) = 0$, S^n is a Γ-manifold with unit, and*

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*) Cf [9], Theorem 2 and Theorem 3. See, also [2].

$$\pi^r(G) \approx \pi^r(U) + \pi^r(S^n) \quad (r \geq 1).$$

Consider, as an example, R^3 over $R^3/R^2 = S^3$. Since $\pi^2(R^2) = 0$, we have

$$\pi^r(R^3) \approx \pi^r(R^2) + \pi^r(S^3) \quad (r \geq 1).$$

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