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On the Theory of Algebraic Correspondences and its Application to the Riemann Hypothesis in Function-Fields

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Since the publication of ARTIN's monumental Quadratische Körper im Gebiete der höheren Kongruenzen, one of the central problems in the theory of algebraic function-fields has been to prove an analogy of RIEMANN hypothesis for "congruence ζ-functions." Although in "elliptic" case this problem had been solved by HASSE, the general case was beyond the scope of his arithmetical method; and it was André WEIL who first insighted the deep connection between the HURWITZ' formula in the theory of algebraic correspondences and the RIEMANN hypothesis in function-fields. By this discovery a new way to the solution of the RIEMANN hypothesis was opened; he sketched the outline of it in a C. R.-note in 1940, and a year later he published the outline of another proof in a P. N. A. S. -note, depending only on the SEVERI's theory of algebraic correspondences. In this paper we shall develope the algebraic theory of correspondences, centering around the SEVERI's formula on the "virtual degree" of divisors on the product of two algebraic curves, in a most general form based on the WEIL's *Foundations of algebraic geometry* (A. M. S. Coll., v. XXIX, 1946), and as its application, we shall prove the RIEMANN hypothesis following the idea of WEIL's P. N. A. S. -note.[1]

During the whole period of my investigation, I was encouraged by Prof. Y. AKIZUKI and by Dr. K. IWASAWA, to whom I express my sincere gratitude, and I wish to dedicate this paper to Prof. S. IYANAGA.

I. Multiplication Ring.

1. *The curve.* We shall fix once for all a "universal domain" K[2] of characteristic p (p is therefore either zero or a positive prime number). Let K be a regular extension of dimension 1 over a *perfect field* k_0, or,

1) After my investigation had been completed (in November 1948), I noticed by another P. N. A. S. -note of WEIL (in 1948) that he had also published a detailed proof of his notes in Pub. Inst. Strassbourg (N. S., no. 2), pp. 1–85 (1948).

2) We shall use the same terminologies and notations in WEIL's book; the results in the same book will be used without mentioning.

in the usual terminology, an algebraic function-field of one variable over k_0; since K is separably generated over k_0, there exists a variable x over k_0 in K such that K is separably algebraic over $k_0(x)$; set

$$[K : k(x)] = n.$$

Let K_x be the subring of K consisting of all elements y, which are finite over every finite specialization of x with reference to k_0; K_x is nothing but the set of all integral elements of K over the ring $k_0[x]$. By a well-known elementary procedure, we can find a minimal base $y_1, \ldots\ldots, y_n$ of K_x over $k_0(x)$; set

$$[K : k(x)] = n$$

Let K_x be the subring of K consisting of all elements y, which are finite over every finite specialization of x with reference to k_0; K_x is nothing but the set of all integral elements of K over the ring $k_0[x]$. By a well-known elementary procedure, we can find a minimal base $y_1, \ldots\ldots, y_n$ of K_x over $k_0[x]$; put then

$$\mathrm{P} = (x, y) = (x, y_1, \ldots\ldots, y_n).$$

Since $k_0(\mathrm{P}) = K$ is regular over k_0, the point P in $(n+1)$-space has a locus C of dimension 1 over k_0 such that for every point $\bar{\mathrm{P}}$ of C, the specialization ring of $\bar{\mathrm{P}}$ in $k_0(\mathrm{P})$ is integrally closed; since k_0 is a perfect field, this implies that every point of C is simple. Consider similarly the ring $K_{x'}$ for $x' = 1/x$ and take a minimal base $y_1', \ldots\ldots, y_n'$ of K_x' over $k_0[x']$, then the point

$$\mathrm{P}' = (x', y') = (x', y_1, \ldots\ldots, y_n')$$

in $(n+1)$-space has a locus C' of dimension 1 over k_0 such that every point of C' is simple on C'. Now since $k_0(\mathrm{P}) = K = k_0(\mathrm{P}')$, the point $\mathrm{P} \times \mathrm{P}'$ in the product $\mathrm{C} \times \mathrm{C}'$ has a locus T over k_0, which is a birational correspondence between C and C' over k_0; since C and C' have no multiple points, if a point $\bar{\mathrm{P}} \times \bar{\mathrm{P}}'$ in $\mathrm{C} \times \mathrm{C}'$ is on T, P and $\bar{\mathrm{P}}'$ are regularly corresponding points of C and C' by T. Therefore this T and the varieties C and C' together with their empty frontiers define a Variety C of dimension 1 and without multiple Point. Moreover if $\bar{\mathrm{P}} = (\bar{x}, \bar{y})$ is a pseudopoint of C, we have $\bar{x} = \infty$, and P' is finite over the specialization $\mathrm{P} \to \bar{\mathrm{P}}$ with reference to k_0; this shows that C is a complete Variety. In this paper, every complete Variety of dimension 1 and without multiple Point will be called a *Curve*.

Now if a Curve C' is given a priori, C' has a perfect field of definition k_0, e. g. the "perfect closure" of any field of definition of C'; take generic Point P' of C' over k_0, then $k_0(P')=K$ is a regular extension of dimension 1 over k_0, to which belongs therefore a Curve C as above. If P is the generic Point of C over k_0 (as defined above), we have $k_0(P')=K=k_0(P)$, so that the Point $P'\times P$ in $C'\times C$ has a Locus T over k_0 in $C'\times C$, which is an everywhere biregular birational correspondence between C' and C. It follows that the algebraic geometry on C' is the same as that on C; thus we may assume, if necessary, that every Curve is constructed as our C.

2) *Correspondences.* Let C_1 and C_2 be two Curves, then their Product $C_1\times C_2$ is a (special type of) *Surface*, by which we mean a complete Variety of dimension 2 and without multiple Point. As in the classical case, every divisor X on this Surface is called a *correspondence between C_1 and C_2*; in particular if X is reduced to a Variety, it is called an *irreducible correspondence*. The irreducible correspondence may have a finite number of multiple Points on it, so that it is not Curve in general. Now to every correspondence X between C_1 and C_2, we shall attach two rational integers $\mathrm{l}(X)$ and $\mathrm{r}(X)$ by

$$\mathrm{pr}_{C_1}(X)=\mathrm{l}(X)\,C_1,\qquad \mathrm{pr}_{C_2}(X)=\mathrm{r}(X)\,C_2.$$

If X is irreducible and if $\mathrm{l}(X)=0$ or $\mathrm{r}(X)=0$, there exists a Point $\bar{P}$ of C_1 or a Point $\bar{Q}$ of C_2 such that

$$X=\bar{P}\times C_2 \quad\text{or}\quad X=C_1\times\bar{Q};$$

more generally a correspondence X of the form $A\times C_2$, or of the form $C_1\times B$, where A or B is a C_1-divisor or C_2-divisor, is called the *d. l.* (degenerate on the left-hand side)-*correspondence* or the *d. r.-correspondence* respectively.

Now let X be any irreducible correspondence between C_1 and C_2, and $\bar{P}$ a Point of C_1 such that the intersection-product $X\cdot(\bar{P}\times C_2)$ is defined on $C_1\times C_2$, then we shall define the C_2-divisor $X(\bar{P})$ by

$$X(\bar{P})=\mathrm{pr}_{C_2}(X\cdot(\bar{P}\times C_2));$$

$X\cdot(\bar{P}\times C_2)$ is not defined if and only if $X=\bar{P}\times C_2$. Since we have $(\bar{Q}\times C_2)(\bar{P})=0$ for every Point $\bar{Q}\neq\bar{P}$ of C_1, it will be natural to set

$$(\bar{P}\times C_2)\ (\bar{P})=0;$$

and in general we shall define $X(A)$ for any correspondence X between C_1 and C_2, and for any C_1-divisor A by linearity in X and in A, i.e. we put

$$X(A)=\sum_{i,j} a_i\, b_j\, X_1(\bar{P}_j)$$

for the expressions $X=\sum a_i X_i$ and $A=\sum_j b_j \bar{P}_j$; it holds

$$X(A)=\mathrm{pr}_{C_2}(X\cdot(A\times C_2)),$$

whenever the intersection-product $X\cdot(A\times C_2)$ is defined on $C_1\times C_2$. Moreover by the "principle of conservation of number", we have

$$\deg.\ X(A)=l(X)\ \deg. A;$$

thus the 'correspondence'

$$A\longrightarrow X(A)$$

is an additive operator of the group of all C_1-divisors (of degree zero) into the group of all C_2-divisors (of degree zero). Let P be a generic Point of C_1 over some common field of definition for C_1 and C_2, over which X is rational, then $X(P)$ has a uniquely determined specialization $X(\bar{P})$ over every specialization $P\rightarrow\bar{P}$ with reference to above mentioned field; thus $X(A)$ is uniquely determined by $X(P)$ only.

3) *Product of correspondences.* Now let X be an irreducible correspondence between two Curves C_1 and C_2, and Y an irreducible correspondence between two Curves C_2 and C_3, then we shall define their *product* $Y\circ X$ by

$$Y\circ X=\mathrm{pr}_{C_1\times C_3}((C_1\times Y)\cdot(X\times C_3)),$$

whenever the intersction-product $(C_1\times Y)\cdot(X\times C_3)$ is defined on $C_1\times C_2\times C_3$; $Y\circ X$ is a correspondence between C_1 and C_3, whenever it is defined.

PROPOSITION 1. *The intersection product $(C_1\times Y)\cdot(X\times C_3)$ is not defined on $C_1\times C_2\times C_3$, if and only if X is of the form $C_1\times\bar{P}$ and Y of the form $\bar{P}\times C_3$ with the same Point $\bar{P}$ of C_2.*

Proof. $(C_1\times Y)\cdot(X\times C_3)$ is not defined, if and only if $C_2\times Y=X\times C_3$; our condition is thus sufficient. Conversely if $C_1\times Y=X\times C_3$, we have

$$C_1\times \mathrm{pr}_{C_2}(Y)=\mathrm{pr}_{C_1\times C_2}(C_2\times Y)$$
$$=\mathrm{pr}_{C_1\times C_2}(X\times C_3)=0,$$

whence $\mathrm{pr}_{C_2}(Y)=0$, $Y=\bar{P}\times C_3$ with some Point $\bar{P}$ of C_2, and $X=C_1\times\bar{P}$ with this $\bar{P}$.

Now since we have $(\bar{Q}\times C_3)\circ(C_1\times\bar{P})=0$ for every points $\bar{P}$ and $\bar{Q}$ $(\bar{P}\neq\bar{Q})$ of C_2, it will be natural to set

$$(\bar{P}\times C_3)\circ(C_1\times\bar{P})=0;$$

and in general we shall define the product $Y\circ X$ for any correspondence X between C_1 and C_2, and for any correspondence Y between C_2 and C_3 by linearity in X and Y, i. e. we put

$$Y\circ X=\sum_{i,j}a_i\, b_j\,(Y_j\circ X_i)$$

for the expressions $X=\sum_i a_i X_i$ and $Y=\sum_j b_j Y_j$; it holds

$$Y\circ X=\mathrm{pr}_{C_1\times C_3}((C_1\times Y)\cdot(X\times C_3)),$$

whenever the intersection-product $(C_1\times Y)\cdot(X\times C_3)$ is defined on $C_1\times C_2\times C_3$. We shall show that our product $Y\circ X$ induces the 'product' of two operators X and Y between the groups of divisors on the corresponding Curves (cf. 2)). To this purpose take a common field of definition for C_1, C_2 and C_3, over which both X and Y are rational; and take a generic Point P of C_2 over this field; we only to show

$$(Y\circ X)(P)=Y(X(P)).$$

By linearity we may assume that both X and Y are irreducible; if Y is a d. l.-correspondence, $Y\circ X$ is also a d. l.-correspondence, and both sides are zero; otherwise we have

$$\begin{aligned}(Y\circ X)(P)&=\mathrm{pr}_{C_3}[[\mathrm{pr}_{C_1\times C_3}\{(C_1\times Y)\cdot(X\times C_3)\}]\cdot(P\times C_3)]\\ &=\mathrm{pr}_{C_3}[(C_1\times Y)\cdot(X\times C_3)]\cdot[(P\times C_2\times C_3)]\\ &=\mathrm{pr}_{C_3}[(C_1\times Y)\cdot\{(X\times C_3)\cdot(P\times C_2\times C_3)\}]\\ &=\mathrm{pr}_{C_3}[Y\cdot(X(P)\times C_3)]\\ &=Y\{X(P)\}\cdot\end{aligned}$$

Since the interchange of factors in $C_1\times C_2$ is an everywhere biregular birational correspondence between $C_1\times C_2$ and $C_2\times C_1$, it transforms the correspondence X into a correspondence between C_2 and C_1, which we shall denote by X^*; similarly Y^* and $(Y\circ X)^*$ are defined as correspondences between C_3 and C_2, and between C_3 and C_1 respectively; we shall show

$$(Y\circ X)^*=X^*\circ Y^*$$

By linearity we may assume that both X and Y are irreducible; if $X=C_1\times\bar{P}$ and $Y=\bar{P}\times C_3$ with the same Point $\bar{P}$ of C_2, both sides are zero; otherwise we have

$$\begin{aligned}(Y\circ X)^*&=(\mathrm{pr}_{C_1\times C_3}((C_1\times Y)\cdot(X\times C_3)))^*\\ &=\mathrm{pr}_{C_3\times C_1}((Y^*\times C_1)\cdot(C_3\times X^*))\\ &=\mathrm{pr}_{C_3\times C_1}((C_3\times X^*)\cdot(Y^*\times C_1))\\ &=X^*\circ Y^*\end{aligned}$$

4) *Associativity of the product.* X, Y being as before, let Z be a correspondence between two Curves C_3 and C_4; we shall prove the associativity of our product:

$$Z\circ(Y\circ X)=(Z\circ Y)\circ X.$$

By linearity we may assume that X, Y and Z are all irreducible. If the intersection-products $(C_1\times Y\times C_4)\cdot(X\times C_3\times C_4)$ and $(C_1\times C_2\times Z)\cdot\{(C_1\times Y\times C_4)\cdot(X\times C_3\times C_4)\}$ are defined on $C_1\times C_2\times C_3\times C_4$, the intersection-products $(C_1\times C_2\times Z)(C_1\times Y\times C_4)$ and $(C_1\times C_2\times Z)\cdot(C_1\times Y\times C_4))\cdot(X\times C_3\times C_4)$ are also defined on $C_1\times C_2\times C_3\times C_4$, and by the "principle of associativity", we have

$$(C_1\times C_2\times Z)\cdot((C_1\times Y\times C_4)\cdot(X\times C_3\times C_4))$$
$$=((C_1\times C_2\times Z)\cdot(C_1\times Y\times C_4))\cdot(X\times C_3\times C_4).$$

Since it holds

$$\begin{aligned}&\mathrm{pr}_{C_1\times C_4}((C_1\times C_2\times Z)\cdot((C_1\times Y\times C_4)\cdot(X\times C_3\times C_4)))\\&=\mathrm{pr}_{C_1\times C_4}((C_1\times Z)\ \mathrm{pr}_{C_1\times C_3\times C_4}((C_2\times Y\times C_4)\cdot(X\times C_3\times C_4)))\\&=\mathrm{pr}_{C_1\times C_4}((C_1\times Z)\ (\mathrm{pr}_{C_1\times C_3}((C_1\times Y)\cdot(X\times C_3))\times C_4))\\&=Z\circ(Y\circ X)\end{aligned}$$

on one hand and

$$\begin{aligned}&\mathrm{pr}_{C_1\times C_4}(((C_1\times C_2\times Z)\cdot(C_1\times Y\times C_4))\cdot(X\times C_3\times C_4))\\&=(\mathrm{pr}_{C_4\times C_1}((C_4\times C_3\times X^*)\cdot(C_4\times Y^*\times C_1)\cdot(Z^*\times C_2\times C_1))^*\\&=X^*\circ(Y^*\circ Z^*))^*=(Z\circ Y)\circ X\end{aligned}$$

on the other, we have $Z\circ(Y\circ X)=(Z\circ Y)\circ X$.

Next, if the intersection-product $(C_1\times Y\times C_4)\cdot(X\times C_3\times C_4)$ is not defined on $C_1\times C_2\times C_3\times C_4$, the intersection-product $(C_1\times Y)\cdot(X\times C_3)$ is not defined on $C_1\times C_2\times C_3$; by prop. 1, X is of the form $C_1\times\bar{P}$, Y of the form $\bar{P}\times C_3$ with the same Point $\bar{P}$ of C_2, and we have

$$\begin{aligned}&Z\circ(Y\circ X)=0,\\&(Z\circ Y)\circ X=\mathrm{pr}_{C_2\times C_4}((C_2\times Z)\cdot(\bar{P}\times C_3\times C_4))\circ X\\&\qquad=\mathrm{r}(Z)\cdot(\bar{P}\times C_4)\circ(C_1\times\bar{P})=0.\end{aligned}$$

Thus, as easily seen, we have only to examine the case, where the intersection-products $(C_1\times Y)\cdot(X\times C_3)$ and $(C_2\times Z)\cdot(Y\times C_4)$ are defined on $C_1\times C_2\times C_3$ and on $C_2\times C_3\times C_4$ respectively, but the intersection-products $(C_1\times C_2\times Z)((C_1\times Y)\cdot(X\times C_3)\times C_4)$ and $(C_1\times(C_2\times Z)\cdot(Y\times C_4))\cdot(X\times C_3\times C_4)$ are not defined on $C_1\times C_2\times C_3\times C_4$. It follows that $X\times C_3\times C_4$ must contain some component of the cycle $(C_1\times(C_2\times Z)\cdot(Y\times C_4))$, so that X

must contain the projection of that component on $C_1 \times C_2$; since every such component is either $C_1 \times C_2$ or or of the form $C_1 \times \bar{P}$ with some Point $\bar{P}$ of C_2, X itself must be of the form $C_2 \times \bar{P}$, In the same way Z must be of the form $\bar{Q} \times C_4$ with some Point $\bar{Q}$ of C_3. Now if it happens $\mathrm{l}(Y)=0$, we have $(C_1 \times Y)\ (X \times C_3)=0$, so that the intersection-product $(C_1 \times C_2 \times Z) \cdot ((C_1 \times Y)\ (X \times C_3) \times C_4)$ is defined on $C_1 \times C_2 \times C_3 \times C_4$; it must be $\mathrm{l}(Y) \neq 0$, and, by the same reason, $\mathrm{r}(Y) \neq 0$. It follows

$$Z \circ (Y \circ X) = (\bar{Q} \times C_4) \circ (C_1 \times Y(\bar{P})) = 0,$$
$$(Z \circ Y) \circ X = (\overset{*}{X} \circ (\overset{*}{Y} \circ \overset{*}{Z}))^{*} = 0;$$

which completes the proof of our associativity formula.

Now let Δ_{11} be the diagonal of $C_1 \times C_1$, i. e. the Locus of the Point $P \times P$ over some field of definition k of C_1, where P is a generic Point of C_1 over k; similarly let Δ_{22} be the diagonal of $C_2 \times C_2$; then for every correspondence X between C_1 and C_2, it holds

$$\Delta_{22} \circ X = X \circ \Delta_{11} = X.$$

In order to prove this fact, we may assume that X is irreducible; let k_1 be a field of definition for X, containing k, and let $\bar{P} \times \bar{P} \times \bar{Q}$ be a generic Point of some component W of the intersection $(C_1 \times X) \cap (\Delta_{11} \times C_2)$ over $\bar{k}_1$, then the Locus of the Point $\bar{P} \times \bar{Q}$ in $C_1 \times C_2$ over $\bar{k}_1$ is nothing but X. It follows that

$$(C_1 \times X) \cdot (\Delta_{11} \times C_2) = aW$$

with some rational integer a. Since however

$$\begin{aligned} \mathrm{pr}\ c_1\ {\scriptstyle(\text{second factor})} \times c_2\ ((C_1 \times X)\ (\Delta_{11} \times C_2)) &= X \cdot (C_1 \times C_2) = X \\ &= \mathrm{pr}\ c_1 \times c_2\ (a\ W) = a\ X, \end{aligned}$$

we must have $a=1$; and it holds

$$\begin{aligned} X \circ \Delta_{11} &= \mathrm{pr}\ c_1\ {\scriptstyle(\text{first factor})} \times c_2\ ((C_1 \times X) \cdot (\Delta_{11} \times C_2)) \\ &= \mathrm{pr}\ c_1 \times c_2\ (W) = X; \end{aligned}$$

moreover it holds

$$\Delta_{22} \circ X = (X^{*} \circ \Delta_{22})^{*} = (X^{*})^{*} = X.$$

5) *Correspondences with valence zero.* Every correspondence X_0 between C_2 and C_3, which is linearly equivalent to a sum of d. l.-and of d. r.-correspondences, is called a *correspondence with valence zero* (between C_2 and C_3); to every such correspondence X_0, there exist therefore a C_2

-divisor A and a C_3-divisor B such that

$$X_0 \equiv (A \times C_3) + (C_2 \times B);$$

$\equiv$ means thereby the linear equivalence.

PROPOSITION 2. *Let X_0 be a correspondence with valence zero, then X_0^* is also a correspondence with valence zero.*

Proof. We have only to show that the linear equivalence on $C_2 \times C_3$ is invariant by $*$; we shall prove more generally the following proposition :

An "equivalence theory" being given, let U and V be two complete Varieties without multiple Point, which correspond by some everywhere biregular birational correspondence T; and let X and Y be two "equivalent" U-cycles, then $T(X)$ and $T(Y)$ are equivalent on V.

In fact since we have

$$T(X) - T(Y) = T(X-Y)$$
$$= \mathrm{pr}_V \{T(X-Y) \times V)\},$$

$T(X) - T(Y)$ is equivalent to zero on V by the propositions (A), (C) of the equivalence theory, (Weil's book)

PROPOSIFION 3. *X_0 being as before, let X be a correspondence between C_1 and C_2, and Y a correspondence between C_3 and C_4, then both $X_0 \circ X$ and $Y \circ X_0$ are correspondences with valence zero.*

Proof. By prop. 2, we have only to show that $Y \circ X_0$ is a correspondence with valence zero; thereby we may assume that Y is irreducible. Let A be a C_2 -divisor and B a C_3-divisor such that

$$X_0 \equiv (A \times C_3) + (C_2 \times B);$$

then we have

$$Y \circ (A \times C_3) = \mathrm{r}(Y) \cdot (A \times C_4);$$

and if $\mathrm{l}(Y)=0$, we have $Y \circ (C_2 \times B) = 0$, otherwise we have

$$Y \circ (C_2 \times B) = C_2 \times Y(B).$$

Thus we have only to show for every function φ on $C_2 \times C_3$, $Y \circ (\varphi)$ is a correspondence with valence zero. By what we have proved above and by prop. 2, we may assume that $\mathrm{l}(Y)$, $\mathrm{r}(Y) \neq 0$; then we shall show that $Y \circ (\varphi)$ is linearly equivalent to zero. Since $\mathrm{r}(Y) \neq 0$, $C_2 \times Y$ has the projection $C_2 \times C_4$ on $C_2 \times C_4$; since $\mathrm{l}(Y) \neq 0$, the intersection-product $(C_2 \times Y) \cdot ((\varphi) \times C_4)$ is defined on $C_2 \times C_3 \times C_4$; it follows from postulates (A), (C') of the equivalence theory that

$$Y \circ (\varphi) = \mathrm{pr}_{C_2 \times C_4} ((C_2 \times Y) \cdot ((\varphi) \times C_4)) \equiv 0.$$

Now if we consider only the correspondences between a Curve C and itself, they form a module $\mathfrak{X}$ over the ring of rational integers; moreover $\mathfrak{X}$ forms a non-commutative ring by $\circ$-multiplication, and the mapping $X \longrightarrow X^*$ is an involution of $\mathfrak{X}$ such that

$$(Y \circ X)^* = X^* \circ Y^*$$

The ring $\mathfrak{X}$ has a unit $\varDelta$, and the correspondences with valence zero form a two-sided ideal $\mathfrak{X}_0$ in $\mathfrak{X}$; thus we may consider the residue-class ring $\mathfrak{R}$ of $\mathfrak{X}$ modulo $\mathfrak{X}_0$; we shall denote its elements by the small Germann letters. Historically $\mathfrak{R}$ arose from the study of the "sigular multiplication" of RIEMANN matrices so that we may call it as the *multiplication ring attached to C*. $\mathfrak{R}$ has a unit $\mathfrak{e}$, which is the class of $\varDelta$. Let X be a representative of an arbitrary element $\mathfrak{x}$ of $\mathfrak{R}$; since the ideal $\mathfrak{X}_0$ is Invariant by $*$, the class of X^* is uniquely determined by $\mathfrak{x}$; we shall denote this class by $\mathfrak{x}^*$. The mapping $\mathfrak{x} \longrightarrow \mathfrak{x}^*$ is an involution of $\mathfrak{R}$ such that

$$(\mathfrak{y} \cdot \mathfrak{x})^* = \mathfrak{x}^* \cdot \mathfrak{y}^*$$

II. Linear Series.

6) *Definition of the linear series.* Let C be an arbitary Curve, and let

$$\varphi_0, \varphi_1, \ldots\ldots, \varphi_r \ (\mathrm{r} \geqq 0)$$

be $(\mathrm{r}+1)$ functions on C, all defined over a field k; we shall consider the r-dimensional projective space L_r. Let $\Pi \times P$ be a generic Point of the Product $L_r \times C$ over k, and put

$$\varphi = x_0\ \varphi_0 + x_1\ \varphi_1 + \ldots\ldots + x_r\ \varphi_r,$$

where $(x) = (x_0, x_1, \ldots\ldots, x_r)$ is a representative of Π; then φ is a function on C, defined over the field $k(\Pi)$. Consider a function Φ on $L^r \times C$, defined over k by

$$\Phi\ (\Pi \times P) = \varphi\ (P),$$

then the $(L^r \times C)$-divisor $(\Phi)_0$ and the C-divisor $(\varphi)_0$ are independent of the choice of the representative (x) of Π, and it holds

$$(\varphi)_0 = \mathrm{pr}\ c((\Phi)_0\ (\Pi \times C)) = (\Phi)_0\ (\Pi \times C)) = (\Phi)_0\ (\Pi);$$

moreover $(\varphi)_0$ is a rational C-divisor over $k\,(\Pi)$.

PROPOSITION 1. *The functions*

$$\varphi_0, \varphi_1, \ldots\ldots, \varphi_r$$

are linearly independent (over the abstract field of constants), *if and only if* $(\Phi)_0$ *does not contain the Variety* $\bar{\Pi} \times C$ *for every Point* $\bar{\Pi}$ of L^r.

Proof. Take a generic Point P of C over the field $k(\bar{\Pi})$, and a representative $(\bar{x}) = (\bar{x}_0, \bar{x}_1, \ldots\ldots, \bar{x}_r)$ of $\bar{\Pi}$, then $\bar{\Pi} \times C$ is contained in $(\Phi)_0$ if and only if it holds

$0 = \Phi(\bar{\Pi} \times P) = \bar{x}_0 \varphi_0(P) + \bar{x}_1 \varphi_1(P) + \ldots\ldots + \bar{x}_r \varphi_r(P)$; but this is equivalent to

$$\bar{x}_0 \varphi_0 + \bar{x}_1 \varphi_2 + \ldots\ldots + \bar{x}_r \varphi_r = 0 .$$

In the following we set the additional assumption that $\varphi_0, \varphi_1, \ldots\ldots, \varphi_r$ are linearly independent.

Now we shall determine the components of $(\Phi)_0$; let $\bar{\Pi} \times \bar{P}$ be a generic point of some component of $(\Phi)_0$ over $\bar{k}$. If $\bar{P}$ is algebraic over k, this component must be $L^r \times \bar{P}$; if $\bar{P}$ is is a generic Point of C over k, such components will be denoted by W; W has the projection L^r on L^r, for otherwise W must have the form $L_{r-1} \times C$ with some linear Subvariety L_{r-1} of L^r, which contradicts to our assumption (cf. prop. 1). It follows that

$$(\Phi)_0 = \sum a (L^r \times \bar{P}) + \sum b\, W,$$

where a and b are positive integers. Let P be a generic Point of C over k, and let Γ_Φ be the graph of Φ; let D be the so-called " projective straight line ", then we have

$$\begin{aligned}
&((\Phi)_0 \times (0)) \cdot (L^r \times P \times D) \\
&\quad = (\Gamma_\Phi \cdot (L^r \times C \times (0)) \cdot (L^r \times P \times D) \\
&\quad = (\Gamma_\Phi \cdot (L^r \times P \times (D))) \cdot (L^r \times C \times (0)) \\
&\quad = L^{r-1} \times P \times (0),
\end{aligned}$$

where L^{r-1} is the linear Subvariety of L^r, defined over the field $k(P)$ by the equation

$$\varphi_0(P) X_0 + \varphi_1(P) X_1 + \ldots\ldots + \varphi_r(P) X_r = 0.$$

It follows that

$$\begin{aligned}
(\Phi)_0 \cdot (L^r \times P) &= \sum b \{W (L^r \times P)\} \\
&= L^{r-1} \times P;
\end{aligned}$$

since $W \cdot (L^r \times P) \succ 0$, and since $b > 0$, there exists only one W and $b = 1$; in particular W must be defined over k. Thus if we put

$$A = \sum a \bar{P}, \quad W(\Pi) = \mathrm{pr}_C \{W \cdot (\Pi \times C)\};$$

we have

$$(\varphi)_0 = A + W(\Pi),$$

where $W(\Pi)$ is a prime rational C-divisor over $k(\Pi)$. Moreover since $A=(\varphi)_0 - W(\Pi)$ is algebraic over k, and is rational over the regular extension $k(\Pi)$ of k, it must be rational over k. We have thus proved the following theorem; in the classical case, the latter part of which is known as the *first* BERTINI's *theorem*:

THEOREM 1. We have

$$(\Phi)_0 = (L^r \times A) + W^r,$$

where A is a rational C-divisor over k, and W is a Subvariety of $L^r \times C$, defined over k, such that it has the projection L^r on L^r and C on C. It follows that

$$(\varphi)_0 = A + W(\Pi),$$

where $W(\Pi)$ is a prime rational C-divisor over $k(\Pi)$.

Now let B be an arbitary C-divisor; take a common field of definition k_1 of C and W, over which B is rational, and a generic Point Π of L^r over k_1; then the set of all the specializations of the C-divisor

$$B + W(\Pi)$$

over k_1 constitutes a *linear series of dimension r on C* with a *generic element* $B+W(\Pi)$ over k_1; k_1 is called a *field of definition* of this linear series. Since all the elements of such linear series have the same degree, i.e. $(B + W(\Pi))$, it is called the *degree* of the linear series; moreover B is called the fix-part of the linear series.

7) *Properties of the linear series.* Let k be a common field of definition of C and W, over which A is rational; let Π be a generic Point of L^r over k, and $\bar{\Pi}$ any Point of L^r then $A+W(\Pi)$ has a uniquely determined specialization

$$A + W(\bar{\Pi}) = A + \mathrm{pr}_C(W \cdot (\bar{\Pi} \times C))$$

over the specialization $\Pi \to \bar{\Pi}$ with reference to k, which is rational over $k(\bar{\Pi})$. We shall show

THEOREM. 2. *Let* $(\bar{x}) = (\bar{x}_0, \bar{x}_1 \ldots\ldots \bar{x}_r)$ be a representative of $\bar{\Pi}$, and *set*

$$\bar{\varphi} = \bar{x}_0\, \varphi_0 + \bar{x}_1\, \varphi_1 + \cdots\cdots + \bar{x}_r\, \varphi_r,$$

then we have

$$A + W(\bar{\Pi}) = (\bar{\varphi}) + (\varphi)_\infty ,$$

and $(\varphi)_\infty$ *is rational over* k.

Proof. Let $\{P_i\}$ be the set of Points, which appear in some $(\varphi_0)_\infty$, $(\varphi_1)_\infty, \cdots, (\varphi_r)_\infty$, then we may put

$$(\varphi)_\infty = \sum_i a_i P_i$$

with some rational integers a_i; $(\varphi)_\infty$ is thus an algebraic C-divisor over k, so that it is rational over k. Now let $\bar{\Pi} \times Z$ be a component of the intersection $\Gamma_\Phi \cap (\bar{\Pi} \times C \times D)$, and $\bar{\Pi} \times \bar{P} \times (\bar{z})$ a generic Point of $\bar{\Pi} \times Z$ over the field $\overline{k(\Pi)}$. If $\bar{P} \neq P_i$ for every i, we have

$$\bar{z} = \bar{x}_0 \varphi_0 (\bar{P}) + x_1 \varphi_1 (\bar{P}) + \cdots\cdots + \bar{x}_r \varphi_r (\bar{P}) = \bar{\varphi} (\bar{P})$$

so that $Z = \Gamma\bar{\varphi}$; on the other hand, if $\bar{P} = P_i$, we have $Z = P_i \times D$. Thus we have

$$\Gamma_\Phi \cdot (\bar{\Pi} \times C \times D) = a (\bar{\Pi} \times \Gamma\bar{\varphi}) + \sum_i b_i (\bar{\Pi} \times P_i \times D),$$

where a and b_i are rational integers; taking the algebraic projection on $L^r \times C$, we get $a = 1$. It follows

$$\begin{aligned}(\Gamma_\Phi \cdot (\bar{\Pi} \times C \times D)) \cdot (L^r \times C \times (0)) \\ = \bar{\Pi} \times (\varphi)_0 \times (0) + \sum_i b_i (\Pi \times P_i \times (0))\end{aligned}$$

on one hand and

$$\begin{aligned}(\Gamma_\Phi \cdot (\bar{\Pi} \times C \times D)) \cdot (L^r \times C \times (0)) \\ = (\Gamma_\Phi (L^r \times C \times (0)) \cdot (\bar{\Pi} \times C \times D) \\ = ((\Phi)_0 \times (0)) \cdot (\bar{\Pi} \times C \times D) \\ = ((\Phi)_0 (\bar{\Pi} \times C)) \times (0)\end{aligned}$$

on the other; whence we hane

$$(\Phi)_0 (\bar{\Pi}) = (\bar{\varphi})_0 + \sum_i b_i P_i .$$

In the same way we have

$$(\Phi)_\infty (\bar{\Pi}) = (\bar{\varphi})_\infty + \sum_i b_i P_i ;$$

since $(\Phi)_\infty(\bar{\Pi})$ is the specialization of $(\Phi)_\infty (\Pi) = (\varphi)_\infty$ over k, we have

$$(\varphi)_\infty = (\bar{\varphi})_\infty + \sum_i b_i P_i$$

It follows that

$$(\Phi)_0 (\bar{\Pi}) = A + W (\bar{\Pi}) = (\bar{\varphi})_0 - (\bar{\varphi})_\infty + (\bar{\varphi})_\infty = (\bar{\varphi}) + (\bar{\varphi})_\infty$$

which completes our proof.

COROLLARY 1. *The fix-part A is the "G. C. D." of $(\varphi_0)_0$, $(\varphi_1)_0$, ..., $(\varphi_r)_0$, and $(\varphi)_\infty$ is the "L. C. M." of $(\varphi_0)_\infty$, $(\varphi_1)_\infty$, ..., $(\varphi_r)_\infty$.*

In fact since $x_0, x_1, \ldots\ldots x_r$ are linearly independent over k, and since A is rational over k, we have

$$\text{G. C. D. } \{(\varphi_0)_0, (\varphi_1)_0, \ldots\ldots, (\varphi_r)_0\} \succ A;$$

on the other hand it holds in general

$$A \succ \text{G. C. D. } \{(\varphi_0)_0, (\varphi_1)_0, \ldots\ldots, (\varphi_r)_0\}.$$

By the same reason we have

$$-(\varphi)_\infty = \text{G. C. D. } \{-(\varphi_0)_\infty, -(\varphi_1)_\infty, \ldots\ldots, -(\varphi_r)_\infty\}$$

COROLLARY 2. *Any two elements of a linear series are always linearly equivalent.*

This is an immediate consequence of th. 2.

COROLLARY 3. *Let $P_1, \ldots\ldots P_d$ be all the distinct Points in $W(\Pi)$, then we have*

$$\dim_k (P_1, \ldots\ldots, P_d) = r;$$

in particular it must be

$$r \leqq d \leqq \deg.\ W(\Pi).$$

Since $P_1, \ldots\ldots, P_d$ are algebraic ove $k(\Pi)$, we have

$$\dim_k (P_1, \ldots\ldots P_d) \leqq \dim_k (\Pi) = r.$$

On the other hand let $\bar{P}_1, \ldots\ldots \bar{P}_r$ be r independent generic Points of C over k, then we can find a Point $\bar{\Pi}$ of L^r with a representative $(\bar{x})$ such that for every i it holds

$$\bar{\varphi}(\bar{P}_i) = \bar{x}\ \varphi_0\ (\bar{P}_i) + \bar{x}_1\ \varphi_2\ (\bar{P}_i) + \ldots\ldots + \bar{x}_r\ \varphi_r\ (\bar{P}_i) = 0;$$

then $\bar{P}_1, \ldots\ldots, \bar{P}_r$ are contained in $(\bar{\varphi})_0$, hence in $A + W(\bar{\Pi})$ (cf. th. 2); since A is rational over k, and since $\bar{P}_1, \ldots\ldots \bar{P}_r$ are not algebraic over k, they must be contained in $W(\bar{\Pi})$. It follows that $(\bar{P}_1, \ldots\ldots, \bar{P}_r)$ is a specialization of some $(P_{i1}, \ldots\ldots, P_{ir})$ over k, so that we have

$$\begin{aligned}\dim_k (P_1, \ldots\ldots, P_d) &\leqq \dim_k (P_{i1}, \ldots\ldots, P_{ir}) \\ &\leqq \dim_k (\bar{P}_1, \ldots\ldots, \bar{P}_r) = r.\end{aligned}$$

Now we shall show

THEOREM 3. *Let $B + W(\Pi)$ be a generic element over a field k_1 (containing k) of a linear series, defined over k_1; let Π be a generic Point over*

k_1 of a linear Subvariety L^s of L^r, defined over k_1, then

$$B+W(\bar{\Pi})$$

is a generic element over k_1 of some linear series of dimension s, which is also defined over k_1.

Proof. Let $(\bar{x})=(\bar{x}_0, \bar{x}_1, \ldots \bar{x}_r)$ $(\bar{x}_{i0}=1)$ be a representative of $\bar{\Pi}$ then we may set

$$\bar{x}_i=\sum_{j=1}^{s} a_i^{(j)} y_j + a_i^{(0)} \quad (i \neq i_0),$$

where $a_i^{(j)}$ $(0 \leqq j \geqq s,\ i \neq i_0)$ are elements of k_1, and y_i $(1 \geqq j \geqq s)$ are independent variables over k_1; above equation holds also for $i=i_0$, if we set

$$a_{i0}^{(j)}=0\ (1 \leqq j \leqq s),\ a_{i0}^{(0)}=1,\ y_0=1.$$

Now consider the Point Π_j in L_r with the "homogeneous coordinates"

$$(a_0^{(j)}, a_1^{(j)}, \ldots\ldots, a_r^{(j)}) \quad (0 \leqq j \leqq s),$$

and consider the s-dimensional projective space L'^s; then $(y_0, y_1, \ldots\ldots, y_s)$ can be considered as a representative of a generic Point Π' of L'^s over k_1. We put

$$\phi_j=\sum_{i=0}^{r} a_i^{(j)} \varphi_i \qquad (0 \leqq j \leqq s)$$

then $\phi_0, \phi_1, \ldots, \phi_s$ are linearly independent functions on C, all defined over k_1; we put further

$$\phi=y_0\, \phi_0+ y_1\, \phi_1+\ldots\ldots+y_s\, \phi_s,$$
$$\bar{\varphi}=\bar{x}_0\, \varphi_0 + \bar{x}_r\, \varphi_1 + \ldots\ldots + \bar{x}_r\, \varphi_r.$$

Let P be a generic Point of C over the field $k_1(\Pi')=k_1(\bar{\Pi})$, and consider a function Φ' on $L'^s \times C$, defined over k_1 by

$$\Phi'(\Pi' \times P)=\phi(P);$$

as in 6), there exists a rational C-divisor A' over k_1, and a Subvariety W'^s of $L'^s \times C$, defined over k_1, such that

$$(\Phi')_0\,(\Pi') = A' + W'(\Pi').$$

On the other hand, since $\phi(P)=\bar{\varphi}(P)$, we have

$$(\Phi')_0\,(\Pi')=(\phi)_0=(\bar{\varphi})_0=A-\{(\varphi)_\infty-(\bar{\varphi})_\infty\}+W(\bar{\Pi});$$

it follows

$$((\varphi)_\infty-(\bar{\varphi})_\infty)+A'+W'\ (\Pi')=A+\ W(\bar{\Pi})\ .$$

Since $(\varphi)_\infty-(\bar{\varphi})_\infty\succ 0$ and since W' (Π') has no Points in common with A, $((\varphi)_\infty-(\bar{\varphi})_\infty)+A'-A=\bar{A}$ is a positive C-divisor over k_1, and we have $W(\bar{\Pi})=\bar{A}+W'$ (Π') ; it follows

$$B+\ W(\bar{\Pi})=(\bar{A}+B)+W'\ (\Pi')\ ,$$

which proves our therem.

COROLLARY 1. *Let $B+W$ (Π) be a generic element over k_1 of a linear series, defined over k_1, and let P be a Point of W (Π), then $B+W$ (Π) is a generic element over k_1 (P) of some linear series of dimension $r-1$, which is defined over k_1 (P).*

Let $(x)=(x_0, x_1, \ldots\ldots x_r)$ be a representative of Π, then we have

$$x_0\ \varphi_0\ (P)+x_1\ \varphi_1\ (P)+\ldots\ldots+x_r\ \varphi_r\ (P)=0\ ;$$

and the linear equation φ_0 (P) $x_0+\varphi_1$ (P) $x_1+\ldots\ldots+\varphi_r$ (P) $x_r=0$ defines a linear Subvariety L^{r-1} of L^r with a generic Point Π over k_1 (P) .

Since we shall not use the following result, we omit its proof;

COROLLARY 2. *Let $\bar{P}$ be a Point of C, then the set of all elements of a linear series, which contains $\bar{P}$, forms also a linear series. Moreover if k_1 is a field of definition of the original linear series, the field k_1 $(\bar{P})$ is one for the 'derived' linear series.*

8) *Complete linear series.* In the following we shall consider only the linear series with positive fix-part. If a linear series is 'large' enough such as every positive C-divisor, which is linearly equivalent to some (then to all) element of the linear series, is already contained in it, such linear series is called to be *complete.* If G is an arbitrary element of a complete linear series on C, it is uniquely determined by G; so we may denote it by $|G|$. Now the following proposition is an immediate consequence of the "principle of conservation of number".:

PROPOSITION 2. *Let P be a generic Point of C over its field of definition k, and let φ be a non-constant function on C, defined by*

$$z=\varphi\ (P)$$

over k, then we have

$$\text{deg.}\ (\varphi)_0=\text{deg.}\ (\varphi)_\infty=[k\ (P):k\ (z)]\ .$$

Now let G be a positive C-divisor, which is rational over a field k, then the set of functions on C satisfying

$$(\varphi) \succ -G$$

form a K-module $L(G)$, where K means the "universal domain" of our algebraic geometry. Let

$$\varphi_0, \varphi_1, \ldots\ldots \varphi_r \ (\varphi_0=1)$$

be $(r+1)$ linearly independent functions in $L(G)$, all defined over k; we shall apply the results in 6) and 7) to this set of functions.

Since $\varphi_0=1$, we have $A=0$ (cf. cor. 1 to th. 2); since $G \succ (\varphi)_\infty$, we have deg. $G \geq$ deg. $(\varphi)_\infty$, so that

$$r \leqq \text{deg. } W(\Pi) = \text{deg. } (\varphi)_0 = \text{deg. } (\varphi)_\infty \leqq \text{deg. } G.$$

It follows that there exists a maximal set of such functions; we may assume that our set is already maximal; then this set forms a base for the K-module $L(G)$. Now set

$$B = G - (\varphi)_\infty \succ 0,$$

and consider the linear series on C with the generic element

$$B + W(\Pi)$$

over k; let $\bar{G}$ be a positive C-divisor, which is linearly equivalent to G. Then there exists a function $\bar{\varphi}$ on C such that $(\bar{\varphi}) = \bar{G} - G$. we can find a Point $\bar{\Pi}$ of L_r with a representative $(\bar{x})$ such that

$$\bar{\varphi} = \bar{x}_0 \ \varphi_0 + \bar{x}_1 \ \varphi_1 + \ldots\ldots + \bar{x}_r \ \varphi_r \ ;$$

by th. 2 we have then

$$W(\bar{\Pi}) + B = (\bar{\varphi}) + (\varphi)_\infty + B = \bar{G}$$

Thus the following result is obtained:

THEOREM 4. *The complete linear series is uniquely determined by its arbitrary element; for any positive C-divisor G, there exists a complete linear series, which contains G.*

We shall denote the dimension of $|G|$ by $l(G)$; we have always

$$\text{deg. } G - l(G) \geqq 0 ;$$

we shall show

LEMMA 1. *Let G_0 and G be two positive C-divisors satisfying $G \succ G_0$, then we have*

$$l(G_0) + \text{deg. } (G - G_0) \geqq l(G) \geqq l(G_0)$$

Proof. Since $G \succ G_0$, we have $-G_0 \succ -G$ so that $L(G) \supset L(G_0)$, $l(G) \geqq l(G_0)$. Assume now that $l(G) > l(G_0) + \text{deg. } (G - G_0)$ and take a field of definition k of C, over which both G_0 and G are rational; then a generic element of $|G|$ over k contains at least deg. $(G-G_0)+(l(G_0)+1)$ independent generic Points of C over k. We fix $l(G_0)+1$ of them, and specialize deg. $(G-G_0)$ of them to $G-G_0$ over k; the rest of Points among the generic element of $|G|$ over k has at least one specialization over that specialization. It follows that there exists a positive C-divisor G_0', which contains at least $l(G_0)+1$ independent generic Points of C over k, and which is linearly equivalent to G_0; this is impossible.

LEMMA 2. *There exists an integer g' such that, for every positive C-divisor G, we have*

$$\text{deg. } G-l(G)\leqq g'.$$

Proof. By a remark in 1), § I (and in the proof of prop. 2, § I), we may use the same notation as in that place. Let $\varphi_x, \varphi_{y1},\dots\dots, \varphi_{yn}$ be $n+1$ functions on C, defined over k_0 by

$$x=\varphi_x(P),\ y_1=\varphi_{y1}(P),\dots\dots,\ y_n=\varphi_{yn}(P),$$

there exists a positive integer m_0 such that

$$(\varphi_{y1}),\dots\dots,(\varphi_{yn})\succ-(m_0+1)\cdot(\varphi_x)_\infty .$$

Then for any integer m greater than m_0, the functions

$$\varphi_x^e\cdot\varphi_{yi}\ (0\leqq e<m-m_0;\ 1\leqq i\leqq n)$$

are linearly independent and belong to $L(m(\varphi_x)_\infty)$; it follows

$$l(m(\varphi_x)_\infty)\leqq n(m-m_0).$$

Set $g'=n\ m_0$; we shall show that g' satisfies the requirement of our lemma.

A positive C-divisor G being given, take so large as

$$n\cdot m\geqq n\cdot m_0+\deg\cdot G;$$

then a generic element of $|m(\varphi_x)_\infty|$ contains at least deg. G independent generic Points of C over k_0, so that that there exists a positive C-divisor R such that

$$m(\varphi_x)_\infty\equiv G+R.$$

It follows

$$\begin{aligned}\text{deg. } G-l(G) &\leqq \text{deg. } (G+R)-l(G+R)\ (cf.\ \text{lem. } 1)\\ &=\text{deg. } (m(\varphi_x)_\infty)-l(m(\varphi_x)_\infty)\\ &\leqq n.\ m_0=g',\end{aligned}$$

as asserted.

Now the following definition of the "genus" of a Curve is due to WEIERSTRASS:

DEFINITION If. G runs over all positive C-divisors, the non-negative integer deg. - $l(G)$ *has a maximal value g, which is called the genus of C; every positive C-divisor G, which attains g, is called to be non-special. Moreover a Curve C with $g=0$ is called to be rational; a Curve C with $g=1$ is called elliptic.*

We shall use the celebrated *RIEMANN-ROCH's theorem* in the following special form:

PROPOSITION 3. *If G is non-special, every C-divisor G such that $G \succ G_0$ is also non-special. Thus every positive C-divisor G such that*

$$\text{deg. } G \geqq \text{deg. } G_0 + g$$

is non-special.

Proof. Since we have

$$g \geqq \text{deg. } G - l(G) \geqq \text{deg. } G_0 - l(G_0) = g,$$

G is non-special. If deg. $G \geqq$ deg. $G_0 + g'$, we have $l(G) \geqq$ deg. G_0, so that $|G|$ contains a G' with $G' \succ G_0$; thus G is non-special, which completes the proof.

Now for every non-special C-divisor G_0, it holds

$$\text{deg. } G_0 = l(G_0) + g \geqq g;$$

we shall see later (in § V) that there exists a non-special C-divisor G_0, which attains this minimal degree g. If we assume this fact, we have the following corollary:

COROLLARY. If a positive C-divisor G satisfies

$$\text{deg. } G > 2g-2$$

we have always

$$l(G) = \text{deg. } G - g.$$

In fact if deg. $G \geqq 2g$, our relation follows from prop. 3; if deg $G = 2g-1$, and if deg. $G - l(G) < g$, we have $l(G) \geqq g$, so that G is equivalent to a non-special C-divisor, this is a contradiction.

In particular every positive divisor on the rational or elliptic Curve is non-special; clearly the converse is also true.

III. Faithful Representation of the Multiplication Ring.

9) *Primitive divisor.* Let C be a Curve and G a positive C-divisor, then a Point $\bar{P}$ of G is contained in the fix-part of $|G|$ if and only if we have

$$\mathrm{L}(G) = \mathrm{L}\ (G-\bar{P})\,,\ \text{i. e. } l\ (G) = l\,(G-\bar{P})\,.$$

In fact if $\mathrm{L}\ (G)=\mathrm{L}\ (G-\bar{P})$, every function φ in $\mathrm{L}\ (G)$ satisfies $-P \succ (\varphi)_\infty$ so that we have

$$G-\mathrm{L.\ C.\ M.}\ (\varphi) \succ -G\{(\varphi)_\infty\} \succ \bar{P}\,,$$

and conversely; this proves our assertion (cf. 8), II).

Now a positive C-divisor G is called *primitive*, if $|G|$ has no fix-part; every positive C-divisor is surely primitive only if its degree is sufficiently large (cf. prop. 31 § II).

LEMMA. Let $W_1, \ldots\ldots, W_n$ be Subvarieties of a projective space L^r, none of which coincide with L^r, then for any non-finite field k, we can find a rational Point Π of L^r over k, which does not lie on

$$W_1 \cup \ldots\ldots \cup W_n.$$

This is a consequence of the following statement, which can be proved by induction on r:

Let $F(x)=F(x_1, \ldots\ldots, x_r) \neq 0$ be a polynomial with coefficients in some field, then for any non-finite field k, there exists a set of r quantities $(\bar{x}) = (\bar{x}_1, \ldots\ldots, \bar{x}_r)$ of k such that $F\ (\bar{x}) \neq 0$.

The following proposition is useful in some occasion:

PROPOSITION 1. Let G be a primitive C-divisor and $\{\bar{P}_1, \ldots\ldots, \bar{P}_m\}$ a set of m Points of C, then there exists a positive C-divisor $\bar{G}$, which is linearly equivalent to G, and which does not contain any of the $\bar{P}_i$ $(1 \leqq i \leqq m)$. Moreover if

$$\deg.\ G \equiv 0\ (\mathrm{mod.}\ p)\ ,$$

$\bar{G}$ can be taken such that it consists of distinct Points. Thereby if G is rational over a non-finite field of definition k of C, $\bar{G}$ can be taken as rational over k.

Proof. Let $W\ (\Pi) = Q_1 + \ldots\ldots + Q_d$ (d=deg. G) be a generic element of $|G|$ over the field $k\,(\bar{P}_1, \ldots\ldots, \bar{P}_m)$ then $W\ (\Pi)$ satisfies all the requirements except the last one. Let U^r $(r=l(G))$ be the Locus of the Point $\Pi\times Q_1\times\ldots\ldots\times Q_d$ over $\bar{k}$

$$L^r\times C^{(d)}=L^r\times C\times\ldots\ldots\times C\ ((d+1)\text{- factors});$$

then U is not cotained in any of the Varieties $L^r\times Z_{ij}$ and $L^r\times Z_\pi$, if d $\not\equiv 0$ (mod. p), where

$$Z_{ij}=C^{(j-1)}\times \bar{P}_i\times C^{(d-j)}\ (1\leq i\leq m;\ 1\leq j\leq d),$$

and where Z_π is the Subvariety of $C^{(d)}$, which corresponds to $\Delta\times C^{(d-2)}$ by the possible interchange π of factors in $C^{(d)}$. Let $W_1,\ldots\ldots, W_n$ be the Subvarieties of L^r, which are the projections on L^r of the components of the intersections $(L^r\times Z_{ij})\cap U$, and $(L^r\times Z_\pi)\cap U$, if $d\not\equiv 0$ (mod. p); then $W_1,\ldots\ldots W_n$ are all different from L^r. It follows that there exists a rational Point $\bar{\Pi}$ of L^r, which does not lie on $W_1\cup\ldots\ldots\cup W_n$; then the rational C-divisor $W(\bar{\Pi})$ over k satisfies all the requirements.

COROLLARY 1. *Let $\{\bar{P}_1,\ldots\ldots, \bar{P}_m\}$ be a set of m Points of C, then we can find a C-divisor A_i, which does not contain any of the $\bar{P}_i$ and which satisfies*

$$A_i\equiv\bar{P}_i\ (1\leq i\leq m).$$

Let $\bar{Q}$ be a Point of C other than $\bar{P}_i$, then for a sufficiently large m, $\bar{P}_i+m\bar{Q}$ is primititive; thus there exists a C-divisor $\bar{G}$, which does not contain any of the $\bar{P}_i$ such that $\bar{G}_i\equiv\bar{P}_i+m\bar{Q}$; set

$$A_i=\bar{G}_i-m\cdot\bar{Q}.$$

COROLLARY 2. *Let G be a rational C-divisor and $\bar{Q}$ a rational Point of C over a non-finite field of definition k_1 of C, then we can find a positive integer m and a positive C-divisor $\bar{G}$, consisting of distinct Points, which is rational over k_1 and satisfies*

$$G+m\bar{Q}\equiv\bar{G}.$$

In fact let $G=G_1-G_2$ $(G_1, G_2\succ 0)$ be the reduced expression of G, then we can take m_0 as

$$l(G_1+m_0\bar{Q})\geq\deg.\ G_2;$$

there exists a positive C-divisor G_3, which we can assume to be rational over k_1, such that

$$G_1+m_0\bar{Q}\equiv G_2+G_3, \text{ i. e. } G+m_0\bar{Q}\equiv G_3 .$$

Now if m_1 is sufficiently large, $G_3+m_1\bar{Q}$ is primitive and we may assume that deg $\cdot (G_3 + m_1\bar{Q}) \not\equiv 0$ (mod. p); there exists a rational C-divisor $\bar{G}$ ($\bar{G}\succ 0$) over k_1 consisting of distinct Points such that $G_3+m_1\bar{Q} = \bar{G}$; put $m=m_0+m_1$.

10) *Properties of the correspondences.* In the following we shall consider the correspondences between two Curves C_1 and C_2. The following proposition is known as the "Homomorphiesatz" in HASSE's school:

PROPOSITION 2. Let X be any correspondence, then for every C_1-divisor A such that $A\equiv 0$, we have $X(A)\equiv 0$.

Proof. By the linearity of $X(A)$ in X, we may assume that X is irreducible. If X is a d. l.-correspondence, we have $X(A)=0$ for every C_1-divisor A; if X is of the form $C_1\times\bar{P}$ with some Point $\bar{P}$ of C_2, we have

$$X(A)=\mathrm{pr}C_2((C_1\times\bar{P})\cdot(A\times C_2))=(\deg. A)\bar{P}=0 ;$$

and if $l(X)$, $r(X)\neq 0$, we have $X(A)\equiv 0$ by the postulates (A), (C') of the equivalence theory.

It follows from prop. 2 and from

$$\deg. X(A)=l(X)\deg. A ,$$

where A is an arbitary C-divisor, that X induces an operator of the divisor-class group of degree zero on C_1 into the similar group on C_2.

Now we shall prove the "Additionssatz":

PROPOSITION 3. Let X be any correspondence with valence zero, then for every C_1-divisor A of degree zero, we have $X(A)\equiv 0$.

Proof. At first we may assume that X does not contain any d. l.-correspoudence; since X is of valence zero, we may put

$$X=(\varphi)+\textstyle\sum_i a_i(\bar{P}_i\times C_2)+(C_1\times B).$$

Let A be a C_1-divisor, which does not contain any of the $\bar{P}_i$, then the intersection-product $(\varphi)\cdot(A\times C_2)$ is defined on $C_1\times C_2$. It follows from the postulates (C) of the equivalence theory that

$$X\ (A)=\mathrm{pr}_{C_2}\ ((\varphi)\cdot(A\times C_2))+(\deg.\ A)\cdot B$$
$$\equiv(\deg.\ A)\cdot B\,.$$

Now let A_i be the C_1-divisor as stated in cor. 1 to prop. 1, then by prop. 2 and by what we have proved above, we have

$$X\ (\bar{P}_i)\equiv X\ (A_i)\equiv B\ ;$$

so that $X\ (A)\equiv(\deg.\ A)\cdot B$ for every C_1-divisor A, and this completes our proof.

We conclude from prop. 3 that every correspondence with valence zero induces a zero operator of the divisor-class group of degree zero on C_1 into the similar group on C_2; we shall prove the converse:

PROPOSITION 4. *Let X be any correspondence such that $X\ (\bar{P})\equiv X\ (\bar{P}_0)$ for infinitely many Points $\bar{P}$ and for fixed Point $\bar{P}_0$ of C_1, then X is a correspondence with valence zero.*

Proof. Let k be a common field of definition of C_1 and C_2, over which both $\bar{P}_0$ and X are rational. We put

$$Y=X-(C_1\times X(\bar{P}_0)),$$

then Y is also rational over k such that $Y(\bar{P})\equiv 0$ for infinitely many Points $\bar{P}$ of C_1; we shall show that Y is a correspondence with valence zero:

If there exists a generic Point P of C_1 over k among the $\bar{P}$, we can find a function θ on C_2, defined over the field $k\ (P)$ such that $Y(P)=(\theta)$ Let Q be a generic Point of C_2 over $k\ (P)$, then we can find a function φ on $C_1\times C_2$, defined over k, by $\varphi\ (P\times Q)=\theta\ (Q)$; we have

$$(\varphi)\cdot(P\times C_2)=P\times(\theta)=Y\cdot(P\times C_2)\ .$$

It follows that $(\varphi)-Y$ is a d. l.-correspondence, and Y is a correspondence with valence zero; this is the trivial case of our proposition (and later we shall use only this case). In the general case, all the Points $\bar{P}$ are algebraic over k; taking if necessary a suitable extension of k, we may assume that there exists a rational Point $\bar{Q}$ of C_2 over k. Let P be a generic Point of C_1 over k, then since the field $k_1=k\ (P)$ is not a finite field, we can find a positive integer m and a positive C_2-divisor $Q_1+\dots\dots+Q_m$, which is rational over $k\ (P)$ such that

$$Y(P)+m\bar{Q}\equiv Q_1+\dots\dots+Q_m\,,$$

and such that

$$Q_i \neq Q_j \ (i \neq j),\ Q_i \neq Q \ (1 \leq i \leq m)$$

(cf. cor 2 to prop. 1). There exists a correspondence Z, which is rational over k and which does not contain any d. l.-correspondence such that

$$Z(P) = Q_1 + \cdots\cdots + Q_m;$$

now if we can show

$$Z(P) \equiv m\bar{Q},$$

we have $Y(P) \equiv 0$, and by the trivial case, Y is a correspondence with valence zero.

In order to prove that linear equivalence, let $\varphi_0, \varphi_1, \cdots\cdots, \varphi_r$ be $r+1$ functions on C_2, which are defined over k and which form a base for $L(m\bar{Q})$. If $Z(P) \not\equiv mQ$, we have rank $\| \varphi_i(Q_j) \| > r$ and the set of $m(r+1)$ quantities $\| \varphi_i(Q_j) \|$ has dimension 1 over k. Let $\bar{P}$ be a specialization of P over any specialization

$$\| \varphi_i(Q_j) \| \rightarrow \| \overline{\varphi_i(Q_j)} \|$$

with reference to k, then we have $\overline{\varphi_i(Q_j)} = \varphi_i(\bar{Q}_j)$, where

$$Z(\bar{P}) = \bar{Q}_1 + \cdots\cdots + \bar{Q}_m$$

Since every specialization $\| \varphi_i(Q_j) \| \rightarrow \| \varphi_i(\bar{Q}_j) \|$ over k such that rank $\| (\varphi_i(\bar{Q}_j) \| \leq r$ is not generic, whence algebraic over k, such specializations are in finite number. By a similar reason, the specializations of P over every one of such specializations are in finte number. Since every Point $\bar{P}$ of C_1 such that

$$Z(\bar{P}) \equiv m\bar{Q}, \text{ i. e. } Y(\bar{P}) \equiv 0$$

is one of such specializations, there are only a finite number of such $\bar{P}$; this is a contradiction.

Now if we consider only the correspondences between a Curve C and itself, i. e. the elements of the ring $\mathfrak{X}$, and if we note that the product in $\mathfrak{X}$ induces the product of the corresponding operators on the divisor-class group of degree zero on C, we have proved again that $\mathfrak{X}_0$ is a two-sided ideal in $\mathfrak{X}$ together with the following result:

THEOREM. The multiplication ring attached to a Curve C can be represented faithfully in the operator-ring of the divisor-class group of degree zero on C.

In the classical case this representation is explicitely obtained by the *Abelian integrals of the first kind* attached to C.

IV. A Generalization of Schubert's Formula.

11) *A general remark.* Let P be a simple Point of a Variety V, defined over a field k, then P determines uniquely a prime rational V-cycle with a generic Point P over k; we shall denote this V-cycle by $(P; k)$. Moreover it is sometimes convenient to generalize the symbol

$$[k(x):k]$$

for non-algebraic set of quantities (x) over k, as being zero.

PROPOSITION 1. *Let U, V be two Varieties, defined over a field k, then for every simple Point $P\times Q$ of $U\times V$, it holds*

$$\mathrm{pr}_U(P\times Q;k)=[k(P,Q):k(P)]\cdot(P;k).$$

Proof. The set of all generic specializations of the Point P over k splits into several classes over $\bar{k}$: two such specializations are put into the same class if and only if they are also generic specializations of each other over $\bar{k}$. Take a representative P^{σ} from each one of these classes; then the Varieties $(P^{\sigma};\bar{k})$ are exactly all the distinct conjugates of the Variety $(P;\bar{k})$ over k. Let Q^{σ} be a geniric specialization of Q over the specialization $P\to P^{\sigma}$ with reference to k; then the set of all generic specializations of Q^{σ} over $k(P^{\sigma})$ splits, as above, into several classes over $\bar{k}(P^{\sigma})$; take a representative $Q^{\sigma\tau}$ from each one of them. It is easily seen that every generic specialization $P'\times Q'$ of $P\times Q$ over k is a generic specialization of some $P^{\sigma}\times Q^{\sigma\tau}$ over $\bar{k}$; so the Varieties $(P^{\sigma}\times Q^{\sigma\tau};\bar{k})$ are exactly all the distinct conjugates of the Variety $(P\times Q;\bar{k})$ over k. It follows

$$(P;k)=[k(P):k]_{\iota}\cdot\textstyle\sum_{\sigma}(P^{\sigma};\bar{k}),$$

$$(P\times Q;k)=[k(P,Q):k]_{\iota}\cdot\textstyle\sum_{\sigma\tau}(P^{\sigma}\times Q^{\sigma\tau};\bar{k})$$

so that we have

$$\mathrm{pr}_U(P\times Q;k)=[k(P,Q):k]_{\iota}$$

$$\cdot\textstyle\sum_{\sigma,\tau}[\bar{k}(P^{\sigma},Q^{\sigma\tau})]:\bar{k}(P^{\sigma})](P^{\sigma};\bar{k}).$$

Now if it holds $[k(P,Q):k(P)]=0$, Q is not algebraic over $k(P)$; since $P^{\sigma}\times Q^{\sigma\tau}$ is a generic specialization of $P\times Q$ over k, $Q^{\sigma\tau}$ is not algebraic over $k(P^{\sigma})$, hence also over its algebraic extension $\bar{k}(P^{\sigma})$. It follows $[\bar{k}(P^{\sigma},Q^{\sigma\tau}):\bar{k}(P^{\sigma})]=0$ for every σ, τ, so that we have $\mathrm{pr}_U(P\times Q;k)$

$=0$. On the other hand if Q is algebraic over $k(P)$, we have

$$[k(P, Q) : k(P)] = [k(P^{\sigma}, Q^{\sigma\tau}) : k(P^{\sigma})]$$
$$= \sum_{\tau} [\bar{k}(P^{\sigma}, Q^{\sigma\tau}) : \bar{k}(P^{\sigma})]_s \; [k(P, Q) : k(P)]_i$$
$$= ([k(P, Q) : k]_i / [k(P) : k]_i) \cdot \sum_{\tau} [\bar{k}(P^{\sigma}. Q^{\sigma\tau}) : \bar{k}(P^{\sigma})];$$

so that we have

$$\mathrm{pr}_U(P \times Q ; k) = [k(P, Q) : k(P)] \; [k(P) : k]_i \; \sum_{\sigma} (P^{\sigma} ; k)$$
$$= [k(P, Q) : k(P)] \cdot (P ; k);$$

which completes our proof.

We note that prop. 1. is nothing but the definition of the algebraic projection of $(P \times Q ; k)$, if this prime rational cycle over k is reduced to a Variety; in the same manner some other formulae in WEIL's book remain to hold, if we replace the Varieties by the prime rational cycles (over some field).

12) *Algebraic series.* Let C_1, C_2 be two Curves, defined over a field k; since C_1 plays merely a subordinate part, C_2 will be denoted also as C. Let X be a correspondence between C_1 and C_2, which does not contain any d. l.-correspondence, and which is rational over k; let P be a generic Point of C_1 over k, then the set of all the specializations of the C-divisor $X(P)$ over k constitutes an *algebraic series on* C with a *generic element* $X(P)$ over k; the integer

$$m = \deg. X(P) = 1(X)$$

is called the degree of the algebraic series. The linear series of dimension 1 on C is a special type of the algebraic series. In the following we shall restrict our algebraic series by the following condition:

(A I) *X is a positive divisor on $C_1 \times C_2$, and if we put the Point-set $\{Q_1, \ldots\ldots, Q_m\}$ is composed of distinct Points.*

It follows from $(A\ I)$ that X is of the form

$$X = X_1 + \ldots\ldots + X_d,$$

where X_i $(1 \leq i \leq d)$ are mutually distinct prime rational divisors on $C_1 \times C_2$ (which are all different from d. l.-correspondences) such that the field $k(P, Q_1, \ldots\ldots, Q_m)$ is separable over $k(P)$; conversely this condition implies $(A\ I)$.

Now at first we shall consider the special case, where $d=1$; in this case it holds

$$X=(P\times Q_1\,;\ k),$$

and the Point-set $\{Q_1,\ldots\ldots, Q_m\}$ is a complete set of conjugates of Q_1 over $k\,(P)$ so that we have

$$[k\,(P,\,Q_1):k\,(P)]=m\,.$$

Let $\mathfrak{G}$ be the Galois group of $k\,(P,\,Q_1,\ldots\ldots, Q_m)$ over $k\,(P)$; we shall consider $\mathfrak{G}$ as a (transitive) permutation group on $\{Q_1\ldots\ldots, Q_m\}$.

The elements σ in $\mathfrak{G}$ such that $Q_1^{\sigma}=Q_1$ constitute a dubgroup $\mathfrak{g}$ of $\mathfrak{G}$ which is a Galois group of $k\,(P,\,Q_1,\ldots\ldots, Q_m)$ over $k\,(P,\,Q_1)$. Every element of $\mathfrak{G}$ induces an automorphism of the field $k\,(Q_1,\,\ldots\ldots, Q_m)$ over k; Let L be its "invariant subfield", i. e. the largest subfield of $k\,(Q_1,\ldots\ldots, Q_m)$, every element of which is invariant by all automorphism of $\mathfrak{G}$. Then $\mathfrak{G}$ and $\mathfrak{g}$ are the Galois groups of $k\,(Q_1,\ldots\ldots, Q_m)$ over L and over $L\,(Q_1)$, respectively; it follows from this the following identities:

$$[k\,(Q_1,\ldots\ldots, Q_m):L]=[k\,(P,\,Q_1,\ldots\ldots, Q_m):k\,(P)]=(\mathfrak{G}),$$
$$[L\,(Q_1):L]=[k\,(P,\,Q_1):k\,(P)]=(\mathfrak{G}:\mathfrak{g})=m$$

We shall denote the common value

$$[k\,(P,\,Q_1,\ldots\ldots, Q_m):k\,(Q_1,\ldots\ldots, Q_m)]$$
$$=[k\,(P,\,Q_1):L\,(Q_1)]=[k(P):L]$$

by $\mu\,(X)$, and we put

$$[L\,(Q_1):k\,(Q_1)]=\nu\,(X)\,;$$

then we have

$$\mu(X)\cdot\nu(X)=[k(P,\,Q_1):k(Q_1)]=\mathrm{r}(X)$$

(cf. prop. 1). We note that $\mu\,(X)$ may be zero, but $\nu(X)$ is always a positive integer; we shall call $\nu(X)$ the *index* of the algebraic series, determined by X on C.

Now in the general case, we put

$$X_i\,(P)=Q_{i1}+\ldots\ldots+Q_{im_i}\ \ (1\leqq i\leqq d),$$

then we have

$$Q_1+\ldots\ldots+Q_m=\sum_{i=1}^{d}\sum_{j=1}^{m_i}Q_{ij}$$
$$m=m_1+\ldots\ldots+m_d.$$

Let $\mathfrak{G}$ be the Galois group of the field $k(P,\,Q_1\ldots\ldots, Q_m)$ over $k(P)$, and let $\mathfrak{G}_i$ be the Galois group of the subfield $k(P,\,Q_{i1},\ldots\ldots, Q_{im_i})$ of $k(P,\,Q_1,\ldots\ldots, Q_m)$ over $k(P)$ $(1\leqq i\leqq \mathrm{d})$. Let L be the inveriant subfield of $k(Q_1,\ldots$

Q_m) by $\mathfrak{G}$, and put

$$[k(P):L]=\mu(X).$$

Now $\mu(X_i)$ and $\nu(X_i)$ being defined for X_i as before $(1 \leqq i \leqq d)$, we shall assume that

$$(A\ II) \quad \mu(X)=\mu(X_i) \quad (1 \leqq i \leqq d);$$

this condition is equivalent to the assumption that L is at the same time the iuvariant subfield of $k(Q_{i1},\dots\dots, Q_{im_i})$ by $\mathfrak{G}_i$ $(1 \leqq i \leqq d)$. In this case we shall define the *index* of the algebraic series, determined by X on C, by

$$\nu(X)=\nu(X_1)+\dots\dots+\nu(X_d).$$

13) *Multiple divisor of the algebraic series.* Conserving the notations in 12), we shall consider the cycle U^1 on $C^{(m)}$:

$$U=\sum(Q_1^{\pi}\times\dots\dots\times Q_m^{\pi};\ k),$$

where π runs over all permutations of m Points $Q_1,\dots\dots, Q_m$ mudulo $\mathfrak{G}$; we put

$$S=\mathrm{pr}_{C\times C}\ (\text{first and second faetors})\ (U).$$

Now the m_j Points $Q_{j1},\dots\dots, Q_{jm_j}$ are, in general, no more conjugate to each other over $k(P, Q_{i1})$, but they split into several complete sets of conjugates over $k(P, Q_{i1})$:

$$\{\overset{(i)}{Q}_{j\ 1+a_{\rho-1}},\dots\dots, \overset{(i)}{Q}_{ja_\rho}\}\ (1 \leqq \rho \leqq d_{ij}),$$

where $a_0=0$ $(1 \leqq i, j \leqq d)$. Thereby if $i=j$, Q_{i1} has only one conjugate, i. e. Q_{i1} itself, over $k(P, Q_{i1})$; we put

$$Q_{ia_1}^{(i)}=Q_{i1} \quad (1 \leqq i \leqq d).$$

Let $\mathfrak{G}^{(i)}$be the Galois group of $k(P, Q_2,\dots\dots, Q_m)$ over $k(P, Q_{i1})$ $(1 \leqq i \leqq d)$, then U can be writtn as

$$U=\sum_{i,\ \pi i}(Q_{i1}\times Q_{11}^{\pi i}\times\dots\dots\times \hat{Q}_{i1}\times\dots\dots\times Q_{dm_d}^{\pi i};\ k),$$

where π_i runs over all permutations of $(m-1)$ Points

$$Q_{11},\dots\dots, \hat{Q}_{i1},\dots\dots, Q_{dm_d}$$

modulo $\mathfrak{G}^{(i)}$($\wedge$ means to omit that element). Let $\mathfrak{G}^{ij\rho}$ and $\mathfrak{G}^{(i\rho)}$ be the Galois groups of $k(P, Q_1,\dots\dots, Q_m)$ over $k(P, Q_{i1}, Q_{ja_\rho}^{(i)})$ and over $k(P, Q_{i1}, Q_{ia_\rho}^{(i)})$ respectively, then we have

$$\sum_{\pi i}(Q_{i1}\times Q_{11}^{\pi i}\times\dots\dots\times \hat{Q}_{i1}\times\dots\dots\times Q_{dmd}^{\pi i};\ k)$$

$$=\sum_{j\neq i,\rho,\pi ij\rho}(Q_{i1}\times Q^{(i)}_{ja\rho}\times Q^{\pi ij\rho}_{11}\times\ldots\times\hat{Q}_{i1}\times\ldots\hat{Q}^{(i)}_{ja_\rho}\times\ldots\times Q^{\pi ij\rho}_{dmd};\ k)$$

$$+\sum_{\rho\neq i,\ \pi i\rho}(Q_{i1}\times Q^{(i)}_{ia\rho}\times Q^{\pi i\rho}_{11}\times\ldots\times\hat{Q}_{i1}\times\ldots\times\hat{Q}^{(i)}_{ia}\times\ldots\times Q^{\pi i\rho}_{dmd};\ k)$$

Since we have

$$(\mathfrak{G}:\mathfrak{G}^{(ij\rho)})=[k\,(P,\ Q_{i1},\ Q^{(i)}_{ja_\rho}):k\,(P)],$$

$$(\mathfrak{G}:\mathfrak{G}^{(i\rho)})=[k(P,\ Q_{i1},Q^{(i)}_{ia\rho}):k(P)],$$

it follows from prop. 1 that

$$S=\sum_{i,j\neq i,\rho}[k(Q_1,\cdots,Q_m):k(Q_{i1},\ Q^{(i)}_{ja\rho})]$$

$$\cdot\{(m-2)!/(\mathfrak{G}^{(ij\rho)}:1)\}(Q_{i,1}\times Q^{(i)}_{ja_\rho};\ k)$$

$$+\sum_{i,\ \rho\neq 1}[k\ (Q_1,\cdots,Q_m):k(Q_{i1},\ Q^{(i)}_{ia_\rho})]$$

$$\cdot\{(m-2)!/(\mathfrak{G}^{(i\rho)}:1)\}\cdot(Q_{i1}\times Q^{(i)}_{ia_\rho};\ k)$$

$$=\{(m-2)!/\mu(X)\}\sum_{i,\ j\neq i,\ \rho}[k(P,\ Q_{i,1}\ Q^{(i)}_{ja_\rho}):k(Q_{i1},\ Q^{(i)}_{ja\rho})]$$

$$\cdot(Q_{i1}\times Q^{(i)}_{ja_\rho};\ k).$$

$$+\{(m-2)!\sum_{i,\ \rho\neq 1}[L(Q_{i1},\ Q^{(i)}_{ia_\rho}):k(Q_{i1},\ Q^{(i)}_{ia_\rho})](Q_{i1}\times Q^{(i)}_{ia\rho};\ k)$$

$$=\{(m-2)!/\mu(X)\}\sum_{i,\ j\neq i,\ \rho}\mathrm{pr}_{C\times C}(P\times Q_{i1}\times Q^{(i)}_{ja_\rho};\ k)$$

$$+(m-2)!\ \sum_i 1/(m_i-2)!\ S_i,\ .$$

$$S_i=(m_i-2)!\ \sum_{\rho\neq 1}[L(Q_{i1},\ Q^{(i)}_{ia\rho}):k\ (Q_{i1},\ Q^{(i)}_{ia\rho})](Q_{i1}\times Q^{(i)}_{ia\rho};\ k)$$

and where the first sum must be omited if $\mu(X)=0$; S will be reduced to S, if X is reduced to X $(1\leq i\leq d)$.

Let R be a generic Point of C over $k(P)$, then we have

$$(P\times Q_{i1}\times R;\ k)\cdot(P\times R\times Q_{j1},\ k)$$

$$=\sum_\rho x_\rho\ (P\times Q_{i1}\times Q^{(i)}_{ja\rho};\ k)$$

with $x_\rho\geq 1$; taking the algebraic projection on $C_1\times C_2$, we get

$$m_jX_i\ (C_1\times C_2)=\sum_\rho x_\rho\ (a_\rho-a_{\rho-1})\cdot X_i$$

(cf. prop. 1), so that

$$m_j=\sum_\rho x_\rho\ (a_\rho-a_{\rho-1})\geq\sum_\rho\ (a_\rho-a_{\rho-1})=m_j;$$

since $a_\rho>a_{\rho-1}$ we have $x_\rho=1$ for every ρ. It follows

$$\mathrm{pr}_c\ (\sum_\rho\{\mathrm{pr}_{c\times c}\ (P\times Q_{i1}\times Q^{(i)}_{ja\rho};\ k\}\cdot\Delta)$$

$$=\mathrm{pr}_c\{\sum_\rho(P\times Q_{i1}\times Q^{(i)}_{ja\rho};\ k)\cdot(C_1\times\Delta)\}$$

$$=\mathrm{pr}_c[\{(P\times Q_{i1}\times R;\ k)\cdot(P\times R\times Q_{j1};\ k)\}\cdot(C_1\times\Delta)]$$

$$=\mathrm{pr}_c[(P\times Q_{i1}\times R;\ k)\cdot\{(P\times R\times Q_{j1};\ k)\cdot(C_1\times\Delta)\}]$$

$$=\mathrm{pr}_c\{(P\times Q_{i1}\times R;\ k)\cdot(P\times Q_{j1}\times Q_{j1};\ k)\}$$

$$=\mathrm{pr}_c\{(P\times Q_{i1};\ k)\cdot(P\times Q_{j1};\ k)\}$$

$$=\mathrm{pr}_c(X_i\cdot X_j);$$

thus we have

$$\mathrm{pr}_c\,(S\cdot\Delta)=\{(m-2)\,!\,/\mu(X)\}\cdot\textstyle\sum_{i,\,j\neq i}\mathrm{pr}_c\,(X_i\cdot X_j)$$
$$+(m-2)\,!\,\textstyle\sum_i 1/(m_i-2)\,!\cdot\mathrm{pr}_c(S_i\cdot\Delta).$$

Now the C-divisor

$$J=\textstyle\sum_{i,\,j\neq i}\mathrm{pr}_c\,(X_i\cdot X_j)+\sum\ \mu(X)/(m_i-2)\,!\cdot\mathrm{pr}_c(S_i\cdot\Delta)$$

may be called the *multiple divisor* of the algebraic series, difined by X on C; if X is reduced to X_i, J will be reduced to J_i;

$$J_i=\{\mu(X_i)/(m-2)\,!\,\}\mathrm{pr}_c\,(S_i\cdot\Delta),$$

so that we may write

$$J=\textstyle\sum_{i,\,j\neq i}\mathrm{pr}_c(X_i\cdot X_j)+\sum_i J_i.$$

14) *Jacobian divisor.* We shall consider a special case, where

$$d=1,\quad \nu(X)=1\,;$$

in such case we have

$$[L(Q_{i1},\,Q_{ia_p}^{(i)})\,;\,k\ (Q_{ia_p}^{(i)})]=1,$$

so that

$$S=(m-2)\,!\,\textstyle\sum_{p\neq 1}\,(Q_{i1}\times Q_{iap}^{(i)}\,;\,k).$$

In particular if C is a Curve as defined in 1), § I, and if (x) is a generic Point of the projective straight line D, the correspondenece X

$$X=\{(x)\times P\,;\,k_0\}$$

between D and C is irreducible and satisfies $\mathrm{r}(X)=1$, hence

$$\mu(X)=\nu(X)=1.$$

Let $(P,\,P_2,\ldots\ldots,\,P_n)$ be a complete set of conjugates of P over $k_0(x)$, and

$$S=(m-2)\,!\,E,$$

then we have $E(P)=P_2+\ldots\ldots+P_n$. Since every D-divisor of degree zero is linearly equivalent to zero on D, X^* is a correspondence with valence zero; it follows that X is a correspondence with valence zero. Thus $X(x)=P+P_2+\ldots\ldots P_n$ determines an algebraic series on C, whose elements are linearly equivalent to each other on C. Since

$$(E+\Delta)\ (P)=P+P_2+\ldots\ldots+P_n,$$

$E+\Delta$ is a correspondence with valence zero; the correspondence E is called an *elementary correspondence*, and the multiple C-divisor

$$J=\{\mu(X)/(m-2)\,!\}\cdot\mathrm{pr}_c\,(S\cdot\Delta)=\mathrm{pr}_c\,(E\cdot\Delta)$$

is called a *Jacobian divisor* on C.

15) *SCHUBERT's formula.* Comming back to our general case, we shall omit the trivial cases, where C is a rational Curve or $\mu(X)=0$. We shall consider a linear series of dimension r, defined over k, on C; let $P_1+\dots\dots+P_n$ be a generic element of it over k, where $P_1,\dots\dots P_r$ are independent generic Points of C over k; then the C-divisor $P_{r+1}+\dots\dots+P_n$ is rational over $k\ (P_1,\dots\dots P_r)$. We shall assume that

(L I) $P_{r+1}+\dots\dots+P_n$ *consists of distinct Points and is prime rational over* $k\ (P_1,\dots\dots, P_r)$;

or equivalently that $k\ (P_1,\dots\dots, P_n)$ is separable over $k\ (P_1,\dots\dots,P_r)$ and its Galois group $\mathfrak{H}$ is transitive on $P_{r+1},\dots\dots, P_n$. (L I) is independent of the choice of the generic element over k, and the linear series with a generic element $P_1+\dots\dots+\hat{P}_r+\dots\dots+P_n$ over $k\ (P_r)$ satisfies also (L I) (over $k\ (P_r)$). Moreover since C is not a rational Curve, our linear series has no fix-part for $r\geqq 1$, and is "simple" for $r\geqq 2$.

Now consider the following $C^{(n)}$-cycle V^r:

$$V=\textstyle\sum_{\pi}\ (P_1\times\dots\dots P_r\times P^{\pi}_{r+1}\times\dots\dots\times P^{\pi}_n\,;\,k),$$

where π runs over all permutations of $(n-r)$-Points $P_{r+1},\dots\dots, P_n$ modulo $\mathfrak{H}$. Let Q be a generic Point of C over k, then the specialization

$$Q_{i1}\longrightarrow Q$$

over k can be extended in $\nu\ (X_i)$-way (where the same specialization is counted $[L(Q_{i1})\ :\ k(Q_{i1})]_i$-times) to the specializations

$$Q_1+\dots\dots+Q_m\longrightarrow Q+Q_{i\varkappa 2}+\dots\dots Q_{i\varkappa m}$$
$$(1\leqq\varkappa\leqq\nu(X_i)\,;\ 1\leqq i\leqq d)$$

over k. Moreover the set of $\nu(X_i)\cdot(\mathfrak{G}^{(i)})$ Points

$$(Q\times Q^{\pi}_{i\varkappa 2}\times\dots\dots\times Q^{\pi}_{i\varkappa m}\,;\ \pi\in\mathfrak{G}^{(i)},\ 1\leqq\varkappa\leqq\nu(X_i)\}$$

forms a complete set of conjugates of the Point $Q\times Qi\varkappa_2\times\dots\dots\times Qi\varkappa_m$ over $k(Q_{i1})$, so that we have

$$\textstyle\sum_{\pi i}\,(Q_{i1}\times Q^{\pi i}_{11}\times\dots\dots\times \hat{Q}^{\pi i}_{dmi}\,;\ k)\cdot(Q\times C^{(m-1)})$$
$$=\textstyle\sum_{\varkappa,\pi}(Q\times Q^{\pi}_{i\varkappa 2}\times\dots\times Q_{if}\times\dots\times Q^{\pi}_{i\varkappa m})$$

where π runs over all permutations of $(m-1)$ Points $Q_{i\varkappa_2}\dots\dots, Q_{i\varkappa_m}$.

In the following we shall assume that

(L II) *the intersection-product*

$$(C^{(m-r-1)}\times V)\cdot(U\times C^{(n-r-1)})$$

is defined on $C^{(m+n-r-1)}$;

it follows that the intersection-product $V \cdot \{(\mathrm{pr}\ _{C^{(r+1)}}U) \times C^{(n-r-1)}\}$ is defined on $C^{(n)}$; we put

$$\deg\{V(\mathrm{pr}\ _{C^{(r+1)}}U) \times C^{(n-r-1)}\}$$
$$=(r+1)!\ (m-r-1)!\,(n-r-1)!\ Z(r, n).$$

Let W be the Subvariety of $C^{(m+n-r)}$, which corresponds to $C^{(m+n-r-2)} \times \varDelta$ by the interchange of the first and the last factors of $C^{(m+n-r)}$; then the intersection-product $(U \times C^{(n-r)}) \cdot W$ is defined on $C^{(m+n-r)}$. By our assumption $(L\ II)$, the intersection-product $(C^{(m-r)} \times V) \cdot \{(U \times C^{(n-r)}) \cdot W\}$ is defined on $C^{(m+n-r)}$, such that its algebraic projection on $C^{(m+n-r-1)}\{=(\hat{C} \times C^{(m+n-r-1)}\}$ is $(C^{(m-r-1)} \times V) \cdot (U \times C^{(n-r-1)})$; it follows in particular that the intersection-product $(C^{(m-r)} \times V) \cdot (U \times C^{(n-r)})$ is defined on $C^{(m+n-r)}$. Thus if we put

$$T=\mathrm{pr}_{C\times C\,(\text{first and last factors})}\{(C^{(m-r)} \times V)(U \times C^{(n-r)})\}$$

we have

$$\begin{aligned}\deg.\ (T \cdot \varDelta) &= \deg.\ [\{(C^{(m-r)} \times V) \cdot (U \times C^{(n-r)})\} \cdot W] \\ &= \deg.\ \{(C^{(m-r-1)} \times V) \cdot (U \times C^{(n-r-1)})\} \\ &= \deg.\ \{V \cdot (\mathrm{pr}\ _{C^{(r+1)}}U) \times C^{(n-r-1)}\} \\ &= (r+1)! \cdot (m-r-1)! \cdot (n-r-1)! \cdot Z(r,n).\end{aligned}$$

On the other hand since the intersection-product $(C^{(m-r)} \times V) \cdot \{(U \times C^{(n-r)})(Q \times C^{(m+n-r-1)})\} = \{(C^{(m-r)} \times V) \cdot (U \times C^{(n-r)})\}(Q \times C^{(m+n-r-1)})$ is defined on $C^{(m+n-r)}$, we have

$$(C^{(m-r)} \times V) \cdot \{(U \times C^{(n-r)}) \cdot (Q \times C^{(m+n-r-1)})\}$$
$$=\textstyle\sum_{i,\kappa,\pi,\,\bar{\pi}}(Q \times Q^{\pi}_{i\kappa 2} \times \ldots\ldots \times Q^{\pi}_{i\kappa m} \times R^{\bar{\pi}}_{r+1} \times \ldots\ldots \times R^{\bar{\pi}}_{n}),$$

where $\bar{\pi}$ runs over all permutations of $(n-r)$ Points $R_{r+1}, \ldots\ldots, R_n$. In the following we shall calculate the number $Z(r, n)$; at first a generalization of *CHASLES' formula*:

PROPOSITION 2. *Let X be a correspondence with valence zero between a Curve C and itself such that the intersection-product $X \cdot \varDelta$ is defined on $C \times C$, then we have*

$$\deg.\ (X \cdot \varDelta)=\mathrm{l}(X)+\mathrm{r}(X).$$

Proof. Put $X=(\varphi)+(A \times C)+(C \times B)$, then the intersection-product $(\varphi) \cdot \varDelta$ is defined on $C \times C$; it follows

$$\mathrm{pr}_c(X \cdot \varDelta)=A+B+Pr_c\{(\varphi) \cdot \varDelta\} \equiv A+B.$$

Let k be a field of definition of φ, over which both A and B are rational, and let P be a generic point of C over k, then we have

$$X(P)=(\varphi)(P)+B \equiv B,$$

so that deg. $X(P)=\mathrm{l}(X)=\deg B$; in the same way, we have $\mathrm{r}(X)=\deg.$

A; it follows

$$\deg. (X\cdot\Delta)=\deg. pr_c (X\cdot\Delta)=\deg. A+\deg. B$$
$$=1 (X)+r(X).$$

Now we can prove the following theorm, which is a generalization of *SCHUBERT's formula* in the classical case:

THEOREM 1. *The number $Z(r. n)$ has the following value*

$$Z(r, n)=\nu(X)\cdot n\cdot\binom{m-1}{1}-(D(X)/2)\cdot\binom{m-2}{r-1}$$
$$=D(X)=\deg. J/\mu(X).$$

In particular, Z (r, n) depends only upon X, r and n.

Proof. If $r=0$, we have

$$\deg.\{V\cdot(pr_c(U)\times C^{(n-1)}\}=\deg\cdot[V\cdot\{\nu(X)\cdot(m-1).!\text{-}C^{(n)}\}]$$
$$=\nu(X)(m-1)!\ n!=(m-1)!\cdot Z(0.\ n),$$

so that $Z(0, n)=\nu(X)n$; this is the SCHUBERT's formula for $r=0$; we may therefore use induction on κ, assuming $r\leq 1$:

Since the algebraic series determined by $T+r\cdot(n-r-1)!\ S$ on C consists of equivalent C-divisors, it is a correspondence with valence zero. It follows from CHASLES' formula that

$$\deg. (T\cdot\Delta)+r(n-r-1)!\ \deg. (S\cdot\Delta)$$
$$=1 (T)+r(T)+r(n-r-1)!\cdot\{1 (S)+r(S)\},\ i.\ e.$$
$$(r+1)!\cdot(m-\kappa-1)!\cdot(n-r-1)!\cdot Z(r, n)+r(n-r-1)!\ (m-2)!D(X)$$
$$=r(T)+\nu(X)\cdot(m-1)!\cdot(n-r-1)!\cdot 2\cdot(m-1)!\cdot\nu(X).$$

Let $\phi(\pi)$ be the least integer such that $P_{\phi(\pi)}$ is algebraic over $k(P_n^\pi, P_1, \ldots\ldots, P_{\phi(\pi)-1})$; since P_n^π can not be algebraic over k, we have $\phi(\pi)\leq r$, and the Points

$$P_1,\ldots\ldots, P_{\phi(\pi)-1}, P_{\phi(\pi)+1},\ldots\ldots, P_r, P_n^\pi$$

are independent generic Points of C over k. Let

$$P_1\times\ldots\ldots\times P_r\times P_{r+1}^\pi\times\ldots\ldots\times P_n^\pi\longrightarrow$$
$$P_1\times\ldots\ldots\times P_{\phi(\pi)-1}\times P_{r+1}^{\varphi(\pi)}\times P_{\phi(\pi)}\times\ldots\ldots\times P_n^{\varphi(\pi)}\times P_r$$

be a generic spcialization over k, then $\varphi(\pi)$ is a uniquely determined permutation of $(n-r)$ Points $P_{r+1},\ldots\ldots, P_n$ modulo $\mathfrak{H}$; moreover if $\varphi(\pi_1)\equiv\varphi(\pi_2)$ modulo $\mathfrak{H}$, we have $\pi_1\equiv\pi_2$ modulo $\mathfrak{H}$; it follows

$$V=\sum_\pi (P_1\times\ldots\times P_{\phi(\pi)-1}\times P_{r+1}^\pi\times P_{\phi(\pi)}\times\ldots\times P_{r-1}\times P_{r+2}^\pi\times\ldots\times P_n^\pi\times P_r;\ k).$$

Since the intersection-product $\{(C^{(m-r)}\times V)(C^{(m+n-r-1)}\times P_r)\}(U\times C^{(n-r)})$

$=\{(C^{(m-r)}\times V)\cdot(U\times C^{(n-r)})\}\cdot(C^{(m+n-r-1)}\times P_r)$ is defined on $C^{(m+n-r)}$, and since U is invariant by any interchange of factors of $C^{(m)}$, we have

$$T^*(P_r)=\text{pr}_{C\ (\textit{first factor})}\{(C^{(m-r)}\times V)\cdot(U\times C^{(n-r)})\}\cdot(C^{(m+n-r-1)}\times P_r)$$
$$=\text{pr}_c[\{C^{(m-r)}\times\textstyle\sum_\pi(P_1\times\ldots\ldots\times P_{r-1}\times P^\pi_{r+1}\times\ldots\times P^\pi_n\ ;\ k(P_r))\}\cdot(U\times C^{(n-r-1)})].$$

The linear series with a generic element $P_1+\ldots\ldots+\hat{P}_r+\ldots\ldots+P_n$ over the field $k\ (P_r)$ satisfies both $(L\ I)$ and $(L\ II)$; so that if we out

$$\bar{V}^{r-1}=\textstyle\sum_\pi\{P_1\times\ldots\ldots\times P_{r-1}\times P^\pi_{r+1}\times\ldots\ldots\times P^\pi_n\ ;\ k\ (P_r)\},$$

we have

$$\begin{aligned}\text{r}(T)&=\deg\ T^*(P_r)\\&=\deg\ (\bar{V}\{(\text{pr}_{C^{(r)}}U)\times C^{(n-r-1)})\}\\&=r\,!\ (m-r)\,!\cdot(n-r-1)!\cdot Z(r-1,\ n-1),\end{aligned}$$

It follows that

$$(r+1)!\cdot(m-r-1)!\cdot(n-r-1)!\cdot Z(r,n)+r\cdot(m-r-1)!\cdot(m-2)!D(X).$$
$$=\nu(X)\cdot(m-1)!\cdot(n-r-1)!.(n+r)+r!\cdot(m-n)!(n-r-1)!\cdot Z(r-1,n-1);$$

this formula holds also for $r=0$. Thus 'adding' for $r=0,1,\ldots\ldots,r$, we get

$$(r+1)!(m-r-1)!(n-r-1)!\cdot Z(r,n)+(n-r-1)!(m-2)!\cdot D(X)\cdot\tfrac{1}{2}\cdot r\cdot(r+1)$$
$$=\nu(X)(m-1)!\cdot(n-r-1)!\cdot(r+1)\cdot n,\ \ i.\ e.$$
$$Z(r.n)=\nu(X)n(m-1)!/(m-r-1)!\ r\,!-(D(X)/2)(m-2)!/(m-r-1)!(r-1)!$$
$$=\nu(X)\cdot n.\binom{m-1}{r}-(D(X)/2)\cdot\binom{m-1}{r-1}$$

which proves our theorem.

V. Additive Function on the Multiplication Ring.

16) *Abelian function-field.* Let C be a Curve as defined in 1), § I, and let $P_1,\ldots\ldots,P_d$ be a set of d independent generic Points of C over a field k, containig k_0. Let P_i be the representative of P_i in C, and let T be an indeterminate ; we put

$$P_i=(x^{(i)},\ y_1^{(i)},\ldots\ldots y_n^{(i)})\,(1\leqq i\leqq \text{d})\ ;$$
$$\Pi_{i=1}^d(T-x^{(i)})=T^d-s_1\ T^{d-1}+\ldots\ldots+(-1)^d s_d,$$
$$\Pi_{i=1}^d(T-y_j^{(i)})=T^d-t_{j1}\ T^{d-1}+\ldots\ldots+(-1)^d\ t_{jd}\ ;$$
$$(s)=(s_1,\ldots\ldots,s_d),\qquad (t)=(t_{11},\ldots\ldots,\ t_{nd}).$$

Then at first we have

$$[k(P_1,\ldots\ldots,P_d)\ :\ k(x^{(1)},\ldots\ldots,x^{(d)})]=n^d\ ;$$

we shall show

$$k(P_1, \ldots\ldots, P_d) = k(x^{(1)}, \ldots\ldots, x^{(d)}; t).$$

Let $(y_{j1}^{(i)}, \ldots\ldots, y_{j\bar{n}}^{(i)})$ be a complete set of conjugates of conjugates of $y_j^{(i)}$ over the field $k(x^{(i)})$, then two polynomials

$$\Pi_{k=1}^{\bar{n}} (T - y_{jk}^{(i)}), \qquad \Pi_{i=1}^{d} (T - y_j^{(i)})$$

have their coefficients in $k(x^{(1)}, \ldots\ldots, x^{(d)}, t)$. Since $T - y_j^{(i)}$ is the G. C. D. of them, $y_j^{(i)}$ must be in $k(x^{(1)}, \ldots\ldots, x^{(d)} t)$ $(1 \leq j \leq n;\ 1 \leq i \leq d)$.

Now let L be the invariant subfield of $k\ (P_1, \ldots\ldots, P_d)$ by the symmetric group $\mathfrak{S}_d$, operating on $\{P_1, \ldots\ldots, P_d\}$, then we have

$$[k(P_1, \ldots\ldots, P_d) : L] = (\mathfrak{S}_d) = d!;$$

we know that $k(s)$ is the invariant subfield of $k(x^{(1)}, \ldots\ldots x^{(d)})$ by $\mathfrak{S}_d$. It follows that $L = k\ (s, t)$; we shall denote this field by $k\ (P_1 * \ldots\ldots * P_d)$.

In particular if d is equal to the genus g of C, the Point (s, t) in $g(n+1)$-space has a locus A_g over k, which depends only upon g, the abstract field of all functions on A_g is called the *field of Abelian functions* attached to C; the set of all Abelian functions, defined over k, constitutes an abstract field, which is isomorphic to $k\ (P_1 \ldots\ldots P_g)$.

We shall show .

PROPOSITION 1. *Let $P_1, \ldots\ldots, P_g$ be g independent generic Points of C over k, then the C-divisor*

$$G_0 = P_1 + \ldots\ldots + P_g$$

is non-special.

Proof. If d is sufficiently large, the C-divisor

$$G = P_1 + \ldots\ldots + P_g + P_{g+1} + \ldots\ldots + P_d$$

is non-special (cf. prop. 3, § II). Since $k(P_1 * \ldots\ldots * P_d)$ is a field of definition of $|G|$, any one of the Points $P_1, \ldots\ldots, P_d$ can not be contained in the fix-part of $|G|$, unless $l(G) = d - g = 0$. It follows that $l(G - P_d) = d - g - 1$, so that $P_1 + \ldots\ldots P_{d-1} = G'$ is non-special; repeating this reasoning, we see that G_0 is non special.

17) *Complementary correspondence.* Let C_1 and $C_2 = C$ be two Curves with genus g_1 and $g_2 = g$ respectively; we shall assume that C is not a rational Curve, i. e. $g \geq 1$. Let X be a positive divisor on $C_1 \times C_2$, and let $\bar{P}$ be a Point of C_1. Take a common field of definition k of C_1 and C_2, over which both $\bar{P}$ and X are rational; then take a generic Point P of C_1

over k, and a set of g independent generic Points $\{\bar{R}_1,\ldots\ldots,\bar{R}_g\}$ of C over $k(P)$: Consider the complete linear series, determined by

$$X(\bar{P})+\bar{R}_1+\ldots\ldots+\bar{R}_g\equiv\bar{G};$$

since $\bar{R}_1+\ldots\ldots+\bar{R}_g$ is non-special, we have $l(G)=\deg.\ X(\bar{P})=1(X)$. There exists at least one C-divisor $R_1+\ldots\ldots+R$ such that

$$X(P)+R_1+\ldots\ldots R_g\equiv\bar{G}.$$

Let $R_1^*+\ldots\ldots+R_g^*$ be a specialization of $R_1+\ldots+R_g$ over the specialization $P\rightarrow\bar{P}$ with reference to k $(\bar{R}_1*\ldots\ldots*\bar{R}_g)$; since $|\bar{G}|$ is defined over this field, we have $X(\bar{P})+R_1^*+\ldots\ldots+R_g^*=G$, whence $R_1^*+\ldots+R_g^*\equiv\bar{R}_1+\ldots$ $\ldots+\bar{R}_g$; since $\bar{R}_1\ldots\ldots\bar{R}_g$ is non-special, we have $R_1^*+\ldots+R_g^*=\bar{R}_1+\ldots+\bar{R}_g$. It follows that $\{R_1,\ldots\ldots,R_g\}$ is a set of independent generie Points of C over k, so that $R_1+\ldots\ldots+R_g$ is non-special; thus there exists one and only one C-divisor $R_1+\ldots\ldots+R_g$ such that

$$X(P)+R_1+\ldots\ldots R_g\equiv\bar{G}.$$

Moreover $R_1+\ldots\ldots+R_g$ is rational over $k(\bar{R}_1*\ldots\ldots*\bar{R}_g)(P)$, so that there exists a correspondence Y, between C_1 and C_2, which does not contain any d. l.-correspondence, and which is rational over k $(\bar{R}_1*\ldots\ldots*\bar{R}_g)$, such that

$$Y(P)=R_1+\ldots\ldots+R_g$$

Since $(X+Y)(P)\equiv\bar{G}$, $X+Y$ is a correspondence with valence zero, and we have

$$1(Y)=g;$$

the correspondence Y is called a *generic complementary correspondence* to X over k; any correspondence Y, which satisfies the last two conditions is called a *complementary correspondence to X*.

PROPOSITION 2. *The C-divisor $Y(\bar{\bar{P}})$ is non-special for every Point $\bar{\bar{P}}$ of C_1.*

Proof. If $\bar{\bar{P}}$ is a generic Point of C_1 over $k(R_1*\ldots\ldots*\bar{R}_g)$, $Y(\bar{\bar{P}})$ is the generic specialization of $Y(P)$ over the specialization $P\rightarrow\bar{\bar{P}}$ with reference to k $(R*\ldots\ldots*\bar{R}_g)$, so that $Y(\bar{\bar{P}})$ is non-special. If $\bar{\bar{P}}$ is algebraic over k $(R_1*\ldots\ldots*\bar{R}_d)$ and if we have $1(Y(\bar{\bar{P}}))\geqq 1$ there exists a C-divisor $R_1^*+\ldots\ldots+R_g^*$ in $|Y(\bar{\bar{P}})|$ such that

$$X(\bar{P})+\bar{R}_1+\ldots\ldots+\bar{R}_g\equiv X(\bar{\bar{P}})+R_1^*+\ldots\ldots+R_g^*,$$
$$\dim_{k(\bar{R}_1*\ldots\ldots*\bar{R}_g)}(R_1^*,\ldots\ldots,R_g^*)\geqq 1,$$

since $\bar{R}_1+\cdots\cdots+\bar{R}_g$ is non-special, it is rational over $k(\bar{\bar{P}}, R_1^*,\cdots\cdots,R_g^*)$, hence we have

$$k(\bar{\bar{P}}, R_1^*,\cdots\cdots,R_g^*)\supset k(\bar{R}_1*\cdots\cdots*\bar{R}_g).$$

It follows that

$$\dim k(\bar{\bar{P}}, R_g^*,\cdots\cdots,R_g^*)\geqq g+1,$$

so the $\{R_1^*,\cdots\cdots,R_g^*\}$ is a set of g independent generic Points of C over k, which brings a contradiction.

PROPOSITION 3. *Y is prime rational over every field k_1, which is linearly disjoint to $k(\bar{R}_1,\cdots\cdots,\bar{R})$ over $k(\bar{R}_1*\cdots\cdots*\bar{R}_g)$.*

Proof. Let $Y=\sum_i a_iY_i(a_i\leqq 1)$ be the unique expression of Y as a sum of prime rational components Y_i over k_1, and let P be a generic Point of C_1 over k_1; then we have

$$Y_i(P)=R_{i1}+\cdots\cdots+R_{is},$$

where $\{i_1,\cdots\cdots,i_s\}$ is a subset of $\{1,\cdots\cdots,g\}$. Since $Y_i(P)$ has a uniquely determined specialization $Y_i(\bar{P})$ over the specialization $P\to\bar{P}$ with reference to k_1, we have

$$Y_i(\bar{P})=\bar{R}_{j1}+\cdots\cdots+\bar{R}_{js},$$

where $\{j_1,\cdots\cdots j_s\}$ is a subset of $(1,\cdots\cdots,g)$. Since however

$$[k_1(\bar{R}j_1):k_1]=[k(\bar{R}_1*\cdots\cdots*\bar{R}_g)(\bar{R}_{j1}):k(\bar{R}_1*\cdots\cdots*\bar{R}_g)];$$

we must have $s\geqq g$, so that $Y=Y_i$.

PROPOSITION 4. *Let P be a generic Point of C_1 over the field k_1, then $k_1(P, R_1,\cdots\cdots,R_g)$ is separable over $k_1(P)$, and its Galois group is the symmetric group $\mathfrak{S}_g$.*

Proof. Since the C-divisor $R_1+\cdots\cdots+R_g$ is rational over $k_1(P)$, the $C^{(g)}$-cycle

$$\sum_\pi(R_1^\pi\times\cdots\cdots\times R_g^\pi),$$

where π runs over $\mathfrak{S}_g$, is rational over $k_1(P)$; it follows that there exists a rational $(C_1\times C^{(g)})$-cyble W over k_1, every component of which has the projection C_1 on C_1, such that

$$W(P)=\sum_\pi(R_1^\pi\times\cdots\cdots\times R_g^\pi).$$

Since $W(\bar{P})$ is prime rational over k_1, by a similar argument as before, we see that $W(P)$ is prime rational over $k_1(P)$; which proves our assertion.

We shall not repeat the similar proof for the following proposition:

PROPOSITION 5. *Let $P_1+\cdots\cdots+P_{1(X)+g}$ be a generic element of $|\bar{G}|$ over k_1 and let*

$$\{P_1,\cdots\cdots,P_{1(X)}\{$$

be a set of independent generic Points of C over k_1, then $k_1(P_1,\cdots\cdots,P_{1(X)+g})$ is separable over $k_1(P_1,\cdots\cdots,P_{1(X)})$, and its Galois group is the symmetric group $\mathfrak{S}_g$.

It follows that $|\bar{G}|$ satisfies the condition $(L\ I)$ with reference to such field k_1.

18) *Equivalence defect of the algebraic series.* In the following we shall assume that the correspodence X satisfies the conditions $(A\ I)$ and $(A\ II)$ in § IV; we shall also omit the trivial case, where $\mu(X)=0$. Let P be a generic Point of C_1 over the field k_1; we put

$$X(P)=Q_1+\cdots\cdots+Q_m,$$

$$\tilde{U}^1=\sum_\pi(P\times Q_1^\pi\times\cdots\cdots\times Q_m^\pi;\ k_1),$$

where π runs over all permutations of m Points $Q_1,\cdots\cdots,Q_m$ modulo the Galois group of $k_1(P,\ Q_1,\cdots\cdots Q_m)$ over $k_1(P)$; moreover, $P_1+\cdots\cdots+P_{m+g}$ being as in prop. 5, we put

$$\tilde{V}^m=(P_1\times\cdots\cdots\times P_{m+g};\ k_1).$$

Then the intersection-product $\tilde{U}(P\times C^{(m)})$ is defined on $C_1\times C^{(m)}$ such that

$$\tilde{U}\cdot(P\times C^{(m)})=\sum_\pi(P\times Q_1^\pi\times\cdots\cdots\times Q_m^\pi),$$

where π runs over all permutations of m Points $Q_1,\cdots\cdots,Q_m$; thus the intersection-product $(C_1\times\tilde{V})\cdot\{(\tilde{U}\times C^{(g)})\cdot(P\times C^{(m+g)})\}$ is also defined on $C_1\times C^{(m+g)}$ such that

$$(C_1\times\tilde{V})\{(\tilde{U}\times C^{(g)})\cdot(P\times C^{(m+g)})\}$$
$$=\sum_{\pi,\bar{\pi}}(P\times Q_1^\pi\times\cdots\cdots\times Q_m^\pi\times R_1^{\bar{\pi}}\times\cdots\cdots\times R_g^{\bar{\pi}}),$$

where $\bar{\pi}$ runs over all permutations of g Points $R_1,\cdots\cdots,R_g$, which are defined as before. In particular the intersection-product $(C_1\times\tilde{V})\ (\tilde{U}\times C^{(g)})$ is defined on $C_1\times C^{(m+g)}$; we put

$$\tilde{T}=\mathrm{pr}_{C_1\times C(\text{last factor})}\{(C_1\times\tilde{V})\cdot(\tilde{U}\times C^{(g)})\}.$$

By prop. 2, $\tilde{T}$ does not contain any d. l.-correspondence, and we have

$$\tilde{T}\ (P)=m\,!\cdot(g-1)\ !\ \ Y\ (P)\ ;$$

it follows that

$$\tilde{T}=m\,!\cdot(g-1)!\ Y.$$

On the other hand the linear series with a generic element $P_1+\dots\dots+P_{m+g-1}$ over $k_1(P_{m+g})$ satisfies the condition $(L\ I)$ in § IV; moreover if we put

$$V=(P_1\times\dots\dots\times P_{m+g-1};\ k_1\ (P_{m+g})),$$

the intersection-product

$$\begin{aligned}&\{(\tilde{U}\times C^{(g-1)})\cdot(C_1\times V)\}\times P_{m+g}\\ &=\{(\tilde{U}\times C^{(g)})\cdot(C_1\times\tilde{V})\}\cdot(C_1\times C^{(m+g-1)}\times P_{m+g})\end{aligned}$$

is defined on $C_1\times C^{(m+g)}$. Thus if we put

$$U=\sum\nolimits_\pi(Q_1^\pi\times\dots\dots\times Q_m^\pi;\ k_1),$$

where π runs over all permutations of m Points $Q_1,\dots\dots,Q_m$ modulo $\mathfrak{G}$, the intersection-product $V\cdot(U\times C^{(g-1)})$ is defined on $C^{(m+g-1)}$; since U is invariant by the interchange of factors in $C^{(m)}$, we may assume that $P_1,\dots\dots,P_{m-1}$ are independent generic Points over $k_1(P_{m+g})$. It follows that our linear series satisfies also the condition $(L\ II)$ in § IV; and we have

$$\begin{aligned}\mathrm{r}(\tilde{T})&=\deg.\ \{(\tilde{U}\times C^{(g-1)})\cdot(C_1\times V)\}\\ &=\mu(X)\ \deg.\ \{V\cdot(U\times C^{(g-1)})\}\\ &=\mu(X)m\,!\ (g-1)\,!\ Z(m-1,\ m+g-1);\end{aligned}$$

hence

$$\begin{aligned}\mathrm{r}(Y)&=\mu(X)\cdot Z(m-1,\ m+g-1)\\ &=\mathrm{r}(X)(m+g-1)-\tfrac{1}{2}\ \deg.\ J.\end{aligned}$$

The number $Z(m-1,\ m+g-1)$:

$$Z(m-1,\ m+g-1)=\nu(X)(m+g-1)-\tfrac{1}{2}\ D(X)$$

is called the *equivalence defect* of the algebraic series, determined by X on C. In fact we have $Z=0$ if and only if $\mathrm{r}(Y)=0$, i.e. Y is a d. r.-correspondence so that X is a correspondence with valence zero. In particular if X is the correspondence with valence zero, considered in 14), § IV, we have

$$\deg. J-2n=2g-2,$$

where J is a Jacobian divisor on C; this is known as *RIEMANN-HURWITZ'* *relation* in the classical case.

19) *Virtual degree of the correspondence.* Let C_1 and $C_2=C$ be two Curves with genus g_1 and $g_2=g$. We shall prove the following formula, which is known as *HURWITZ' formula* in the classical case:

PROPOSITION 6. *Let X and Y be two correspondences between C_1 and C_2 such that the intersection-product $X\cdot Y$ is defined on $C_1\times C_2$, then the* *intersection-product* $(X \circ Y^*) \cdot \Delta$ *is also defined on* $C_1 \times C_2$ *such that*

$$\mathrm{pr}_c(X \cdot Y) = \mathrm{pr}_c\{(X \circ Y^*)\Delta\}$$

Proof. By linearity we may assume that both X and Y are irreducible. Let k be a common field of definition for C_1, C_2, X and Y; let $P \times Q$ be a generic Point of $C_1 \times C_2$, over k, and let $\bar{P} \times \bar{Q}$ be a generic Point of X over k. Let W be the Locus of the Point $Q \times P \times Q$ in $C_2 \times C_1 \times C_2$ over k; then the intersection-product $(C_2 \times X) \cdot W$ is defined on $C_2 \times C_1 \times C_2$ and we have $W \cdot (C_1 \times X) = Z$ (cf. 4), § I). Since we have $Z \neq Y^* \times C_2$, the intersection-product $Z(Y^* \times C_2)$ is defined on $C_2 \times C_1 \times C_2$; it follows that the intersection-products $\{(C_2 \times X) \cdot (Y^* \times C_2)\}, W \cdot \{(C_2 \times X) \cdot (Y^* \times C_2)\}$ are defined on $C_2 \times C_1 \times C_2$ such that

$$Z \cdot (Y^* \times C_2) = W \cdot \{(C_2 \times X) \cdot (Y^* \times C_2)\}$$

Taking the algebraic projection on $C_2 \times C_2$, we get

$$\mathrm{pr}_{\,c \times c}(Z \cdot (Y^* \times C_2)) = (X \circ Y^*) \cdot \Delta\,;$$

taking one more the algebraic projection, we get

$$\begin{aligned} \mathrm{pr}\{(X \circ Y^*) \cdot \Delta\} &= \mathrm{pr}_c \text{ (first factor) } (\mathrm{pr}_{c \times c}\{Z \cdot (Y^* \times C_2)\}) \\ &= \mathrm{pr}_c[\mathrm{pr}_{c \times c}\{Z(Y^* \times C_2)\}] \\ &= \mathrm{pr}_c(X^* \cdot Y^*) \\ &= \mathrm{pr}_c(X \cdot Y). \end{aligned}$$

The following proposition will be used in the next §:

PROPOSITION 7. *Let* X *be an irreducible correspondence between* C_1 *and* C_2 *such that* $1(X) = 1$, *then we have*

$$X \circ X^* = \mathrm{r}(X)\Delta.$$

Proof. Let k be a common field of definition for C_1, C_2 and X; and let $\bar{Q} \times \bar{P} \times \bar{R}$ be a generic Point of some component Z of the intersection $(C_2 \times X) \cap (X^* \times C_2)$ over $\bar{k}$, then the Points $\bar{P} \times \bar{R}$ and $\bar{P} \times \bar{Q}$ are in X. Since $1(X) = 1$, it holds

$$\bar{Q} = X(\bar{P}) = \bar{R}$$

so that $\bar{P} \times \bar{Q}$ is a generic Point of X over k, and we have

$$(C_2 \times X) \cdot (X^* \times C_2) = aZ.$$

Taking the algebraic projection on $C_2 \times C_1$ we get $a = 1$; taking the algebraic projection on $C_2\times C_2$, we get

$$X\circ X^*=\mathrm{r}(X)\Delta.$$

Now we can develope the most important part of the theory of correspondences between C_1 and C_2:

At first we shall express the "continuous equivalence" by $\backsim$, as the linear equivalence has been expressed by $\equiv$. Then for any Points $\bar{P}$, $\bar{Q}$ of C_1, C_2 respectively, it holds

$$X\backsim \mathrm{l}(X)\cdot(C_1\times\bar{Q})+\mathrm{r}(X)(\bar{P}\times C_2),$$

whenever X is a correspondence with valence zero. Now if C is rational, every X is a correspondence with valence zero, so that for every X it holds $X\backsim \mathrm{l}(X)\cdot(C_1\times\bar{Q})+\mathrm{r}(X)$ $(\bar{P}\times C_2)$; moreover $(C_1\times\bar{Q})$ and $(\bar{P}\times C_2)$ are linearly independent in the sense of $\backsim$ over the ring of rational integers. In the following we shall omit this trivial case, where is nothing to prove; moreover we shall denote by $\mathfrak{x}$, $\mathfrak{y}$,......the classes of the correspondences by $\backsim$, and by $\mathfrak{x}$, $\mathfrak{y}$,......the classes of the correspondences modulo the correspondences with valence zero.

LEMMA. Every $\mathfrak{x}$ *contains a representative* X *such that* $X\succ 0$; *hence every* $\mathfrak{x}$ *contains a representative* X_1 *such that*

$$X_1=X+b\,(C_1\times\bar{Q})+a(\bar{P}\times C_2),\ \ X\succ 0.$$

Proof. Let k be a common field of definition for C_1 and C_2 over which a representative X' of $\mathfrak{x}$ and a Point $\bar{Q}$ of C_2 are rational; let P be a generic Point of C_1 over k. Then for a sufficiently large m, there exists a rational C_2-divisor $\bar{G}$ over $k(P)$ such that

$$X'(P)+m\bar{Q}\equiv\bar{G},\quad \bar{G}\succ 0;$$

we may put $\bar{G}=X(P)$ with some correspondence X such that $X\succ 0$; and X is a representative of $\mathfrak{x}$.

Now let X_1, X_2 be any two representatives of $\mathfrak{x}$ such that X_1, $X_2\succ 0$; let Y_i be a generic complementary correspondence to X_i for $i=1, 2$. Let k be a common field of definition of C_1 and C_2 over which both Y_1 and Y_2 are rational, and let P be a generic Point of C_1 over k; then there exists a rational C_2-divisor B (of degree zero) over k^* such that

$$(Y_1-Y_2)(P)\equiv B.$$

It follows that

$$Y_1-Y_2\equiv C_1\times B;$$

and in particular $r(Y_1)=r(Y_2)$; we shall denote this non-negative integer, which depends only upon $\mathfrak{x}$, by $\delta(\mathfrak{x})$; we have $\delta(\mathfrak{x})=0$ if and only if $\mathfrak{x}=0$.

THEOREM 1. *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be arbitary given, then we can find a representative X, Y of $\boldsymbol{x}$, $\boldsymbol{y}$ such that $\chi(X, Y)$ is defined. Moreover $\chi(X, Y)$ is independent of the choice of such representatives, so that we may put*

$$\chi(\boldsymbol{x}, \boldsymbol{y})=\chi(X, Y),$$

whenever χ (X, Y) is defined; we shall put

$$\chi(X, Y)=\chi(\boldsymbol{x}, \boldsymbol{y}),$$

even if $\chi(X, Y)$ is not defined.

Proof. Let

$$X=X_1+b_x(C_1\times \bar{Q}_x)+a_x(\bar{P}_x\times C_2).$$
$$Y_2=Y_1+b_y\cdot(C_1\times \bar{Q}_y)+a_y(\bar{P}_y\times C_2)$$

be two representatives of $\boldsymbol{x}$, $\boldsymbol{y}$, as stated in the lemma; we may assume that $\bar{P}_x \doteqdot \bar{P}_y$, $\bar{Q}_x \doteqdot \bar{Q}_y$, and that $(C_1\times \bar{Q}_y)$ and $(\bar{P}_y\times C_2)$ are not contained in X_1. Let k be a common field of definition for C_1 and C_2, over which X_1, Y_1 and four Points $\bar{P}_x$, $\bar{P}_y$, $\bar{Q}_x$, $\bar{Q}_y$ are all rational; and let Y be a generic complementary correspondence to Y_1 over k, and let Y'' be a generic complementary correspondence to Y' over $k(\bar{R}_1* \ldots\ldots *\bar{R}_h)$, Then we have

$$Y_1 \sim Y''+b_y'(C_1\times \bar{Q}_y) + a_y'(\bar{P}_y\times C_2)$$

with some rational integers b_y' and a_y' ; moreover the intersection-product $X\cdot Y''$ is surely defined on $C_1\times C_2$. Thus if we put

$$Y=Y''+(b_y+b_y')\cdot(C_1\times \bar{Q}_y)+(a_y+a_y')\cdot(\bar{P}_y\times C_2),$$

the intersection-product $X\cdot Y$ is defieed on $C_1\times C_2$, and Y is also a representative of $\boldsymbol{y}$.

Now let X, X_1 and Y, Y_1 be the representatives of $\boldsymbol{x}$ and $\boldsymbol{y}$ snch that χ (X, Y) and $\chi(X_1, Y_1)$ are defined; by a similar argument as above, we can find a representative Y_2 of $\boldsymbol{y}$ such that the intersection-products $X\cdot Y_2$ and $X_1\cdot Y_2$ are defined on $C_1\times C_2$. It follows from the postulates (D), (S) of the equivalence theory that

$$\chi(X\cdot Y)=\chi(X\cdot Y_2)=\chi(X_1\cdot Y_2),=\chi(X_1, Y_1),$$

which proves our assertions.

Now let X_1 and X_2 be two represntives of $\boldsymbol{x}$, then we have $l(X_1)=l(X_2)$, $r(X_1)=r(X_1)$; thus $l(x)$ and $r(x)$ have a definite meaning. We shall prove the following theorem, which is known as *SEVERI's formula* in the classical case:

THEOREM 2. *Let X be an arbitary correspondence between C_1 and C_2 and let $\mathfrak{x}$ and $\mathfrak{x}$ be the classes of X, then it holds*

$$\chi(\mathfrak{x}, \mathfrak{x})=2\,\mathrm{l}(\mathfrak{x})\mathrm{r}(\mathfrak{x})-\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}.$$

Proof. We may assume that X is of the form

$$X=X_1+b\,(C_1\times\bar{Q})+a\,(\bar{P}\times C_2)$$

with $X_1 \succ 0$; take a common field of definition k of C_1 and C_2, over which X_1, $\bar{P}$, $\bar{Q}$ are rational. Let P be a generic Point of C_1 over k, and let $\{\bar{R}_1,\ldots\ldots,\bar{R}_{2g}\}$ be a set of independent generic Points of C over $k(P)$; let Y be a generic complementary correspondence to X_1, defined over k by

$$X_1(\bar{P})+\bar{R}_1+\ldots\ldots+\bar{R}_g\equiv X_1(P)+Y(P):$$

let Z be a generic complementary correspondence to Y, defined over $k(\bar{R}_1*\ \ldots\ldots\ *\bar{R}_g)$ by

$$Y(\bar{P})+\bar{R}_{g+1}+\ldots\ldots+\bar{R}_{2g}\equiv Y(P)+Z(P).$$

Then

$$X'=Z+b'(C_1\times\bar{Q})+a'(\bar{P}\times C_2)$$

is also a representative of $\mathfrak{x}$ for some rational integers b' and a'. Now if we put

$$k_1=k(\bar{R}_1*\ \ldots\ldots*\ \bar{R}_g,\ \bar{R}_{g+1}*\ \ldots\ldots*\ \bar{R}_{2g}\ ),$$

k_1 is linearly disjoint to $k(\bar{R}_1,\ldots\ldots,\bar{R}_g)$ over $k(\bar{R}_1*\ \ldots\ldots\ *\bar{R}_g)$; and if we put

$$Y(P)=R_1+\ldots\ldots+R_g,\ Z(P)=R_{g+1}+\ldots\ldots+R_{2g}\ ,$$

the Galois group of $k_1(P,\ R_1,\ldots\ldots R_{2g})$ over $k_1(P)$ is the direct product of the symmetric groups, operating on $\{R_1,\ldots\ldots R_g\}$ and on $\{R_{g+1},\ldots\ldots R_{2g}\}$; moreover the invariant subfields of $k_1(R_1,\ldots\ldots,R_g)$, $k_1(R_{g+1},\ldots\ldots R_{2g})$ and $k_1(R_1,\ldots\ldots R_{2g})$ are the same:

$$\begin{aligned} k_1(R_1*\ \ldots\ldots\ *R_g)&=k_1(R_{g+1}*\ \ldots\ldots\ *R_{2g}) \\ &=k_1(R_1*\ \ldots\ldots\ *R_g,\ R_{g+1}*\ \ldots\ldots\ *R_{2g}). \end{aligned}$$

It foollows that the correspondence $Y+Z$ satisfies the conditions $(A\ I)$ and $(A\ II)$ in § IV (over k_1); thereby we may assume that

$$\mu(Y+Z)\neq o,$$

for otherwise X_1 is a correspondence with valence zero, and our formula holds trivially. Thus we can apply to $Y+Z$ the results on the equivalence defect of the algebraic series; it is a correspondence with valence zero, we have

$$r(Y+Z)(3g-1)-\tfrac{1}{2}\deg. J=o;$$

thereby

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{dg}.\ J&=\tfrac{1}{2}\deg. J_Y+\tfrac{1}{2}\deg. J_Z+\chi(Y,Z)\\&=(\delta(\mathfrak{x})(2g-1)-\delta(-\mathfrak{x}))\\&\quad+\{\delta(-\mathfrak{x})(2g-1)-\delta(\mathfrak{x})\}+\chi(Y.Z)\\&=\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}(2g-2)\\&\quad+\{\chi(Y+Z,Z)-\chi(Z,Z)\}\end{aligned}$$

On the other hand, since we have

$$Y+Z \sim 2g\ (C_1\times\overline{Q})+\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}(\overline{P}\times C_2),$$

by HURWITZ' formula we get

$$\begin{aligned}\chi(Y+Z,Z)&=\chi\{(Y+Z)\circ Z,^{*}\Delta\}\\&=2g.\ \delta(-\mathfrak{x})+\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}.\ g;\end{aligned}$$

moreover since $Z=X'-b'\ (C_1\times\overline{Q})-a'(\overline{P}\times C_2)$, we get

$$\begin{aligned}\chi(Z,Z)&=\chi(\mathfrak{x}\mathfrak{x})-2r(\mathfrak{x})b'-21(\mathfrak{x})a'+2a'b'\\&=\chi(\mathfrak{x}\mathfrak{x})-2\delta(-\mathfrak{x})b'-21(\mathfrak{x})a'.\end{aligned}$$

It follows

$$\begin{aligned}\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}(3g-1)&=\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}.\ (2g-2)\\&\quad+2g\cdot\delta(-\mathfrak{x})+\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}\cdot g+\delta(-\mathfrak{x})b'\\&\quad+21(\mathfrak{x})a'-\chi(\mathfrak{x}\mathfrak{x});\end{aligned}$$

hence

$$\begin{aligned}\chi(\mathfrak{x},\mathfrak{x})&=2\ \delta(-\mathfrak{x})\cdot(g+b')+2a'\cdot 1(\mathfrak{x})\\&\qquad-\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}\\&=2\ 1(\mathfrak{x})\{\delta(-\mathfrak{x})+a'\}-\{\delta(\mathfrak{x})+\delta(-\mathfrak{x})\}\\&=2\ 1(\mathfrak{x})\cdot r(\mathfrak{x})-\{\delta(\mathfrak{x})-\delta(-\mathfrak{x})\}.\end{aligned}$$

COROLLARY. *X is a correspondence with valence zero, if and only if it holds*

$$X\sim b\ (C_1\times\overline{Q})+\ a\ (\overline{P}\times C_2).$$

In fact if we put

$$Y=X-b(C_1\times\overline{Q})-\ a\ (\overline{P}\times C_2),$$

we have $Y\sim 0$, so that

$$\chi(\mathfrak{y},\mathfrak{y})=\ -\{\delta(\mathfrak{y})+\delta(-\mathfrak{y})\}=0,$$

whence $\delta(\mathfrak{y})=\delta(-\mathfrak{y})=0$, and $\mathfrak{y}=0$.

We can derive an interesting consequence from this corollary, as it will be given in the appendix.

Now if we consider the special case of $C_1=C_2=C$, the elements $\mathfrak{x}$, $\mathfrak{y}$, ...form the multiplication ring attached to C. For every correspondence X between C and itself, we put

$$\sigma(Z)=\mathrm{l}(X)+\mathrm{r}(X)-\chi(X,\Delta);$$

since $\sigma\{b(C_1\times Q)+a(P\times C_2)\}=a+b-(a+b)=0$, σ depends only upon the class $\mathfrak{x}$ of X, therefore we may put

$$\sigma(\mathfrak{x})=\sigma(X);$$

σ is an additive function $\mathfrak{R}$, and it holds

$$\sigma(\mathfrak{x}^*)=\sigma(\mathfrak{x}).$$

By HURWITZ' formula, we have

$$\begin{aligned}\sigma(X\circ Y^*)&=\mathrm{l}(X)\mathrm{r}(Y)+\mathrm{l}(Y)\mathrm{r}(X)-\chi(X,\ Y)\\&=\mathrm{l}(Y)\mathrm{r}(X)+\mathrm{l}(X)\mathrm{r}(Y)-\chi(Y^*,X^*)\\&=\sigma(Y^*\circ X),\end{aligned}$$

so that it holds

$$\sigma(\mathfrak{x},\ \mathfrak{y})=\sigma(\mathfrak{y},\ \mathfrak{x})$$

for exery $\mathfrak{x}$, $\mathfrak{y}$ in $\mathfrak{R}$.

Moreover by RIEMANN-HURWITZ' relation, we get

$$\begin{aligned}\sigma(m\,\mathfrak{e})&=m\,\sigma(\mathfrak{e})=-m\,\sigma(E)\\&=-m\{2(n-1)-\deg.\ J\}=2mg\,;\end{aligned}$$

this is known as *CAYLEY-BRILL'S formula* in the classical case.

Furthermore since we have

$$\begin{aligned}\sigma(\mathfrak{x}\mathfrak{x}^*)&=\sigma(X\circ X^*)\\&=2.\,\mathrm{l}(X)\ \mathrm{r}\ (X)-\chi(X,X)\\&=\delta(\mathfrak{x})+\delta(-\acute{\mathfrak{x}}),\end{aligned}$$

$\sigma(\mathfrak{x}\mathfrak{x}^*)$ is a positive definite function on $\mathfrak{R}$. It follows

$$\sigma\{2g\mathfrak{x}-\sigma(\mathfrak{x})\mathfrak{e}\}\{2g\mathfrak{x}^*-\sigma(\mathfrak{x})\mathfrak{e}\}\geqq 0.$$

so that it holds

$$\sigma(\mathfrak{x}\mathfrak{x})^*\geqq\sigma(\mathfrak{x})^2/2g\geqq 0.$$

VI. Riemann Hypothesis.

20) *ζ-functions.* In this §, we shall assunme exclussively that our Curve C is defined as in 1), § I over a finite field $k_1=k_0$ with q elments. Let k_m for $m=1,2,\ldots\ldots$, be the cyclic field of degree m over k_1, and let P be a generic Point C over k_1, then the ζ-function $\zeta_m(s)$ of the function-field $k_m(P)$ over k_m is defined by

$$\zeta_m(s)=\sum_{A\succ 0}(q^m)^{-s\ \deg\cdot(A)},$$

where A runs over all positive rational C-divisors over k_m. We know that $\zeta_m(s)$ is "divisible" by the ζ-function ζ_1 (s) of the rational function-field $k_m(x)$over k_m:

$$\zeta_m(s)=\zeta_0(s)L_m(s),\quad 1/\zeta_0(s)=(1-(q^m)^{-s})\cdot(1-(q^m)^{1-s}),$$

and if we put

$$q^s=Z,$$

$(q^m)^{2gs}.\ L_m(s)$ is a polynomial of degree $2g$ in Z^m with integral rational coefficients of the form

$$G_m(Z^m)=(q^m)^{2gs}L_m(s)=Z^{2mg}-(1+q^m-\nu_m)\cdot Z^{m(2g-1)}+\ldots+q^{mg},$$

where g means the geuns of C, and ν_m means the number of rational Points of C over k_m · Since $\zeta_0(s)\neq o$ for every complex value of s, the RIEMANN hypothesis, i. e. the statement that "all the zero points of $\zeta_1(s)$ are of the torm

$$\tfrac{1}{2}+\sqrt{-1}\cdot T\quad (T \text{ real number})\text{''}$$

is equivalent to the assertion that "all the roots

$$\pi_1\ldots\ldots,\pi_{2g}$$

of $G_1(Z)$ satisfy

$$|\pi|^2=q.\text{''}$$

From the functional equation

$$(q^m)^{gs}\cdot L_m(s)=(q^m)^{g(1-s)}\cdot L_m(1-s)$$

of $L_m(s)$, me conclude the following equation

See e. g. HASSE, H., Ueber die Kongruenzzetafunktionen, Sitz. Ber. Berlin, 1934.

$$\Pi_{i=1}^{2g}(Z-\pi_i)=\Pi_{i=1}^{2g}(Z-q/\pi_i);$$

we put

$$\pi_i^*=q/\pi_i \quad (1\leqq i\leqq 2g).$$

Since $L_m(s)$ can be expressed as a product of L-functions of the "basic field" $k_1(P)$ in the form

$$L_m(s)=\Pi_{i \bmod m}L(s-2\pi\sqrt{-1}\cdot i/m\cdot\log q),$$

we conclude

$$G_m(Z^m)=\Pi_{i=1}^{2g}(Z^m-\pi_i^m),$$

hence

$$\mathrm{Sp}\cdot(\pi^m)=\textstyle\sum_{i=1}^{2g}\pi_i^m=1+q^m-\nu_m$$
$$(m=1,2,\ldots\ldots).$$

22) *Frobeniusean correspondence.* Now let

$$P=(x,' y_1,\ldots\ldots,y_n),\ P'=(x', y_1,'\ldots\ldots, y_n')$$

be the representatives of P in C, C' respectively (cf. 1), § I, then the points

$$\mathrm{P}_m=(x^{q^m}, y_1^{q^m}\ldots\ldots, y_n^{q^m}),\ \mathrm{P}_m'=(x'^{q^m}, y_1'^{q^m},\ldots\ldots, y_n'^{q^m})$$

in $(n+1)$-space are the specializations of P, P' over k_1 respectively, since every element of k_1 is equal to its q-th power. By the same reason $\mathrm{P}_m\times\mathrm{P}_m'$ is a spcciaiization of $\mathrm{P}\times\mathrm{P}'$ over k_1 so that it defines a Point P_m on C; furthermore $k_1(P_m)$ is a subfield of $k_1(P)$ such that

$$[k_1(P):k_1(P_m)]=[k_1(x):k_1(x^{q^m})]=q^m.$$

Let F_m be the Locus of the Point $P\times P_m$ in $C\times C$ over k_1, then F_m is an irreducible correspondence between C and itself, defined over k_1, such that

$$\mathrm{l}(F_m)=1,\ \mathrm{r}(F_m)=q^m;$$

it follows

$$F_m\circ F_m^*=q^m\cdot\Delta,\ F_m=F\circ\ldots\ldots\circ F\ (m\text{-factors}).$$

In its close connection with the "Frobeniusean substitution" in algebraic number field, it may be called the *Frobeniusean correspondence* between C and itself.

PROPOSITION I. Let

$$\mathrm{Pr}_C(F_m\cdot\Delta)=\textstyle\sum e_{\bar P}\bar P$$

be the reduced expression for the C-divisor on the left-hand side, then $\bar P$ *are* *exactly all the rational Points of C over k_m; moreover it holds $e_{\bar{P}}=1$ for every $\bar{P}$ so that we haxe*

$$\chi(F_m, \Delta)=\nu_m \quad (m=1,2 \ldots\ldots).$$

Proof. Let $\bar{P}=(a, b_1,\ldots\ldots, b_n)$ be a representative of the Point $\bar{P}$, then $\bar{\mathrm{P}}\times\bar{\mathrm{P}}$ is a specialization of $\mathrm{P}\times\mathrm{P}_m$ or of $\mathrm{P}'\times\mathrm{P}'_m$ over k_1 so that we have

$$a^{q^m}=a,\ b_1^{q^m}=b_1,\ldots\ldots,\ b_n^{q^m}=b_n,$$

this meas however $k_m(\bar{P})=k_m(\bar{\mathrm{P}})=k_m$; and conversely.

In order to show $e_{\bar{P}}=1$, we may assume e.g. that $\bar{\mathrm{P}}$ is the representative of $\bar{P}$ in C; let F_m and Δ' be the representatives of F_m and Δ in $\mathrm{C}\times\mathrm{C}$ respectively. We shall prove

$$i(\mathrm{F}_m\cdot\Delta,' \bar{\mathrm{P}}\times\bar{\mathrm{P}};\ \mathrm{C}\times\mathrm{C})=1$$

by the "criterion of multiplicity one". Let

$$(X,\ Y,\ X',\ Y')=(X,\ Y_1,\ldots\ldots,\ Y_n,\ X',\ Y_1',\ldots\ldots Y_n')$$

be the indeterminates for $(2n+2)$–space, and let

$$P_i(X,\ Y)=0 \quad (1\leqq i\leqq N)$$

be a set of equations for C over k_1, then the set of equations

$$\begin{aligned} &P_i(X,\ Y)=0 \quad (1\leqq i\leqq N),\\ &Q_0(X,\ Y,\ X',\ Y')=X^{q^m}-X'=0\\ &Q_j(X,\ Y,\ X',\ Y')=Y_j^{q^m}-Y_j'=0 \quad (1\leqq j\leqq n)\end{aligned}$$

is contained in the ideal defining F_m over k_1, and

$$\begin{aligned} &P_i(X,\ Y)=0 \quad (1\leqq i\leqq N)\\ &R_0(X,\ Y,\ X',\ Y')=X-X'=0\\ &R_k(X,\ Y,\ X',\ Y')=Y_k-Y_k'=0 \quad (1\leqq k\leqq n)\end{aligned}$$

is a set of equatiens for Δ' over k Since $\bar{\mathrm{P}}$ is simple on C, the rank of the matrix

$$\|(\partial P_i/\partial X_a)\ (\partial P_i/\partial Y)_b\|$$

is n, so that the rank of the matrix

$$\begin{Vmatrix} (\partial P_i/\partial X)_a & (\partial P_i/\partial Y)_b & 0 & 0 \\ (\partial Q_j/\partial X)_a & (\partial Q_j/\partial Y)_b & (\partial Q_j/\partial X')_b & (\partial Q_j/\partial Y')_b \end{Vmatrix}$$

$$=\begin{Vmatrix} (\partial P_i/\partial X)_a & (\partial P_i/\partial Y)_b & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\begin{array}{|ccc} -1 & & \\ & \ddots & \\ & & -1 \end{array}} \\ 0 & 0 & & \end{Vmatrix}$$

is $(2n+1)$. It follows that $\overline{\mathrm{P}}\times\overline{\mathrm{P}}$ is simple on F_m; since it is simple on $\mathrm{C}\times\mathrm{C}$ and on $\Delta,'$ and since the rank of the matrix

$$\left\|\begin{matrix} (\partial P_i/\partial X)_a & (\partial P_i/\partial Y)_b & 0 & 0 \\ (\partial Q_j/\partial X) & (\partial Q_j/\partial Y)_b & (\partial Q_i/\partial X')_a & (\partial Q_j/\partial Y')_b \\ (\partial R_k/\partial X)_a & (\partial R_k/\partial Y)_b & (\partial R_k/\partial X')_a & (\partial R_k/\partial Y')_b \end{matrix}\right\|$$

$$=\left\|\begin{array}{cc|cc} (\partial P_i/\partial X)_a & (\partial P_i/\partial Y)_b & 0 & 0 \\ \hline 0 & 0 & -1 & \\ 0 & 0 & & \ddots \quad -1 \\ \hline 1 & & -1 & \\ & \ddots & & \ddots \\ & 1 & & -1 \end{array}\right\|$$

is $(2n+2)$, F_m and Δ' are transversal to each other at $\overline{\mathrm{P}}\times\overline{\mathrm{P}}$ on $\mathrm{C}\times\mathrm{C}$, which proves our assertion.

Now let $\mathfrak{f}$ be the element of $\mathfrak{R}$ with the representative F, then F_m is a representsive of $\mathfrak{f}^m$, and we have

$$\sigma(\mathfrak{f}^m)=\sigma(F_m)=1+q^m-\chi(F_m,\Delta)$$
$$=1+q^m-\nu_m=\mathrm{Sp}\ (\pi^m)\ (m=1,2,\ldots\ldots).$$

22) *Solution of the RIEMANN hypophesis.* Let

$$F(x)=\sum a_\mu X^\mu$$

be any polynomial with integral rational coefficients, then

$$F(\mathfrak{f})=\sum a_\mu \mathfrak{f}^\mu$$

is defined as an element of $\mathfrak{R}$; since $\mathfrak{f}^*\cdot\mathfrak{f}=\mathfrak{f}\cdot\mathfrak{f}^*=q\mathfrak{e}$, we have

$$F(\mathfrak{f})\cdot F(\mathfrak{f})^*=(\textstyle\sum_\mu a_\mu^2\ q^\mu)\cdot\mathfrak{e}+\sum_{\mu<\nu}a_\mu a_\nu q^\mu\{(\mathfrak{f}^*)^{\nu-\mu}+\mathfrak{f}^{\nu-\mu}\}.$$

Furthermore since $\sigma(m\cdot\mathfrak{e})=2mg$, $\sigma\{(\mathfrak{f}^*)^m\}=\sigma(\mathfrak{f}^m)=\mathrm{Sp}(\pi^m)$, we have

$$\sigma\{F(\mathfrak{f})\cdot F(\mathfrak{f})^*\}=\textstyle\sum_\mu a_\mu^2 q^\mu 2g+\sum_{\mu<\nu}2a_\mu a_\nu q^\mu\cdot\mathrm{Sp}\cdot(\pi^{\nu-\mu}).$$

On the other hand we have

$$F(\pi)\cdot F(\pi^*)=\textstyle\sum_\mu a_\mu^2 q^\mu+\sum_{\mu<\nu}a_\mu a_\nu q^\mu\{(\pi^*)^{\nu-\mu}+\pi^{\nu-\mu}\};$$

siece $\mathrm{Sp}\cdot\{(\pi^*)^m\}=\mathrm{Sp}\cdot(\pi^m)$, we have

$$\mathrm{Sp}\cdot\{F(\pi)F(\pi^*)\}=\sigma\{F(\mathfrak{f})\ F(\mathfrak{f})^*\}.$$

Since $\sigma(\mathfrak{x}\cdot\mathfrak{x}^*)$ is non-negative for every $\mathfrak{x}$ in $\mathfrak{R}$, we have

$$\sigma\{F(\mathfrak{f})\cdot F(\mathfrak{f})^*\}\geqq 0,\ \text{whence}\ \mathrm{Sp}\{F(\pi)\cdot F(\pi^*)\}\geqq 0$$

for all integral rational a_μ; since $\mathrm{Sp}\{F(\pi)\cdot F(\pi^*)\}$ is a quadratic form in a_μ, this implies $\mathrm{Sp}\{F(\pi)\cdot F(\pi^*)\}\geqq 0$ for all rational valees of a_μ, hence also for all real values af a_μ.

Now we shall prove the RIEMANN hypothesis in the form that "if we have $|\pi_i|^2 \neq q$ for some root π_i of $G_1(Z)$, we can find a polynomial $F(X)$ (of degree at most $2g$) wsth real coefficients such that

$$\mathrm{Sp}\{F(\pi)\cdot F(\pi^*)\} < 0."$$

At first the following trivial lemma:

LEMMA 1. Let ω_1 and ω_2 be two different real numbers, then we can find a polynomial

$$J(X)=A_0+A_1X$$

with real coefficients such that $J(\omega_1)$ and $J(\omega_2)$ take two preassigned real values.

We shall prove a similar statent:

LEMMA 2. Let $\omega_1, \omega_2, \omega_3$ and ω_4 be four different complex numbers satisfying

$$\bar{\omega}_1=\omega_3, \quad \bar{\omega}_2=\omega_4,$$

then we can find a polynomoal

$$J(X)=A_0+A_1X+A_2X^2+A_3X^3+A_4X^4$$

with real coefficients such that $J(\omega_1)$ and $J(\omega_2)$ take two preassigaed complex values.

Proof. We have only to show that the rank of the matrix

$$\left\|\begin{matrix} 1 & R(\omega_1) & R(\omega_1^2) & R(\omega_1^3) & R(\omega_1^4) \\ 1 & I(\omega_1) & I(\omega_1^2) & I(\omega_2^3) & I(\omega_1^4) \\ 1 & R(\omega_2) & R(\omega_2^2) & R(\omega_2^3) & R(\omega_2^4) \\ 1 & I(\omega_2) & I(\omega_2^2) & I(\omega_2^3) & I(\omega_2^4) \end{matrix}\right\|$$

is four. By 'elementary operations', which leave invariant the rank in question, we can transform this matrix in the form

$$\frac{1}{4}\left\|\begin{matrix} 1+\sqrt{-1} & \omega_1 & \omega_1^2 & \omega_1^3 & \omega_1^4 \\ 1+\sqrt{-1} & \omega_2 & \omega_2^2 & \omega_2^3 & \omega_2^4 \\ 1+\sqrt{-1} & \omega_3 & \omega_3^2 & \omega_3^3 & \omega_3^4 \\ 1+\sqrt{-1} & \omega_4 & \omega_4^2 & \omega_4^3 & \omega_4^4 \end{matrix}\right\|$$

and the determinant

$$\begin{vmatrix} \omega_1 & \omega_1^2 & \omega_1^3 & \omega_1^4 \\ \omega_2 & \omega_2^2 & \omega_2^3 & \omega_2^4 \\ \omega_3 & \omega_3^2 & \omega_3^3 & \omega_3^4 \\ \omega_4 & \omega_4^2 & \omega_4^3 & \omega_4^4 \end{vmatrix} = \omega_1\omega_2\omega_3\omega_4 \Pi_{i>j}(\omega_i-\omega_j)$$

can not be zero by our assumptions.

Now assume that $|\pi_i|^2 \neq q$ for some π_i and put

$$\omega_1=\pi_i, \quad \omega_2=\pi_i^*, \quad \omega_3=\bar{\pi}_i, \quad \omega_4=\bar{\pi}_i^*,$$

then according to $\omega_1=\omega_3$ or $\omega_1\neq\omega_3$, the assumptions in lem. 1 or in lem. 2 are satisfied by ω_1, ω_2 or by ω_1, ω_2, ω_3, ω_4. Let $a_1, \ldots\ldots, a_t$ be all the distinct roots of $G_1(Z)$, which are equal to none of the ω_i, then the polynomial

$$H(X)=\Pi_{i=1}^{t}(X-a_i)$$

has real coefficients. Let $J(X)$ be the polynomial of degree one $(\omega_1=\omega_3)$ or four $(\omega_1\neq\omega_3)$ with real coefficients such that

$$J(\omega_1)=1/H(\omega_1), \quad J(\omega_2)=-1/H(\omega_2),$$

and put

$$F(X)=J(X)\cdot H(X),$$

then $F(X)$ has also real coefficients and we have

$$\mathrm{Sp}\,\{F(\pi).\,F(\pi^*)\}=\begin{cases}-2\rho & (\omega_1=\omega_3),\\ -4\rho & (\omega_1\neq\omega_3),\end{cases}$$

where ρ means the common multiplicity of the roots ω_1, ω_2, ω_3 and ω_4 in $G_1(Z)$; it follows that

$$\mathrm{Sp}\{F(\pi)\cdot F(\pi^*)\}<0,$$

which proves the RIEMANN hypothesis.

Appendix.

We shall now give an application of our theory to the algebraic geometry on the Snrface

$$S=C_1\times C_2$$

where C_1 and C_2 are two Curves with respective genus g_1 and g_2. Let U be any complete Variety without multiple Point; let $\mathfrak{G}_0(U)$, $\mathfrak{G}_c(U)$ and $\mathfrak{G}_l(U)$ be respectively the group of all U–divisors, the group of the U–divisors which are continuously equivalent to zero, and the group of the U–divisors which are linearly equivalent to zero. If an S–divisor X belongs to $\mathfrak{G}_c$

(S), it foollows from SEVERI's formula (th. 2, § V) that X is a correspondence with valence zero:

$$X = (\varphi) + (A \times C_2) + (C_1 \times B),$$

where $A \in \mathfrak{G}_C(C_1)$; $B \in \mathfrak{G}_C(C_2)$ conversely such X belongs to $\mathfrak{G}_C(S)$. Thereby X belongs to $\mathfrak{G}_i(S)$ if and only if we have

$$A \in \mathfrak{G}_i(C_1), \quad B \in \mathfrak{G}_i(C_2) ;$$

it follows that the factor-group $\mathfrak{G}_C(S)/\mathfrak{G}_i(S)$ is the direct product of the factor-groups $\mathfrak{G}_C(C_1)/\mathfrak{G}_i(C_1)$ and $\mathfrak{G}_C(C_2)/\mathfrak{G}_i(C_2)$. Since the elements of the latter groups are in a one-to-one correspondence with the Points of the "Jacobian Varieties" $V(C_1)$, $V(C_2)$ of C_1, C_2 respectively, the "Picard Variety" $V(S)$ of S is the Product of $V(C_1)$ and $V(C_2)$:

$$V(S) = V(C_1) \times V(C_2);$$

in particular we have

$$\dim V(S) = \dim V(C_1) + \dim V(C_2)$$
$$= g_1 + g_2.$$

Moreover the module of the differential forms of degree one of the first kind on S is the direct sum of the similar modules attached to C_1 and C_2; it follows that the dimension of $V(S)$ is equal to the number of linearly independent differential forms of degree one of the first kind on S. Furthermore from the arithmetical structure of the Jacobian Variety, and from the results in § III, we can prove that the factor-group $\mathfrak{G}_0(S)/\mathfrak{G}_C(S)$ has a finite number of *independent* generators. This number is usually denoted by $\rho(S)$, and is called the "Picard number" of the Surface S ; $\rho(S)$ is 'in general', i. e. for 'general' Curves C_1, C_2 equal to 3, and at most $4g_1 \cdot g_2 + 2$:

$$2 \leqq \rho(S) \leqq 4g_1 g_2 + 2.$$

Thus we can solve the WEIL's conjecture on Picard Variety in the case of this special type of Surface ; this might be regarded as a first step in this important conjecture.