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On the Algebraic geometry of Chevalley and Weil

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During the War CHEVALLEY and WEIL have developed independently two theories of intersections of algebraic varieties (CHEVALLEY [1]-[3], WEIL [1][1]). Although in most cases we can do without the so-called " loally analytic " theory of CHEVALLEY, it would be sometimes convenient to use this powerful theory to the study of " truely local " properties of algebraic varieties or of differential forms on them. Since we can nowhere find the discussion about the connections between two theories, we shall give it in the following.

At first the word " variety " in the theory of CHEVALLEY is synonymous as the " variety " defined over a fixed algebraically closed field k in the sense of WEIL. If we use the word variety only in this meaning, the notions of a " simple " subvariety in both theories are shown to be the same (cf. CHEVALLEY [3] prop. 4 of Part III, § 3). Furthermore other formal notions in both theories can be seen to be synonymous, and the statement " U and V are subvarieties of a variety Ω and M is a proper component of the intersection $U\cap V$ on Ω " has the same meaning in both theories; thus the symbols $i_\Omega(M;\,U\cdot V)$ and $i(U\cdot V,\,M;\,\Omega)$ are at the same time defined.

THEOREM. *It holds* $i_\Omega(M;\,U\cdot V)=i(U\cdot V,\,M;\,\Omega)$.

Proof. It can be easily seen that we have only to show $i(P;\,V\cdot L)=j(V\cdot L,\,P)$, where P is a proper point of intersection of a variety V and of a linear variety L in S^n. We shall use n letters $X_1,\ldots\ldots,X_r,\ Y_1,\ldots\ldots,Y_{n-r}$ ($r=\dim(V)$) in describing the equations in S^n. By a suitable affine correspondence, which is everywhere biregular, we may assume that L is defined by the set of equations $X_1=0,\ldots\ldots,X_r=0$ in S^n, and P has the coordinates $(0,\ldots\ldots,0)$ (cf. CHEVALLEY [3] th. 7 of Part III, § 5; WEIL [1] th. 10 of Chap. VI, § 3). We denote by

$$m((y)\rightarrow(\bar{y})/(x)\rightarrow(\bar{x})(k))$$

the multiplicity of a specialization $(y)\rightarrow(\bar{y})$ over the specialization $(x)\rightarrow(\bar{x})$ with reference to k, whenever it is defined. Now if $(x,\,y)$ is a generic point

1) Numbers in [ ] refer to the references cited at the end of the paper.

of V over k, (x) is a set of independent variables over k and (y) is algebraic over the field $k(x)$; furthermore it holds

$$\begin{aligned} j(V\cdot L,\ P) &= m((x,\ y)\longrightarrow(0,\ 0)/(x)\longrightarrow(0)(k)) \\ &= m((y)\longrightarrow(0)/(x)\longrightarrow(0)(k)) \end{aligned}$$

(cf. WEIL [1] cor. to th. 3 of Chap. V, § 1). On the other hand it holds

$$i(P;\ V\cdot L)=e(\mathfrak{R}_V(P);\ x_1,\ldots\ldots,\ x_r)$$

(cf. CHEVALLEY [3] prop. 1 of Part III, § 8), so that we have only to show that $e(\mathfrak{R}_V(P);\ x_1,\ldots\ldots,\ x_r)=m((y)\rightarrow(0)/(x)\longrightarrow(0)(k))$.

In the following we shall fix an abstract field $k((x))$ and its algebraic closure $\overline{k((x))}$; we may assume that $k(x)$ and $\overline{k(x)}$ are subfields of $k((x))$ and $\overline{k((x))}$ respectively. Let $\mathfrak{P}$ be the prime ideal in $k[x,\ y]$, which defines V, then we have

$$\begin{aligned} e(\mathfrak{R}_V(P);\ x_1,\ldots\ldots,x_r) &= e(\overline{\mathfrak{R}}_V(P);\ x_1,\ldots\ldots x_r) \\ &= e(\overline{\mathfrak{R}}(P)/\mathfrak{P}\cdot\overline{\mathfrak{R}}(P);\ x_1,\ldots\ldots,x_r). \end{aligned}$$

Let $\overline{V}_\alpha$ be the sheets of V at P and let $\overline{\mathfrak{P}}_\alpha$ be the prime ideal in $\overline{\mathfrak{R}}$ (P) $=k[[x,\ y]]$, which corresponds to $\overline{V}_\alpha$, then we have

$$\mathfrak{P}\cdot k[[x,\ y]]=\cap_\alpha\overline{\mathfrak{P}}_\alpha$$

(cf. CHEVALLEY [3] def. 1 of Part III, § 2). Since $\overline{V}_\alpha$ has the same dimension as V, the residue-class $(\overline{x}^\alpha)$ of (x) modulo $\overline{\mathfrak{P}}_\alpha$ form a system of parameters in $\overline{\mathfrak{R}}(P)/\overline{\mathfrak{P}}_\alpha$, whence the residue-class $(\overline{y}^\alpha)$ of (Y) modulo $\overline{\mathfrak{P}}_\alpha$ is integral over $k[[\overline{x}^\alpha]]$, so that the subring $k[[\overline{x}^\alpha]]\ [\overline{y}^\alpha]$ of $k[[\overline{x}_\alpha,\ \overline{y}^\alpha]]$ is finite over $k[[\overline{x}_\alpha]]$ Since $k[[\overline{x}_\alpha]]$ is a complete local ring, $k[[\overline{x}^\alpha]]\ [\overline{y}^\alpha]$ is a complete semi-local ring and is a dense subset of $k[[\overline{x}_\alpha,\ \overline{y}_\alpha]]$ (in the semi-local topology); we conclude $k[[\overline{x}^\alpha]]\,[\overline{y}^\alpha]=k[[\overline{x}^\alpha,\overline{y}^\alpha]]$. There exists an isomorphism between $k[[\overline{x}^\alpha]]$ and $k\,[[x]]$, mapping $(\overline{x}^\alpha)$ on (x), over k; this isom orphism can be extended to an isomorphism between the algebraic closure of $k((\overline{x}^\alpha))$ and $\overline{k((x))}$. Let (η_α) be the image of $(\overline{y}^\alpha)$ by this isomorphism, then (η_α) is a specialization of (y) over $k(x)$ so that (η_α) is a conjugate ot (y) over $k(x)$ (cf. WEIL [1] cor. to th. 3 of Chap. II, § 1); furthermore it holds

$$\begin{aligned} e(\mathfrak{A}_V(P);\ x_1,\ldots\ldots,\ x_r) &= \textstyle\sum_\alpha\ e(k[[\overline{x}^\alpha,\overline{y}^\alpha]];\ \overline{x}_1^\alpha,\ldots\ldots,\ \overline{x}_r^\alpha) \\ &= \textstyle\sum_\alpha[k((\overline{x}^\alpha))(\overline{y}^\alpha):\ k((\overline{x}^\alpha))] \\ &= \textstyle\sum_\alpha[k((x))(\eta_\alpha):\ k((x))]. \end{aligned}$$

On the other hand let

$$\mathfrak{P}\cdot k((x))[y]=\cap_{\bar{\alpha}}\ \mathfrak{P}_{\bar{\alpha}}\cap_{\beta}\mathfrak{P}_{\beta}$$

be a irredundant representation of the ideal $\mathfrak{P}\cdot k((x))[y]$ in $k((x))[y]$ as an intersection of primary ideals, then since the primary components $\mathfrak{P}_*$ ($*=\bar{\alpha}, \beta$) are all isolated (cf. CHEVALLEY [2]), this representaion is unique (cf. v. d. WAERDEN [1] Kap. XII, § 88), and since $k((x))$ is separably generated over $k(x)$, the ideals $\mathfrak{P}_*$($*=\bar{\alpha}, \beta$) are all prime (cf. CHEVALLEY [2]), and there exists a set of elements (ξ_*) ($*=\bar{\alpha}, \beta$) in $\overline{k((x))}$ such that

$$k((x))[y]/\mathfrak{P}_*\simeq k((x))[\xi_*]\ (*=\bar{\alpha}, \beta).$$

By the same reason as before, (ξ_*) is a conjugate of (y) over $k(x)$; since it holds

$$k(x, y)_{k((x))}{}^{2)}\simeq\textstyle\sum_{\bar{\alpha}}k((x))(\xi_{\bar{\alpha}})+\sum_{\beta}k((x))(\xi_{\beta})$$

(cf. CHEVALLEY [2]), the complete sets of conjugates of ($\xi_{\bar{\alpha}}$) for all $\bar{\alpha}$ and of (ξ_β) for all β over $k((x))$ in $\overline{k((x))}$ constitute a complete set of conjugates of (y) over $k(x)$ in $\overline{k(x)}$, so that if we assume that ($\xi_{\bar{\alpha}}$) has the specialization (0) and (ξ_β) has not at the center of $k[[x]]$, we have

$$m((y))\rightarrow(0)/(x)\rightarrow(0)(k))=\textstyle\sum_{\bar{\alpha}}[k((x)(\xi_{\bar{\alpha}}):k((x))]$$

(cf. WEIL [1] th. 2, th. 3 and prop. 7 of Chap. III, § 3).

Now since (η_α) is aconjugate of (y) over $k(x)$ and has a specialization (0) at the center of $k[[x]]$, (η_α) is a conjugate of some ($\xi_{\bar{\alpha}}$) over $k((x))$ in $\overline{k((x))}$. We shall show that ($\xi_{\bar{\alpha}}$) is conversely a conjugate of some (η_α) over $k((x))$ in $\overline{k((x))}$. Since ($\xi_{\bar{\alpha}}$) is integral over $k[[x]]$ (cf. WEIL [1] th. 1 of Chap. III, § 3), and since (x, $\xi_{\bar{\alpha}}$) generates the maximal ideal in $k[[x]][\xi_{\bar{\alpha}}]$, the same reason as before shows that $k[[x]][\xi_{\bar{\alpha}}]=k[[x, \xi_{\bar{\alpha}}]]$; the ideal $\bar{\mathfrak{P}}_{\bar{\alpha}}$ of all power series $P(x,y)$ in $k[[x,y]]$, satisfying $P(x, \xi_{\bar{\alpha}})=0$, is prime and it holds $\mathfrak{P}\subset\bar{\mathfrak{P}}_{\bar{\alpha}}$, whence $\bar{\mathfrak{P}}_\alpha\subset\bar{\mathfrak{P}}_{\bar{\alpha}}$ for some α. Comparing the dimension of both ideals, we get $\bar{\mathfrak{P}}_\alpha=\bar{\mathfrak{P}}_{\bar{\alpha}}$ (cf. CHEVALLEY [1]), whence ($\xi_{\bar{\alpha}}$) must be a conjugate of this (η_α) over $k((x))$. Thus we conclude

$$\begin{aligned} e(\mathfrak{R}_V(P); x_1,\dots\dots, x_r) &=\textstyle\sum_\alpha[k((x))(\eta_\alpha):k((x))] \\ &=\textstyle\sum_{\bar{\alpha}}[k((x))(\xi_{\bar{\alpha}}):k((x))] \\ &=m((y)\rightarrow(0)/(x)\rightarrow(0)(k)). \end{aligned}$$

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2) $k(x, y)_{k((x))}$ means the algebra over $k((x))$, which arises from $k(x, y)$ (considered) as a commutatative algebra over $k(x))$ by extending the ground field $k(x)$ to $k((x))$.

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