Journal of the Mathematical Society of Japan Vol. 1, No. 3, Dec. 1949.

A Theorem on compact semi-simple groups

MORIKUNI GOTÔ

(Received Sept. 17, 1948)

Let G be a topological group. We shall denote by $D(G)$ the closure of the commutator subgroup of G, and call a connected compact group *semi-simple* if $D(G)$ coincides with G. A connected compact Lie group is semi-simple in our sense if and only if it is a semi-simple Lie group. We note here the fact that any factor group of a connected compact semi-simple group is also semi-simple. In the present note we shall prove the following.

Theorem. *Let G be a connected compact semi-simple group. Then for any element x of G there corresponds a pair of elements y and z such that*

$$x=y^{-1}z^{-1}yz.$$

Similar results have been obtained by K. Shoda[1] for the special linear group over an algebraically closed field, and recently by H. Tôyama[2] for some types of compact simple Lie groups. Our theorem is an extension of the theorem of Tôyama.

In order to prove our theorem we shall first prove a special case, namely the following

Lemma. *Let G be a connected compact semi-simple Lie group. Then any element x in G is representable in a form $y^{-1}z^{-1}yz$ for suitably chosen elements y and z.*

Proof Let A be a maximal connected commutative subgroup of G. Then A is a closed toroidal group,[3] and any element of G is known to be conjugate with some element of A. Hence it is sufficient to prove the case when x is contained in A.

Now we introduce a system of canonical coordinates in A. Then any a of A is given by its coordinates:

$$a=a(\varphi), \quad \varphi=(\varphi_1,\cdots\cdots, \varphi_n).$$

where φ_i varies over all real numbers mod. 1. Let now H be the normalizer of A in G. The transformation by an element h of H induces a

1) K. Shoda: Einige Saetze ueber Matritzen, Jap. Journ. of Math. v. XIII, 1937.
2) H. Tôyama: On commutators of matrices.
3) See L. Pontrjagin: Topological groups, Princeton, 1939.

continuous automorphism in A:

$$h^{-1}a(\varphi)h=a(\varphi^*),$$
$$\varphi^*=\varphi\ S,$$
$$\varphi=(\varphi_1,\cdots\cdots,\varphi_n) \qquad \varphi^*=(\varphi_1^*,\cdots\cdots,\varphi_n^*),$$

where $S=S(h)$ is a real matrix of degree n. The correspondence

$$h\longrightarrow S(h)$$

gives obviously a linear representation of H/A. Denote by (S) the matric group composed of $S(h)$'s. It is well known that (S) is a finite group isomorphic with H/A: $(S)\cong H/A$, and that (S) is determined by the local structure of G[4].

There exists now an element S_0 of (S) such that

$$(*) \qquad \det.(S_0-1_n)\neq 0,$$

where 1_n denotes the unit matrix of degree n. In fact as all compact simple infinitesimal groups and the corresponding (S)'s are kuown[5], we can readily find such S_0 that satisfies $(*)$. Let h_0 be an element of H such that $S(h_0)=S_0$. Then $(*)$ and the relation

$$a(\varphi)^{-1}h_0^{-1}\ a(\varphi)\ h_0=a(\varphi(S_0-1_n))$$

imply that for a given $a(\psi)$ in A there corresponds an element $a(\varphi)$ of A so that

$$a(\varphi)^{-1}\ h_0^{-1}\ a(\varphi)\ h_0=a(\psi),$$

and this completes our proof. Q.E.D.

Proof of the theorem. Let G be a connected compact semi-simple group. Then there exists a sequence $\{G_\alpha\}$ of compact semi-simple Lie groups such that G is the G_α-adic limit group of $\{G_\alpha\}$: $G=lim\ G_\alpha$. Hence we can easily conclude the existeuce of connected compact semi-simple Lie groups L_λ such that G is homomorphic with the direct product ΠL_λ:

$$(\Pi\ L_\lambda)/D\cong G$$

where D is a closed (0-dimensional) invariant subgroup of ΠL_λ. Accordingly the validity of the theorem for L_λ immediately implies that for G. Q.E.D.

Remark. In connection with the theorem of Shoda mentioned above, we can prove the following results modifying the proof of our lemma:

---

4) See e.g. F. Gantmacher: Canonical representations of automorphisms of a complex semi-simple Lie groups, Rec. Math. v. **5**, 1939.

5) F. Gantmacher: loc. cit.

Let G be a connected complex semi-simple Lie group of complex dimension r, and let C be the set of all elements of G of the form $y^{-1}z^{-1}yz$. Then the complementary set $G-C$ of C in G is contained in a closed set of complex dimension at most $r-1$. Hence C contains an open, connected set, which is everywhere dense in G.

Mathematical Institute, Nagoya University.