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On the Differential Forms of the First Kind on Algebraic Varieties.

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In his book "Foundations of algebraic geometry"[1] A. Weil proposed several problems concerning differential forms on algebraic varieties. In this note we shall take up some of them. Especially we shall discuss differential forms of the first kind which are defined on a complete abstract varieties without multiple point. Here the field of definition is assumed to be arbitrary.

1. Let $K=k(x_1,\ldots\ldots, x_n)=k(x)$ be a field, generated over a field k by a set (x) of quantities; the totality $\mathfrak{D}$ of all derivations in $k(x)$ over k forms a finite K-module. Every element z of K defines a linear function dz from $\mathfrak{D}$ into K; we call this linear function the differential of z, and we can define multiplication between a differential and an element of K as usual. The set $\mathfrak{F}$ of those linear functions, which are sums of the products thus obtained, forms the dual K-module of $\mathfrak{D}$, and therefore the dimensions of $\mathfrak{D}$ and $\mathfrak{F}$ are equal.

As usual we can form the Grassmann algebra from the finite K-module $\mathfrak{F}$. An homogeneous element, of degree m, is called a differential form of degree m, belonging to the extension $k(x)$ of k.

PROPOSITION 1. Let $K=k(x)$ be a separably generated extention of k, and $dim_k\ (x)=n$. If $(u_1,\ldots\ldots,u_n)$ is a set of elements of $k(x)$, such that $k(x)$, such that $k(x)$ is separably algebraic over $k(u)$, then every differential form belonging to the extention $k(x)$ of k can be expressed in one and only one way, as polynomials in $du_1,\ldots\ldots,du_n$ with coefficients in $k(x)$.

PROOF. Let z be an arbitrary element of K; it is sufficient to prove that dz is expressed uniquely as a linear form in $du_1,\ldots\ldots,du_n$ with coefficients in $k(x)$. As z is separably algebric over $k(u)$, there exists a polynomial $P(U, Z)$ in $k[U_1,\ldots\ldots,U_n, Z]$ such that $P(u, z)=0$, $P_Z(u,z)\neq 0$,

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During my investigation I have received kind criticisms from Mr. Igusa to whom I express my hearty thanks.

[1] In this note we shall stick throughout, in terminologies and notations, to Weil, l. c.

$(P_Z(U, Z)=\frac{\partial P(U, Z)}{\partial Z})$. Therefore

$$dP(u,z)=\sum_{i=1}^{n} P_{Ui}(u, z)\ du_i+P_Z(u, z)dz=0$$

$$dz=-\frac{\sum_i P_{Ui}(u, z)\ du_i}{P_Z(u, z)}$$

Considering dimension we see readily that, this expression is unique.

The above definitions can be applied to the case in which k is a field of definitition for an algebraic variety V, and (x) a generic point of V over k; then the above defined differential forms are called differential forms on V. Similarly for abstract varieties.

In order to study the local properties of differential forms at a simple Poit of a Vaiety[2], we shall introduce local uniformizing parameters at that Point of the Variety.

DEFINITION 1. Let k be a field of definition for a Variety $\mathbf{U}^n$, $\mathbf{P}$ be a generic Point of the Variety $\mathbf{U}$ over the field k, and $\mathbf{P}'$ be a simple Point on the Variety $\mathbf{U}$, U_α be a representative of $\mathbf{U}$ in an ambient space S^N in which P' has a representative $P'_\alpha=(x')$, and $P_\alpha=(x)$ be the representative of $\mathbf{P}$ in U_α. We shall call the set of quantities $(t)=(t_1,\ldots\ldots,t_n)$ the set of uniformizing parameters at P', if every t_i is contained in the specialization ring of $\mathbf{P}'$ in $k(\mathbf{P})$ and there exist polynomials $F_i(T_1,\ldots\ldots,T_n;X_1,\ldots\ldots,X_N)$ in $k[T, X]$ $(i=1,\ldots\ldots, N)$, which fulfil the next conditions:

$$F_i(t, x)=0 \qquad (i=1,\ldots\ldots, N)$$

$$det\ (\frac{\partial F_i}{\partial x'_j})\neq 0 \qquad (i, j=1,\ldots\ldots, N)$$

It is evident that the above definition is equivalent to Weil's one. And we can easily see also that: if (t') is the specialization of (t) over $(x)\longrightarrow(x')$ with reference to k, (x') is the proper specialization of (x) over (t)——(t') with reference to k, of multiplicity 1; and vice versa.

The above defintions are independent of the choice of the representative of $\mathbf{U}$. Moreover, as the field $k(x)$ is separably algebraic over the field $k(t)$, every differential form can be expressed in one and only one way as a

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[2] As in Weil, l. c. we distinguish abstract varieties by the use of capitals, and also by the use of bold face capitals to denote them. Similarly for related notions, as points, subvarieties, etc.

homogeneous polynomial in $dt_1,\ldots\ldots,dt_n$ with coefficients in $k(x)$.

PROPOSITION 2. Let k be a field of definition for a variety U^n, $P=(x)$ be a generic point of the variety U over a field k. If the set $(t)=(t_1,\ldots\ldots,t_n)$ is uuiformizing parameters of the veriety U at a simple point $P'=(x')$ on U, then every dx_i $(i=1,\ldots\ldots, N)$ can be expressed as $dx_i=\sum_{\alpha=1}^{n} w_{i\alpha}\, dt_\alpha$ such that $w_{i\alpha}$ is contained in the specialization ring of the point P' in $k(P)$.

PROOF. Suppose that $F_i(T_1,\ldots\ldots,T_n,\ldots\ldots,X_N)$ are the polynomials which are defined in the def. 1. Then we have

$$\sum_{j=1}^{n}\frac{\partial F_i}{\partial x_j}\, dx_j+\sum_{\alpha=1}^{n}\frac{\partial F_i}{\partial t_\alpha}dt_\alpha=0 \quad (i=1,\ldots\ldots,N)$$

and therefore

$$dx_i=\frac{\sum_{\alpha=1}^{n} H_\alpha(t,x)\; dt_\alpha}{det(\dfrac{\partial F_i}{\partial x_j})}\ , \quad H_\alpha(T,\, X)\in k[T,\, X]$$

By the def. 1

$$det\,(\frac{\partial F_i}{\partial x'_j})\neq 0$$

which proves our proposition.

The next proposition is an immediate consequence of the preceeding:

PROPOSITION 3. Under the same assumptions as those in the prop. 2, if z is contained in the specialization ring of the point P' in $k(P)$, then the differential dz can be expressed as $dz=\sum_{\alpha=1}^{n}w_\alpha\, dt_\alpha$ such that $w_\alpha(\alpha=1,\ldots\ldots, n)$ is contained in the specielizetion ring of P' in $k(P)$.

DEFINITION 2. Let k be a field of definition for a Variety $\mathbf{U}^n$, $\mathbf{P}$ be a generic Point of the Variety $\mathbf{U}$ over k, and (t) be a set of uniformizing parameters at a simple Point $\mathbf{P}'$ on the Variety $\mathbf{U}$. Let a differential form ω on $\mathbf{U}$ be expressed as a homogeneous polynomial in $dt_1,\ldots\ldots,dt_n$. When its coefficients are all contained in the specialization ring of $\mathbf{P}'$ in $k(\mathbf{P})$, then we say that ω is finite at the Point $\mathbf{P}'$.

It is easily seen from the prop. 3 that the def. 2 is independent of the choice of a set of uniformizing parameters at $\mathbf{P}'$.

DEFINITION 3. A differential form ω which is finite at every Point on a complete Variety $\mathbf{U}$ without multiple Point, is called a differential form of the first kind on the Variety **U**.

It is desirable that this definition is invariant on the birationally equivalent Variaties. In § 2 we shall prove this proposition, which settles one of Weil's problems.

2. First we shall prove

PROPOSITION. 4. Let U^n and V^n be birationally equivalent varieties defined over a field k, $P=(x)$, $Q=(y)$ be, respectively, their generic points over k, and T a bitational correspondence between U and V with P and Q as mutually corresponding generic points. If a simple $(n-1)$—dimensional subuariety X^{n-1} of U and a simple subvariety $Y^m (m<n)$ correspond by T and a differential form ω belonging to the field $k(P)=k(Q)$ over k is finite at a generic point B of Y^m over k, then ω is also finite at a generic point A of X^{n-1} over k.

PROOF. We shall treat the case when ω is of degree 1. The other cases may be proved similarly.

Let (t), (u) be, respectively, a set of uniformizing parameters at the point B, A on V, U.

$$\omega=\sum^{n} Z_i\, dt_i$$

where z_i are contained in the specialization ring of B in $k(Q)$. As the y_i $(i=1,\ldots\ldots,N)$ are contained in the specialization ring of A in $k(P)$, the specialization ring of A in k(P) contains that of B in $k(Q)$. Therefore z_i, t_i, $(i=1,\ldots\ldots,n)$ are contained in the specialization ring of A in $k(P)$. By the prop. 3 we can easily verify the proposition.

PROPOSITION 5. If a differential form ω on a variety U^n is finite at the generic point of every simple $(n-1)$—dimensional subvariety of U, ω is finite at every simple point of U.

PROOF. Let k be a field of definition for U and $P=(x)$ a generic point of U over k. If $P'=(x')$ is a specialization of P over k and $P''=(x'')$ is that of P' over k, then a set of uniformizing parameters at P'' on U becomes also that at P'.

We shall treat the case when ω is of degree 1.

If we suppose that this proposition is not true, there exists a simple point $Q=(y)$ on U such that ω is not finite at Q. Let (t) be a set of uniformizing parameters at Q. Then ω is expressed in a form $\omega=\sum_{i=1}^{n} z_i\, dt_i$ with at least one z_i, say, z_{i0} not contained in the specializalization ring of

Q in $k(P)$. As (y, ∞) is a specialization of (x, z_{t0}) over k, there exists a component of $(z_{t0})_\infty$ which contains Q. This gives a contradiction.

By the prop. 4 and the prop. 5 we have the following theorem.

THEOREM 1.[3] If ω is a differential form of the first kind on a complete Variety **U** without multiple Point, ω is always finite at every simple Point of a Variety which is birationally equivalent to **U**.

3. In the rest of the paper we shall consider several properties concerning differential forms of the first kind.

PROPOSITION 6. Let k be a field of definition for a Variety $\mathbf{U}^n$, **P** a generic Point of the Variety **U** over k, and **Q** a simple Point on **U**. If a differential form ω on **U** is finite at **Q**, then ω induces uniquely a differential form on a Subvariety **V** which has **Q** as a generic Point over $\bar{k}$, ($\bar{k}$ denotes an algebraic closure of k.)

PROOF. We shall treat the case when ω is of degree 1. The other case may be treated similarly. Without loss of generality we can assume $\bar{k}=k$. In a representative $\mathbf{U}_\alpha$ of **U** two Points **P** and **Q** have, respectively, representatives $P=(x)$ and $\mathbf{Q}=(x')$. Let (t) be a set of uniformizing parameters at **Q**. Then we obtain

$$\omega=\sum_{i=1}^{n} R_i(x)\ dt_i,$$

where $R_i(X) \in k(X)$, and t_i, $R_i(x)$ $(i=1,\ldots\ldots, n)$ are included in the specialization ring of **Q** in $k(\mathbf{P})$. Furthermore ω is represented as follows:

$$\omega=\sum_{\mu=1}^{N} S_\mu(x)dx_\mu,$$

where $S_\mu(X)$ $(\mu=1,\ldots\ldots,N)$ are included in the specialization ring of **Q** in $k(\mathbf{P})$. Next we want to show that $\omega'=\sum_\mu S_\mu(x')dx'_\mu$ and $\omega''=\sum_i R_i(x')\ dt_0'$ (where (t') represents a specialization of (t) over (x)——(x') with reference to k.) are equal. From our proof to the prop. 2

$$dx_\mu=\frac{\sum_i H_{\mu i}\ (x)dt_i}{F(x)}\ , \quad dx'_\mu=\frac{\sum_l H_{\mu i}(x')dt_i'}{F(x')}$$

where $H_{\mu i}(X)$ and $F(X)$ are the rational functions with coefficients in k. Then we obtain

$$\frac{\sum_\mu S_\mu(x)H_{\mu i}(x)}{F(x)}=R_i(x)\,,\frac{\sum\ S_\mu(x')H_{\mu i}(x')}{F(x')}=R_i(x').$$

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[3] This result has been obtained also by Van der Waerden, as was communicated to the writer by K. Kodaira.

Therefore

$$\omega'=\omega''$$

We have proved that ω induces the uniquely determined differential form ω' on **V**.

PROPOSITION 7. Under the same assumptions and notations as in the above proposition, **R** is a simple Point on both **U** and **V**. If ω is finite at **R** on **U**, then ω' is also finite at **R** on **V**.

PROOF. Let (t) be a set of uniformizing parameters at **R** on **U**. It It follows that

$$\omega=\sum_{i=1}^{n} R_i(x)dt_i$$

where $t_i=T_i(x)$ and $R_i(x)$ $T_i(X)$ are rational functions with coefficients in k and $R_i(X)$, $T_i(x)$ are included in the specialization ring of **R** in $k(\mathbf{P})$. Similarly we have

$$\omega'=\sum_{i=1}^{n} R_i(x')\, dt_i'$$

where $t'_i=T_i(x')$, and $R_i(x')$, $T_i(x')$ are included in the specialization ring of **R** in $k(\mathbf{Q})$.

From the prop. 3 follows that $dt'_i(i=1,\ldots\ldots,n)$ are finite at **R** on **V**, and this proves the proposition.

From the above two propositions follows immediately.

THEOREM 2. Let $\mathbf{U}^n$ be a complete Variety without multiple Point, and let $\mathbf{V}^m$ be complete Subvariety of **U** without multiple Point. Then every differential form of the first kind on **U** determines a differential form of the first kind on **V**.

Next we shall consider differential forms on a Product-Variety. The following proposition can easily be seen.

PROPOSITION 8. Let **P** and **Q** be respectively, simple Points on Varieties **U** and **V**. If (t) and (u) are sets of uniformizing parameters at **P** and **Q** on **U** and **V** respectively, then (t, u) is a set of uniformizing parameters at a Point $\mathbf{P}\times\mathbf{Q}$ on a Product-Variety $\mathbf{U}\times\mathbf{V}$.

Every differential form on a Variety **U** or a Variety **V** determines a differential form on a Product-Variety $\mathbf{U}\times\mathbf{V}$, in the natural manner. Then a sum of differential forms of the first kind on **U** and **V** determines a differential form of the first kind on $\mathbf{U}\times\mathbf{V}$. The converse of this statement for the case of differential forms of degree 1, will be proved in the following.

PROPOSITION 9. Let $\mathbf{U}^n$ and $\mathbf{V}^m$ be Varieties defined over a field k

and let **P** and **Q** be algebraically independent generic Points over k of **U** and **V** respectively. ω is a differential form of degree 1 on a Product-Variety $\mathbf{U}\times\mathbf{V}$ and is represented as follows:

$$\omega=\tau+\sigma$$

with $$\tau=\sum_{i=1}^{l}v_i\, dx_i \quad \text{where } x_i \in k(\mathbf{P}),\ v_i \in k(\mathbf{P}, \mathbf{Q})$$

$$\sigma=\sum_{i=1}^{k}w_j\, dy_j \quad \text{where } x_j \in k(\mathbf{Q}),\ w_j \in k(\mathbf{P}, \mathbf{Q})$$

If $\omega=0$, then $\tau=0$ and $\sigma=0$.

PROOF. If $k(\mathbf{P})$ and $k(\mathbf{Q})$ are, respectively, separably algebraic over $k(t_1,\ldots\ldots,t_n)$ and $k(u_1,\ldots\ldots,u_m)$, $k(\mathbf{P}, \mathbf{Q})$ is separably algebraic over $k(t,u)$. Then

$$\tau=\sum_{\mu=1}^{n}v'_\mu\, dt_\mu \qquad \sigma=\sum_{\nu=1}^{m}w'_\nu\, du_\nu$$

$$\omega=\sum_{\mu}v'_\mu\, dt_\mu+\sum_{\nu}w'_\nu\, du_\nu$$

and v'_μ, w_ν' are uniquely determined. Therefore, if $\omega=0$, we have $v_\mu'=0$. $w'_\nu=0$ $(\mu=1,\ldots\ldots,n;\ \nu=1,\ldots\ldots,\ m)$; thus the proposition is proved.

THEOREM 3. Let $\mathbf{U}^n$ and $\mathbf{V}^m$ be complete Varieties without multiple Point. Every differential form ω of the first kind and of degree 1 on a Product-Variety $\mathbf{U}\times\mathbf{V}$ is represented as a sum of those of **U** and **V**.

PROOF. $\mathbf{U}\times\mathbf{V}$ and ω are defined over a field k. Let **P**, **Q**, be generic Points of **U**, **V** over k, and (t), (u) be sets of uniformizing parameters at **P**, **Q** on **U**, **V**. Then

$$\omega=\tau+\sigma$$

$$\tau=\sum_{i=1}^{n}v_i\, dt_i$$

$$\sigma=\sum_{j=1}^{m}w_j\, du_j \qquad \text{where } v_i,\ w_j \in k(\mathbf{P}, \mathbf{Q})$$

If v_i for a certain i is not contained in $k(\mathbf{P})$, $(\mathbf{P}, 0)$ is a specialization of $(\mathbf{P}, \frac{1}{v_i})$ with reference to k. This specialization can be extended to a specialtzation $(\mathbf{P}, 0, \mathbf{Q}')$ of $(\mathbf{P}, \frac{1}{v_i}, \mathbf{Q})$ with reference to k. This means that ω is not finite at a Point $\mathbf{P}\times\mathbf{Q}'$ on $\mathbf{U}\times\mathbf{V}$ which is absurd since ω is of the first kind. The proposition is thus proved.

PROPOSITION 10. Let $\mathbf{U}^n$ be a Variety, defined over a field k, and let K be an overfield of k. Let τ_λ be differential forms on $\mathbf{U}^n$ having k as a common field of definition; and the c_λ be linearly independent quantities over k, which is contained in K. If the differential form $\omega=\sum c_\lambda\tau_\lambda$ is finite at every simple Point on $\mathbf{U}$, then each τ_λ is also finite at every simple Point

PROOF. Let (t) be a set of uniformizing parameters at a simple Point $\mathbf{Q}$ of $\mathbf{U}^n$ in $k(\mathbf{P})$ and $\mathbf{P}$ be a generic Point of $\mathbf{U}$ over K. Then

$$\tau_\nu=\sum_i z_{\lambda i}dt_i$$

where z is quantities in $k(\mathbf{P})$ and therefore

$$\omega=\sum_i\ (\sum_\lambda c_\lambda\ z_{\lambda i})\ dt_i$$

(t) is also a set of uniformizing parameters at each one of the conjugates of $\mathbf{Q}$ over k. As ω is finite at every conjugate of $\mathbf{Q}$ over k, $\sum c_\lambda\ z_{\lambda i}$ are contained in the specialization ring of every conjugate of $\mathbf{Q}$ over k, in $K(\mathbf{P})$. From Weil, l.c. IV, prop. 8 $z_{\lambda i}$ are contained in the specialization ring of $\mathbf{Q}$ in $k(\mathbf{P})$, which proves the proposition.

THEOREM 4. Let $\mathbf{U}^n$ be a complete Variety without multiple Point, defined over a field k, and let K be an overfield of k. Let ω be a differential form of the first kind on $\mathbf{U}$, having K as a field of definition. Then the differential form ω is represented as a linear combination with coefficients in K, of differential forms of the first kind on $\mathbf{U}$, having k as a field of definition.

PROOF. If ω is represented as a linear combination as above, its terms are of the first kind, by the above proposition.

Let $\mathbf{P}$ be a generic Point of $\mathbf{U}$ over K, and (t) be a set of uniformizing parameters (contained) in $k(\mathbf{P})$ at the Point $\mathbf{P}$ on $\mathbf{U}$. Then

$$\omega=\sum_i\ y_i\ dt_i$$

where y_i are contained in $K(\mathbf{P})$. A generic Point of a $(n-1)$-dimensional Subvariety over K being non-algebraic over k is a generic Point of $\mathbf{U}$ over k. Therefore (t) is a set of uniformizing parameters at such Point. As ω is a differential form of the first kind, the quantities y_i are all contained in the specialization ring of such Point in $K(\mathbf{P})$. A divisor $(y_i)_\infty$ is a algebraic $\mathbf{U}$-divisor over k. By Weil, l.c. VIII, theorem 10, it follows

$$y_i=\sum c_\lambda\ z_{i\lambda}$$

where $z_{i\lambda}$ and c_λ are respectively contained in $k(\mathbf{P})$ and K. It follows that $\omega=\sum\ c_\lambda(\sum\ z_{i\lambda}\ dt_i)$. This proves the proposition,