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# On linearly ordered groups.

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A group $G$ is called a *linearly ordered group* (=l. o. group), when in $G$ is defined a linear order $a>b$, preserved under the group multiplication:

$$a>b \text{ implies } ac>bc \text{ and } ca>cb \text{ for all } c \text{ in } G.$$

A typical example is the additive group $R$ of all real numbers with respect to the usual order. Subgroups of $R$ are also linearly ordered and they are, as is well known, characterized among other l. o. groups by the condition that their linear orders are archimedean, that is to say, that for any positive elements[1] $a$, $b$ there is a positive integer $n$, so that it holds

$$a^n>b, \quad b^n>a.$$

Everett and Ulam have proved that we can define a linear order in a free group with two generators, so that it becomes a l. o. group[2]. In the following we shall generalize this theorem in the form that any l. o. group can be obtained by an order homomorphism from a proper l. o. free group, and then study the general character of group- and order-structure of these groups. Finally we shall add some examples which will illustrate our theorems.

We prove first some lemmas.

*Lemma 1. Let $G$ be a l. o. group and $P$ the set of all positive elements in $G$. $P$ has then following properties:*

i) *$e \notin P$, and if $x \neq e$ either $x \in P$ or $x^{-1} \in P$.*

ii) *$x \in P$ and $y \in P$ implies $xy \in P$.*

iii) *if $x \in P$, then $axa^{-1} \in P$ for all $a$ in $G$.*

*Conversely, if a group $G$ contains a subset $P$, having the properties i), ii), iii), we can then introduce a linear order in $G$ by defining,*

$$a>b, \text{ if } ab^{-1} \in P. \tag{1}$$

*Proof.* The former part is almost obviovs. We have only to note that iii) follows from $axa^{-1}>aea^{-1}=e$ for $x>e$. We prove the latter part According to i) and $ba^{-1}=(ab^{-1})^{-1}$ it can be seen that one and only one of the relations

$$a=b,\ a>b,\ b>a$$

holds for any $a$, $b$ in $G$. The transitivity follows then from ii). Moreover $a>b$ implies $(ac)(bc)^{-1}=ab^{-1}\epsilon P$, i e $ac>bc$ and also from iii) $(ca)(cb)^{-1}=c(ab^{-1})c^{-1}\epsilon P$, namely $ca>cb$, which completes the proof.

*Lemma 2. Let* $\{G_\alpha\}$ *be a set of l. o. groups and* $G$ *the (restricted or complete) direct product of* $G_\alpha$. *Then one can introduce a linear order in* $G$ *so that it becomes a l. o. group.*

*Proof.* We may consider that the groups $G_\alpha$ are well-ordered, where $\alpha$ runs over all transfinite numbers $\alpha<\Omega$. Elements of $G$ are then given by their components:

$$x=\{x_\alpha\},\ x_\alpha\epsilon G_\alpha.$$

Now let $P$ be the set of all such $x=\{x_\alpha\}$, that

$$x_\alpha=e_\alpha{}^{3)} \text{ for all } \alpha<\beta \text{ and } x_\beta>e_\beta \text{ in } G_\beta$$

for some transfinite number $\beta(<\Omega)$ We see readily that $P$ satisfies i), ii), iii) of Lemma 1 and thus we can introduce a linear order in $G$[4].

We now consider abelian groups and prove

*Lemma 3. A linear order can be defined in an abelian group A, if and only if A contains no element of finite order except the unit*[5].

*Proof.* The condition is necessary; if $x>e$ and $x^n=e$ with some integer $n>1$, then it would be $x^{-1}=x^{n-1}>e$, what is a contradiction. Now let $A$ be an abelian group with no element of finite order. We can then imbed $A$ in a direct product $G$ of groups $G_\alpha$, which are isomorphic to the additive group of all rational numbers. But as the latter is obviously a l. o. group, so is $G$ according to Lemma 2. The subgroup $A$ af $G$ can be therefore also linearly ordered, q e.d.

We now return to the general case and study the order homomorphism. We mean here by an *order homomorphism (isomorphism)* a homomorphic (isomorphic) mapping $a\rightarrow a'$ between two l. o. groups $G$, $G'$, so that $a\geqq b$ in $G$ implies $a'\geqq b'$ in $G'$. We can then easi.y prove the following two lemmas.

*Lemma 4. Let* $a\rightarrow a'$ *be an order homomorphism between* $G$, $G'$ *and let* $G'\cong G/N$, *where* $N$ *is a normal subgroup of* $G$. *Then*

$$a\geqq b\geqq e \text{ and } a\epsilon N \text{ implies } b\epsilon N \qquad (2)$$

*and the set P of positive elements in G consists of positive elements in N and the elements whose homomorphic maps in G' are positive. Conversely if a normal subgroup N of a l. o. group G has the above property (2), then elements of any coset $aN(\neq N)$ of $G/N$ are all positive or all negative. Defining positive or negative accordingly, $G/N$ becomes a l. o. group and the natural mapping $a\rightarrow aN$ then gives an order homomorphism between G and $G/N$.*

*Lemma 5. Let G be a group and N a normal subgroup of G. We suppose that N and G/N are both linearly ordered, and the order in N is invariant under the inner transformations of G, namely*

$$a\in N,\ a\geqq e \text{ implies } bab^{-1}\geqq e \text{ for all } b\in G. \tag{3}$$

*Then, if we take as positive elements of G positive elements in N and the elements belonging positive cosets in G/N, we can define a linear order in G. The natural mapping $G\rightarrow G/N$ gives then an order homomorphism.*

It is to be noted here particularly that the condition (3) is necessary in order that $G$ becomes a l. o. group, when $N$ and $G/N$ are so. But if $N$ is contained in the centre of $G$, (3) is trivially fulfiled, so that by a central extension we always obtain a l. o. group $G$. This remark will be useful soon afterwards.

Now we prove the following

*Theorem 1. For a given linearly ordered group G there is a linearly ordered free group F, so that G is the image of an order homomorphic mapping from F to G.*

*Proof.* We take a free group $F$ and a normal subgroup $N$ of $F$, so that $G\cong F/N$. By transferring the linear order in $G$, we can make $F/N$ a l. o. group. Now let $F_1=F, F_2, F_3,\ldots\ldots$ be the descending central chain of $F$[6] and put $N_i=F_i\cap N$. These groups are then all normal in $F$ and we have

$$N_1=N\geqq N_2\geqq N_3\geqq\ldots\ldots,\quad \cap_{i=1}^{\infty}N_i=e,$$

as $\cap_{i=1}^{\infty}F_i=e$[7]. From

$$[F,\ N_{i-1}]^{8)}=[F,\ F_{i-1}\cap N]\leqq[F,\ F_{i-1}]\cap[F,\ N]\leqq F_i\cap N=N_i,$$

$$N_{i-1}/N_i=(F_{i-1}\cap N)/(F_i\cap N)=(F_{i-1}\cap N)/(F_i\cap(F_{i-1}\cap N))\cong(F_i(F_{i-1}\cap N))/F_i,$$

we see that $N_{i-1}/N_i$ is contained in the centre of $F/N_i$ and has no element of finite order, because $F_i(F_{i-1} \cap N)/F_i$ is a subgroup of $F_{i-1}/F_i$, which is known to be free abelian[9]. We can therefore define a linear order in $N_{i-1}/N_i$ according to Lemma 3. By making use of the remark stated just before the theorem, a linear order can be defined, starting from $F/N_1 = F/N$, step by step to all factor groups $F/N_i$, so that

$$F/N_i \rightarrow F/N_{i-1} \tag{4}$$

gives always an order homomorphism. Now take an element $x \neq e$ in $F$. According to $\cap_{i=1}^{\infty} N_i = e$ there is an $N_i$, which does not contain $x$. We then call $x$ positive (or negative), if the coset of $x$ in $F/N_i$ is positive (or negative) in the sense of above defined linear order in $F/N_i$. By the successive order homomorphism (4), such a definition is uniquely determined independently of the choice of $N_i$, which does not contain $x$. It is then easy to see that $F$ thus becomes a l. o. group and $F \rightarrow F/N$ gives an order homomorphism (Lemma 4).

The above theorem shows at the same time that a free group $F$ admits various linear orders. If we take as $G$ the unit group in the above proof, we shall have a particular linear order in $F$, perhaps the simplest one. Positive elements in $F$ are then defined as follows. We first define an arbitrary linear order in each central factor group $F_{i-1}/F_i$. For an element $x \neq e$ in $F$ there is an index $i$ such that $x$ is contained in $F_{i-1}$ but not in $F_i$. We call then $x$ positive, if $x$ is in a positive coset in $F_{i-1}/F_i$. Now, when $F$ is thus ordered, we can define a topology in $F$ taking the subsets $\{x;\ a > x > b\}$ as neighbourhoods (order topology), and $F$ becomes then a topological group (every l. o. group can be thus considered as a topological group). It is note-worthy that this topology in $F$ just coincides with the one, which was defined by G. Birkhoff by making use of congruences with respect to $F_i$[10].

Now let $G$ be an arbitrary l. o. group. We define the absolute $|x|$ of $x$ in $G$ by

$$|x| = x, \text{ if } x \geqq e$$
$$= x^{-1}, \text{ if } x < e.$$

It is then easily seen that

$$|x| = |x^{-1}|, \qquad |x|\,|y| \geqq |xy|.$$

We denote by $G(x)$ the set of all $y$ in $G$, for which we have $|x|^n \geqq |y|$ with some positive integer $n$. From (5) it follows immediately that $G(x)$ forms a subgroup of $G$ and if $y \epsilon G(x)$ and $|z| \leqq |y|$, $z$ is also contained in $G(x)$ On the other hand we call $x$ and $y$ equivalent: $x \sim y$, when it holds $|x|^m \geqq |y|$, $|y|^n \geqq |x|$ with some integers $m$, $n$ and define a linear order in the set of these equivalent classes $\lambda$, $\mu$,...... by putting

$$\lambda > \mu, \quad \text{if} \quad x > y \quad \text{for all} \quad x \epsilon \lambda,\ y \epsilon \mu.$$

From the definition it follows immediately that the unit of $G$ forms a class with only one element. We denote by $L$ the set of all equivalent classes, which are different from this unit class.

It is clear that $G(x) = G(y)$ if and only if $x \sim y$ and we may therefore write

$$G(x) = G_\lambda, \text{ if } x \epsilon \lambda.$$

Now let $G_\lambda{}^*$ be the composite of all groups $G_\mu$ with $\mu < \lambda$ (if $\lambda$ is the first element in $L$, we take the unit group as $G_\lambda{}^*$). We can prove easily that if $G_\lambda = G(x)$, $G_\lambda{}^*$ is the set of all elements in $G_\lambda$, which are not equivalent to $x$ and consequently that $G_\lambda{}^*$ is normal in $G_\lambda$. Moreover as $G_\lambda$ and $G_\lambda{}^*$ satisfy (3) in Lemma 5, we can induce the linear order of $G$ into $G_\lambda/G_\lambda{}^*$. But this order in $G_\lambda/G_\lambda{}^*$ is archimedean, for all elements in $G_\lambda$, not contained in $G_\lambda{}^*$, are mutually equivalent. There is accordingly an order isomorphism $I_\lambda$ between $G_\lambda/G_\lambda{}^*$ and a subgroup $R_\lambda$ of the group of all real numbers $R$.

Next take an element $x$ and consider the inner automorphism $I_x$ of $G$ by $x$. If $G_\lambda = G(y)$, then $xG_\lambda x^{-1} = G(xyx^{-1}) = G_{\lambda'}$, so that $\lambda \rightarrow \lambda'$ gives a one-to-one order preserving correspondence in $L$. As $G_\lambda{}^*$ is transformed into $G_{\lambda'}{}^*$ by the same automorphism, $I_x$ induces an order isomorphism between $G_\lambda/G_\lambda{}^*$ and $G_{\lambda'}/G_{\lambda'}{}^*$, which we denote by $I_{x,\lambda}$. $I_{\lambda'} I_{x,\lambda} I_\lambda^{-1}$ then gives an order isomorphism between $R_\lambda$ and $R_{\lambda'}$, so that

$$r' = I_{\lambda'} I_{x,\lambda} I_\lambda^{-1}(r) = r_{x,\lambda} r, \text{ for } r \epsilon R_\lambda,\ r' \epsilon R_{\lambda'},$$

with some positive real number $r_{x,\lambda} > 0$.

We have thus obtained the following

*Theorem 2. For any linearly ordered group $G$, there is a linearly ordered set $L = \{\lambda, \mu, \ldots\ldots\}$ and a corresponding sequence of subgroups $\{G_\lambda;\ \lambda \epsilon L\}$ of*

*G, which have following properties:*

*1)* $G_\lambda > G_\mu$, *if* $\lambda > \mu$ ($G_\lambda \neq e$, *if* $\lambda$ *is the first element in* $L$);

*2) for any* $x$ *in* $G$, *there is an element* $\lambda$ *in* $L$, *such that* $x$ *is contained in* $G_\lambda$, *but not in* $G_\mu$ *for all* $\mu < \lambda$;

3) *let* $G_\lambda^*$ *be the composite of all* $G_\mu$ *with* $\mu < \lambda$ ($G_\lambda^* = e$, *if* $\lambda$ *is the first element in* $L$). $G_\lambda^*$ *is then normal in* $G_\lambda$ *and there is an isomorphism* $I_\lambda$ *between* $G_\lambda/G_\lambda^*$ *and a subgroup* $R_\lambda$ *of the group of all real numbers* $R$;

*4) if we denote by* $I_x$ *the inner automorphism of* $G$ *by the element* $x$, *each* $G_\lambda$ *goes to some other* $G_{\lambda'}$ *under* $I_x$. $\lambda \rightarrow \lambda'$ *then gives an order preserving one-to-one correspondence in* $L$. $G_\lambda^*$ *is transformed consequently to* $G_{\lambda'}^*$ *and* $I_x$ *induces an isomorphism* $I_{x,\lambda}$ *between* $G_\lambda/G_\lambda^*$ *and* $G_{\lambda'}/G_{\lambda'}^*$. *Further the isomorphism* $I_{\lambda'} I_{x,\lambda} I_\lambda^{-1}$ *between* $R_\lambda$ *and* $R_{\lambda'}$ *is given by*

$$r' = I_{\lambda'} I_{x,\lambda} I_\lambda^{-1}(r) = r_{x,\lambda} r, \text{ for } r \in R_\lambda,\ r' \in R_{\lambda'},$$

*where* $r_{x,\lambda}$ *is a certain positive number;*

*5) let* $x \in G_\lambda$, $x \notin G_\mu$ *for* $\mu < \lambda$ (*cf. 2*). $x$ *is then positive if and only if the coset of* $x$ *in* $G_\lambda/G_\lambda^*$ *is mapped by* $I_\lambda$ *into a positive number in* $R_\lambda$.
*For a given linearly ordered group* $G$, *such* $L$ *and* $G_\lambda$ *are uniquely determined by the relations 1)—4) (up to an isomorphism)*

*Conversely if* $G$ *is a group and there exists a sequence of subgroups* $G_\lambda$ *of* $G$, *corresponding to a linearly ordered set* $L$, *and satisfying above conditions 1)—4) and if we define positive elements by 5), then* $G$ *becomes a linearly ordered group.*

As $L$ and the corresponding $R_\lambda$ are uniquely determined for a given l. o. group, we may classify the set of all l. o. groups by grouping together those groups into a class, which have the same $L$ and $R_\lambda$ in the sense of above theorem. We note here some remarks on this classification. First there is no restriction upon $L$ and $R_\lambda$. Namely, if a linearly ordered set $L$ and corresponding subgroups $R_\lambda$ of $R$ are arbitrarily given, there is always a l. o group $G$, to which $L$ and $R$ belong in the above sense: to obtain such $G$, we have only to construct the restricted direct product of all $R_\lambda$ ($\lambda \in L$) and take as $G_\lambda$ the restricted direct product of all $R_\mu$ $\mu \leqq \lambda$. We have thus proved in the same time that every such class contains an abelian group.

Next let us suppose that $L$ is such a linearly ordered set that it admits no order preserving one-to-one correspondence except the identity. This is particularly the case, when $L$ is well-ordered or has an order inverse to a well-ordered set. Then we have from 4) in Therem 2 that each $G_\lambda$ must be normal in $G$ and that $r \to r' = r_{\lambda, x} \cdot r$ ($r_{\lambda, x} > 0$) gives an automorphism of $R_\lambda$. $G$ is thus a solvable group in a generalized sense[11]. If further $R_\lambda$ admits no automorphism of that type except the identity, $G_\lambda / G_\lambda^*$ is contained in the centre of $G/G_\lambda^*$. For example if $L$ is well-ordered (or has an order inverse to it) and all $R_\lambda$ are the group of rational integers (namely free cyclic groups), every group of type $\{L, R_\lambda\}$ is nilpotent in a generalized sense[12]. I may be possible to determine in this manner the group-theoretical structure of l. o. groups of various types more explicitely. Here we do not enter in this problem[13].

We give finally some examples of l. o. groups of various types.

*Example 1.* Let $G^{(1)}$ be the restricted direct product of denumerable number of free cyclic groups $C_1, C_2 \ldots\ldots$ and let $G_n$ be the direct product of $C_1, C_2, \ldots\ldots, C_n$. By theorem 2 we can define a linear order in $G^{(1)}$. $L$ is then of type $\omega = 1, 2, 3, \ldots\ldots$ and $R_\lambda$ are the group of all rational integers $I$.

*Example 2.* Again, let $G^{(2)}$ be the (complete or restricted) direct product of $C_1, C_2, \ldots\ldots$ Now take as $G_n$ the direct product of $C_n, C_{n+1}, \ldots\ldots$ $L$ is here of type $\omega^* = \ldots\ldots, 3, 2, 1$ and $R_\lambda$ are again $I$. The linear order considered here coincides with the one stated in the proof of Lemma 2.

*Example 3.* Let $G^{(3)}$ be generated by three elements $a, b, c$ with relations

$$ab = ba, \quad cac^{-1} = a, \quad cbc^{-1} = ab.$$

An element in $G$ can be written uniquely in the normal form $a^x b^y c^z$ ($x, y, z = 0, \pm 1, \pm 2, \ldots\ldots$). If we put $G_1 = \{a\}$, $G_2 = \{a, b\}$, $G_3 = \{a, b, c\}$, $G^{(3)}$ becomes a l. o. group. Here $L = \{1, 2, 3\}$, $R_1 = R_2 = R_3 = I$. $G^{(3)}$ may be perhaps one of the simplest non-abelian l. o. group.

*Example 4.* As a generalization of $G^{(3)}$ let us consider $G^{(4)} = \{z, a_1, a_2, \ldots\ldots\}$ with relations

$$za_i = a_i z, \quad a_i a_j = a_j a_i, \text{ for } |j - i| \neq 1, \quad a_i a_{i+1} = a_{i+1} a_i z.$$

Putting $G_n = \{z, a_1, \ldots\ldots, a_{n-1}\}$ we have a non-abelian l. o. group $G^{(4)}$ of the same type with $G^{(1)}$.

*Example 5.* A free group $F$ with a finite number of generators has the same type with $G^{(2)}$, if we define the order by making use of $F_i$.

*Example 6.* Let $G^{(6)}$ be the set of all linear functions $f(x)=ax+b$, where $a>0$, $-\infty<b<\infty$. Defining the group composition $h=f\times g$ by $h(x)=f(g(x))$ and the order $f>g$ by the value of these functions for sufficiently large $x$, $G^{(6)}$ becomes a l. o. group. Here $G_1=\{f(x)=x+b\,;\ -\infty<b<\infty\}$, $G_2=G^{(6)}$, $L=\{1, 2\}$ and $R_1=R_2=R$. Moreover we have

$$r_{f,1}=a, \text{ for } f(x)=ax+b,$$

whereas $r_{f,2}=1$ for all $f$.

*Example 7.* Take the additive group $G^{(7)}$ of all such real functions $f(x)$ which are defined on the interval $[0, 1]$ and are represented in the $x$-$y$ plane by a broken line through the origin $(0, 0)$. We define $f(x)>g(x)$, if there is such $a$ $(0\leqq a<1)$, so that $f(x)=g(x)$ for $0\leqq x\leqq a$ and $f(a+\varepsilon)>g(a+\varepsilon)$ for sufficiently small $\varepsilon>0$. Putting

$$G_\lambda=\{f(x)\,;\ f(x)=0 \text{ for } 0\leqq x\leqq 1-\lambda\},\ 0<\lambda\leqq 1,$$

we see that $L$ coincides with the interval $(0, 1]$ and $R_\lambda=R$ for all $\lambda$.

*Example 8.* Let $G^{(8)}$ be the subset of all those functions considered in Example 7, which are moreover monotone increasing (in the strict sense) and go through the point $(1, 1)$. We define the group composition $h=f\times g$ by $h(x)=f(g(x))$. By the same order as in Example 7, $G^{(8)}$ becomes a non-abelian l. o. group. Here $G_\lambda$ is given by

$$G_\lambda=\{f(x)\,;\ f(x)=x \text{ for } 0\leqq x\leqq 1-\lambda\},\ 0<\lambda\leqq 1,$$

and again $L=(0, 1]$. $R=R_\lambda$ for all $\lambda$. But in this case no $G_\lambda$ except $G_1=G^{(8)}$ is normal in $G^{(8)}$. In fact we have

$$fG_\lambda f^{-1}=G_{f(\lambda)}.$$

On the other hand $r_{f,\lambda}$ are here all 1. If we want to obtain an example in which $r_{f,\lambda}$ is not constant, we have only to extend[14] the group $G^{(7)}$ by $G^{(8)}$, putting

$$gfg^{-1}=f(g^{-1}(x)), \text{ for } f\in G^{(7)},\ g\in G^{(8)}.$$

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**References.**

1) An element $a$ is called positive in a l. o. group, if $a > e$, where $e$ is the unit of the group.

2) Everett, C. J. and Ulam, S. On ordered groups, Trans. Amer. Math. Soc. 57 (1945).

3) $e_\alpha$ is the unit of $G_\alpha$.

4) In the case of the restricted direct product we can define a linear order in another way. Cf. the first example at the end of the paper.

5) We can prove more generally (by making use of subsequent lemmas) that a linear order can be defined in such a group, which has no element of finite order and has a finite or transfinite ascending central chain.

6) Cf. H. Zassenhaus, Lehrbuch der Gruppentheorie (1937), p. 119 or E. Witt, Treue Darstellung Liescher Ringe, Crelle Jour., 177 (1937).

7) E. Witt, l. c. 6).

8) The bracket means the commutator group.

9) E. Witt, l. c. 6).

10) G. Birkhoff, Moore-Smith convergence in general topology, Ann. Math., 38 (1937).

11) If $L$ is particularly a finite set, $G$ is surely solvable in the usual sense. Cf. Example 3, 6 below.

12) For the generalization of nilpotent groups cf. R. Baer, Nilpotent groups and their generalizations, Trans. Math. Amer. Math. Soc., 47 (1940).

13) In this connection hold particularly remarkable relations between l. o. groups and nilpotent groups, cf. 5).

14) Cf. Lemma 5.