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On the weak Topology of an infinite Product Space.

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1. Introduction. We shall define a monotonic topology of a space R as a closure operator which assigns to each subset M of R a closure $\bar{M}\subset R$ with following properties

$$\bar{0}=0, \qquad M\supset N\rightarrow \bar{M}\supset \bar{N}.$$

If we assume furthermore

$$\overline{M\cup N}\subset \bar{M}\cup \bar{N},$$

then we say that the topology is additive.

In this note we define a weak monotonic topology and from it a weak additive topology of an infinite product space by means of the closure operator, and show that these topologies are the weakest respectively in all allowable topologies.

2. Let $R=P\{R^x\,|\,X\}$ be the $X=\{x\}$ product space of R^x whose points are $p=\{p^x\,|\,p^x\in R^x,\ x\in X\}$. Usually the topology of R is necessarily to satisfy the condition that the projection $\pi^x: R\rightarrow R^x$ is continuous. This condition is expressed by the closure operator as follows:

$$\pi^x(\bar{M})\subset\overline{\pi^x(M)}=\overline{M^x} \quad \text{for any} \quad M\subset R, \tag{1}$$

where the left side closure means that in R, and the right side closure in R^x.

If we define

$$^m\bar{M}=P\{\overline{M^x}\,|\,x\in X\} \quad \text{for any} \quad M\subset R,$$

this closure determines a monotonic topology of R, for it follows that

$$M\supset N\rightarrow M^x\supset N^x\rightarrow \overline{M^x}\supset\overline{N^x}\rightarrow P\{\overline{M^x}\,|\,X\}\supset P\{\overline{N^x}\,|\,X\}.$$

Clearly this topology $^m\bar{M}$ is the weakest in all topologies of R satisfying (1).

3. We shall define now the weakest additive topology of R. Let μ be a finite subdivision of $M(\subset R)$,

$$\mu: M=M_1\cup\cdots\cdots\cup M_{n(\mu)},$$

and let $\mathfrak{M}=\{\mu\}$ be the set of all finite coverings of M.

If we take as a new closure $\tilde{M}$ of M the set

$$\tilde{M}=\bigcap_{\mathfrak{M}}(\overset{m-}{M_1}\cup\ldots\ldots\cup\overset{m-}{M}_{n(\mu)}),$$

then the topology given by the closure $\tilde{M}$ is weaker than any additive ${}^a\overline{M}$ of R. For an additive topology is monotonic, therefore clearly

$$\overset{a-}{M}\subset\overset{m-}{M},$$

also for any $\mu\in\mathfrak{M}$

$$\overset{a-}{M}=\overset{a-}{M_1}\cup\ldots\ldots\cup\overset{a-}{M}_{n(\mu)}\subset\overset{m-}{M_1}\cup\ldots\ldots\cup\overset{m-}{M}_{n(\mu)},\quad \text{i.e.}\quad \overset{a-}{M}\subset\tilde{M}.$$

We prove next the additivity of $\tilde{M}$.

Let a binary covering of M be $M=A\cup B$, and the sets of all finite coverings of A, B repectively $\mathfrak{A}=\{\alpha\}$, $\mathfrak{B}=\{\beta\}$, then the closure $\tilde{A}$ and $\tilde{B}$ are from definition

$$\tilde{A}=\bigcap_{\mathfrak{A}}(\overset{m-}{A_1}\cup\ldots\ldots\cup\overset{m-}{A}_{n(\alpha)}),$$
$$\tilde{B}=\bigcap_{\mathfrak{B}}(\overset{m-}{B_1}\cup\ldots\ldots\cup\overset{m-}{B}_{n(\beta)}),$$

and

$$\tilde{A}\cup\tilde{B}=\bigcap_{\mathfrak{A},\mathfrak{B}}(\bigcup_{i=1}^{n(\alpha)}\overset{m-}{A_i}\cup\bigcup_{i=1}^{n(\beta)}\overset{m-}{B_i}).$$

Any two elements α and β determine a finite covering $\mu: M=A_1\cup\ldots\ldots\cup A_{n(\alpha)}\cup B_1\cup\ldots\ldots\cup B_{n(\beta)}$ of M, but all pairs (α,β) form a subset of $\mathfrak{M}$. Hence it follows

$$\tilde{A}\cup\tilde{B}\supset\tilde{M}. \tag{2}$$

Conversely we reduce from a covering μ of M to a pair of coverings α and β respectively of A and B such that

$$M=\bigcup M_i,\quad A=\bigcup(M_i\cap A)=\bigcup A_i,\quad B=\bigcup(M_i\cap B)=\bigcup B_i.$$

From monotonic property

$$\bigcup_{n(\mu)}\overset{m-}{M_i}\supset\bigcup\overset{m-}{A_i}\cup\bigcup\overset{m-}{B_i}\supset\tilde{A}\cup\tilde{B},$$

it follows that

$$\bigcap_{\mathfrak{M}} (\bigcup_{n(\mu)} \overset{m-}{M_i}) = \tilde{M} \supset \tilde{A} \cup \tilde{B}. \tag{3}$$

From (2) and (3) $\tilde{M}=\tilde{A}\cup\tilde{B}$ for every binary covering $M=A \cup B$.

The continuity of the projection π^x of R with the topology $\tilde{M}$ on R^x is clear from the fact

$$\pi^x(\tilde{M}) \subset \pi^x(\overset{m-}{M}) = \bar{M}^x.$$

4. We shall now consider the bases of neighborhood systems of these weakest topologies.

Let U^x be a neighborhood of a point p^x in R^x, and $U^{x\prime}$ denote the complement of U^x in R^x, then the subset

$$U = U^x \times P'\{R^y \mid y \in X - x\} \tag{4}$$

is a neighborhood of a point p whose x-coordinate is $\pi^x(p)=p^x$, for $\overset{m-}{U'} = \overset{m-}{\overline{U^{x\prime} \times P'\{R^y \mid y \in X-x\}}} = \overline{U^{x\prime}} \times P'\{R^y\}$ and $\overline{U^{x\prime}} \not\ni p^x$ reduce to $\overset{m-}{U'} \not\ni p$. When x and U^x rnn respectively through all elements of X and all neighborhoods U_i^x of p^x, then the system $\{U\}$ defined by (4) is a neighborhood system of p of R in the monotonic topology.

For let N be a neighborhood of the point p. This means $\overset{m-}{N'} \not\ni p$, i.e. $P\{\overline{N'^x} \mid x \in X\} \not\ni p$. Therefore for some x

$$\overline{N'^x} \not\ni p^x,$$

also $N'^{x\prime}$ is a neighborhood U^x of p^x in R^x, i.e. $N'^x = U^{x\prime}$. Clearly from $N' \subset P\{N'^x \mid X\} \subset U^{x\prime} \times P'\{R^y \mid X-x\} = U'$ the formula $N \supset U$ holds.

Hence N must be included in the neighborhood system $\{U\}$.

Next we consider the system of neighborhoods of the weakest additive topology $\tilde{M}$ which is stronger than the monotonic topology $\overset{m-}{M}$. A neighborhood $U = U^x \times P'\{R^y \mid y \in X-x\}$ of $p=\{p^x\}$ in the topology $\overset{m-}{M}$ therefore must be a neighborhood of p in the additive topology $\tilde{M}$. If U_1, U_2 are two neighborhoods of p thus defined, then from the additive property of the topology, $U_1 \cap U_2$ must be again a neighborhood of p. Hence the set of all the neighborhoods of p

$$U_1 \cap \ldots\ldots \cap U_n \ (n: \text{ finite}) : \begin{cases} U_i = U_i^{x(i)} \times P'\{R^y \mid y \in X - x(i)\}, \\ \overline{U_i^{x(i)\prime}} \not\ni p^{x(i)} \end{cases} \tag{5}$$

gives an additive topology, also the weakest additive. We conclude therefore that the neighborhoods $U=U^x\times P'\{R^y\}$ in $\overset{m-}{M}$ are a subbase of the neighborhood system in $\tilde{M}$. If the topologies of R^x are all additive, then the usual weak topology defined by means of neighborhoods is equivalent to the weakest additive topology $\tilde{M}$.

But when the topology of R^x is only monotonic, and not additive, then the set of subsets

$$U_1 \cap \ldots\ldots \cap U_n: \begin{cases} U_i=U_i^{x(i)}\times P'\{R^y \mid X-x(i)\} \\ \overline{U_i^{x(i)\prime}} \not\ni p^{x(i)} \\ x(i)\neq x(j), \quad \text{if} \quad i\neq j \end{cases}$$

does not form a neighborhood system of p in $\tilde{M}$. For in the formula (5) when $x(1)=x(2)=x$,

$$(U_1^x\times P'\{R^y\})\cap(U_2^x\times P'\{R^y\})=(U_1^x\cap U_2^x)\times P'\{R^y\},$$
$$\overline{U_i^{x\prime}}\not\ni p^x$$

is surely a neighborhood of p, i.e.

$$\overline{(U_1^x\cap U_2^x)'\times P'\{R^y\}}\not\ni p.$$

But $U_1^x\cap U_2^x$ is not necessarily a neighborhood of p^x in R^x. For example, consider two spaces R^1, R^2 defined as follows:

$$R^1=\{a,\ b,\ c\}.$$

$$\text{topology}:\quad \bar{a}=b,\ \bar{b}=a,\ \bar{c}=c,\ \overline{a\cup b}=a\cup b,\ \overline{b\cup c}=a\cup c,$$
$$\overline{a\cup c}=a\cup b\cup c,\ \overline{a\cup b\cup c}=a\cup b\cup c.$$

R^2=segment I $(0\leqq t\leqq 1)$ with the usual topology of real numbers.

In R^1 the neighborhoods of a are $b\cup c$, $a\cup b$, $a\cup b\cup c$, and in $R=R^1\times R^2$ two subsets $(a\cup b)\times I$, $(b\cup c)\times I$ are neighborhoods of a point $p=(a\times t)$ in the weakest additive topology. Hence the meet $((a\cup b)\times I)$ $\cup$ $(b\cup c)$ $\times I)=(b\times I)$ is a neighborhood of p. But b is never a neighborhood of a in R^1.

A final remark. Let X be a partially ordered set, and suppose that if and only if $x>y$, a continuous mapping f_{xy} of R^x in R^y exists, where $\{f_{xy}\}$ satisfies the transitive law. Then an infinite product Pr $\{R^x\}$ with relations is a space of points

$$p=\{p^x \mid x \in X\}, \quad x > y \rightarrow f_{xy}(p^x)=p^y.$$

Then the weak additive topology of $P_r\{R^x\}$ is defined by means of relative topology, for $P_r\{R^x \mid X\}$ is a subset of $P\{R^x \mid X\}$. This topology, for instance, agrees with the one of the projections nets of a compact metric space in the sence of Mr. H. Freudenthal.

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