Tohoku Math. J.
71 (2019), 35–52

RATIONAL ORBITS OF PRIMITIVE TRIVECTORS IN DIMENSION SIX

AKIHIKO YUKIE

(Received July 13, 2016, revised August 17, 2016)

Abstract. Let $G = \mathrm{GL}(1) \times \mathrm{GSp}(6)$ and V be the irreducible representation of G of dimension 14 over a field of characteristic not equal to $2, 3$. This is an irreducible prehomogeneous vector space. We determine generic rational orbits and their stabilizers of this prehomogeneous vector space.

1. Introduction. Let k be a field of characteristic not equal to $2, 3$. We denote the separable closure and the algebraic closure of k by $k^{\mathrm{sep}}, \overline{k}$ respectively. If $\sigma, \tau \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$, we define $(\sigma\tau)(x) = \tau(\sigma(x))$ for $x \in k^{\mathrm{sep}}$ and so $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts on k^{sep} from the right. If $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$ and $x \in k^{\mathrm{sep}}$, we use the notation x^σ for the action of σ. If G is an algebraic group over k, and R is a k-algebra, G_R is the group of R-rational points of G. For algebraic groups, we only consider representations which are algebraic. If G is an algebraic group, we use the notation G° for its identity component. If V is a representation of G over k and $x \in V$, we denote the G-orbit of x by $G \cdot x$ to avoid confusion with the stabilizer G_x. We use similar notations for the G_k-orbit, etc. We denote the group of $n \times n$ invertible matrices by $\mathrm{GL}(n)$. We use the notation $\mathrm{M}(n)$ for the space of all $n \times n$ matrices. $\mathrm{M}(n)_R$ for a k-algebra R is defined similarly as above. We denote the $n \times n$ unit matrix by I_n.

Let $W = k^6$ and $W_1 = \wedge^3 W$. These are irreducible representations of $\mathrm{GL}(6)$. Let $\mathbb{e}_1, \ldots, \mathbb{e}_6$ be the coordinate vectors of W. If $1 \le i_1, \ldots, i_t \le 6$ are distinct, we use the notation $e_{i_1 \cdots i_t} = \mathbb{e}_{i_1} \wedge \cdots \wedge \mathbb{e}_{i_t}$. For $x, y \in W_1$, we define $B(x, y) = x \wedge y \in \wedge^6 W \cong k$. This is a non-degenerate alternating bilinear form on W_1. It is easy to see that $B(gx, gy) = \det g\, B(x, y)$ for $g \in \mathrm{GL}(6)$. Let $\omega = e_{14} + e_{25} + e_{36}$. We put

$$\begin{aligned} \mathrm{Sp}(6) &= \{g \in \mathrm{GL}(6) \mid g\omega = \omega\}, \\ \mathrm{GSp}(6) &= \{g \in \mathrm{GL}(6) \mid {}^\exists c(g) \in \mathrm{GL}(1), g\omega = c(g)\omega\}. \end{aligned} \tag{1.1}$$

These are connected algebraic subgroups of $\mathrm{GL}(6)$. It is well-known that $\mathrm{Sp}(6)$ is a simple group, $\mathrm{GSp}(6)$ is a reductive group and $c(g)$ is a rational character of $\mathrm{GSp}(6)$ with kernel $\mathrm{Sp}(6)$.

Let $G = \mathrm{GL}(1) \times \mathrm{GSp}(6)$. We define an action of $\mathrm{GL}(1)$ on W_1 assuming that $\alpha \in \mathrm{GL}(1)$ acts by multiplication by α. This makes W_1 a representation of G. The subspace $U = \{v \wedge \omega \mid v \in W\} \subset W_1$ is invariant by the action of G. So $V = W_1/U$ is a representation of G defined over k. The reason why we use this G instead of a group like $\mathrm{GSp}(6)$ or $\mathrm{GL}(1) \times \mathrm{Sp}(6)$ is that it

2010 *Mathematics Subject Classification.* Primary 11S90; Secondary 11R34.
Key words and phrases. Prehomogeneous vector spaces, trivectors, rational orbits.

avoids unessential complications regarding rational orbits (the reader should see the comment after (4.1)).

Let $U^{\perp}$ be the orthogonal complement of U with respect to B, i.e.,

$$U^{\perp} = \{v \in W_1 \mid v \wedge w = 0 \ {}^{\forall} w \in U\}.$$

Since U is G-invariant, $U^{\perp}$ is also G-invariant. Since $B(x, y)$ is non-degenerate, the map $U^{\perp} \to V$ induced by the inclusion map $U^{\perp} \to W_1$ is an isomorphism as representations of G. It is known that V is an irreducible representation of G (without the assumption $\mathrm{ch}\, k \neq 2, 3$). We briefly review the irreducibility of V at the end of Section 2.

Obviously, $\dim V = 14$. We use the same notation $e_{i_1 \cdots i_t}$ for its image in V by abuse of notation. We put

$$w = e_{123} + e_{456} \in V. \tag{1.2}$$

Note that $w \in U^{\perp}$.

The pair (G, V) is an example of what we call a prehomogeneous vector space. We review the definition of prehomogeneous vector spaces as follows.

DEFINITION 1.3. Let G be a connected reductive group, V a representation and χ a non-trivial primitive character of G, all defined over k. Then (G, V, χ) is called a *prehomogeneous vector space* if it satisfies the following properties.

(1) There exists a Zariski open orbit.
(2) There exists a non-constant polynomial $\Delta(x) \in k[V]$ such that $\Delta(gx) = \chi(g)^a \Delta(x)$ for a positive integer a.

The polynomial Δ is called a *relative invariant polynomial*.

In [6, p.35, DEFINITION 1], the definition of prehomogeneous vector spaces does not include the reductiveness of G nor the the existence of a relative invariant polynomial. However, we included these assumptions because we only consider those satisfying these conditions.

If V is irreducible, the above χ is unique and if $\Delta(x)$ is a relative invariant polynomial of the lowest degree, any relative invariant polynomial is a constant times a power of $\Delta(x)$. Since we only consider irreducible prehomogeneous vector spaces in this paper, we use the notation (G, V) instead of (G, V, χ). Let $V^{\mathrm{ss}} = \{x \in V \mid \Delta(x) \neq 0\}$. Points in V^{ss} are called *semi-stable points*.

We shall show in Section 3 that (G, V) in the present paper is an irreducible *regular* prehomogeneous vector space in the following sense. We only consider irreducible prehomogeneous vector spaces for simplicity.

DEFINITION 1.4. Suppose that (G, V) is an irreducible prehomogeneous vector space. If there exists $w \in V$ such that $G \cdot w \subset V$ is Zariski open and the scheme-theoretic stabilizer G_w is smooth and reductive, (G, V) is said to be *regular*.

Note that this notion of regularity coincides with that in [6, pp.60,61, DEFINITION 7] if $k = \mathbb{C}$. Also if $k = \mathbb{C}$, (G, V) in this paper is known to be regular (see [6, p.108, PROPOSITION 22]).

The reason why we consider the notion of regularity is that in the situation of Definition 1.4, $V^{\mathrm{ss}}_{k^{\mathrm{sep}}} = G_{k^{\mathrm{sep}}} \cdot w$ (see [8] or [3, Corollary 2.4]). If $x \in V^{\mathrm{ss}}_k$ and $x = g_x w$ where $g_x \in G_{k^{\mathrm{sep}}}$, $\{g_x^{-1} g_x^{\sigma}\}_{\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)}$ determines an element, say c_x, of the first Galois cohomology set $\mathrm{H}^1(k, G_w)$ (we shall review the definition of the first Galois cohomology set in Section 3). This enables us to start a cohomological consideration of orbits because of the following well-known theorem (see [2, pp.268,269] for example).

THEOREM 1.5. *If* (G, V) *is as in Definition* 1.4*, the map*

$$G_k \backslash V^{\mathrm{ss}}_k \ni x \mapsto c_x \in \mathrm{Ker}(\mathrm{H}^1(k, G_w) \to \mathrm{H}^1(k, G)) \tag{1.6}$$

is well-defined and bijective.

Note that it is assumed in [2] that $\mathrm{ch}\, k = 0$. However, the proof of the above theorem works as long as $V^{\mathrm{ss}}_{k^{\mathrm{sep}}}$ is a single $G_{k^{\mathrm{sep}}}$-orbit.

Let Ex(2) be the set of isomorphism classes of extensions of k of degree up to two. Note that since we are assuming $\mathrm{ch}\, k \neq 2, 3$, any quadratic extension of k is a separable extension of k. If k_1/k is a quadratic extension (which is Galois of course), let $\sigma(k_1) \in \mathrm{Gal}(k_1/k)$ be the non-trivial element. If A is a square matrix with entries in k_1, we define $A^* = {}^t A^{\sigma(k_1)}$ where $A^{\sigma(k_1)}$ is the matrix obtained by applying $\sigma(k_1)$ to all entries. If $Q = (q_{ij})$ is a 3×3 matrix with entries in k_1, Q is said to be *Hermitian* if $Q^* = Q$. Let $\mathrm{H}_3(k_1)$ be the k-vector space of 3×3 Hermitian matrices with entries in k_1 and $\mathrm{H}_{3,\mathrm{ns}}(k_1)$ the subset of $\mathrm{H}_3(k_1)$ consisting of non-singular matrices. Let

$$\mathrm{SH}_3(k_1) = \{Q \in \mathrm{H}_{3,\mathrm{ns}}(k_1) \mid \det Q = 1\}\,.$$

If $Q \in \mathrm{H}_{3,\mathrm{ns}}(k_1)$, we define the unitary group $\mathrm{U}(k_1, Q)$ and the special unitary group $\mathrm{SU}(k_1, Q)$ as follows:

$$\begin{aligned} \mathrm{U}(k_1, Q) &= \{g \in \mathrm{GL}(3)_{k_1} \mid gQg^* = Q\}\,, \\ \mathrm{SU}(k_1, Q) &= \{g \in \mathrm{U}(k_1, Q) \mid \det g = 1\}\,. \end{aligned} \tag{1.7}$$

We shall define these groups as algebraic groups over k in Section 2. Let

$$Q_1 = \mathrm{I}_3,\ Q_2 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}. \tag{1.8}$$

If $Q = Q_1, Q_2$, we use the notation $\mathrm{U}(k_1, 3), \mathrm{SU}(k_1, 3)$ and $\mathrm{U}(k_1, 1, 2), \mathrm{SU}(k_1, 1, 2)$ for $\mathrm{U}(k_1, Q), \mathrm{SU}(k_1, Q)$ respectively. The group $\mathrm{GL}(3)_{k_1}$ acts on $\mathrm{H}_3(k_1)$ by

$$\mathrm{GL}(3)_{k_1} \times \mathrm{H}_3(k_1) \ni (g, Q) \mapsto gQg^* \in \mathrm{H}_3(k_1)\,.$$

The action of $\mathrm{GL}(3)_{k_1}$ (resp. $\mathrm{SL}(3)_{k_1}$) on $\mathrm{H}_3(k_1)$ leaves $\mathrm{H}_{3,\mathrm{ns}}(k_1)$ (resp. $\mathrm{SH}_3(k_1)$) stable. Also $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{SH}_3(k_1)$ by $Q \mapsto {}^t Q^{-1}$.

Let

$$\nu = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}. \tag{1.9}$$

$\mathbb{Z}/2\mathbb{Z}$ acts on SL(3) by assuming that the action of the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ is $g \mapsto {}^t g^{-1}$. This action of $\mathbb{Z}/2\mathbb{Z}$ on SL(3) defines a semi-direct product structure $\mathrm{SL}(3)_{k_1} \rtimes (\mathbb{Z}/2\mathbb{Z})$. For the rest of this paper, $\mathrm{SL}(3)_{k_1} \rtimes (\mathbb{Z}/2\mathbb{Z})$ means the semi-direct product in this sense.

The following theorem is the main result of this paper.

THEOREM 1.10. *There is a map* $\gamma_V : G_k\backslash V_k^{\mathrm{ss}} \to \mathrm{Ex}(2)$ *with the following properties* (1)–(4).

(1) $\gamma_V^{-1}(k) = G_k \cdot w$.
(2) $G_w^\circ \cong \mathrm{GL}(1) \times \mathrm{SL}(3)$ *and* $G_w/G_w^\circ \cong \mathbb{Z}/2\mathbb{Z}$. *Also* G_w/G_w° *is represented by* $(1, \nu)$.
(3) *If* k_1 *is a quadratic extension of* k, $\gamma_V^{-1}(k_1)$ *is in bijective correspondence with* $(\mathrm{SL}(3)_{k_1} \rtimes (\mathbb{Z}/2\mathbb{Z}))\backslash \mathrm{SH}_3(k_1)$.
(4) *If* $G \cdot x \in \gamma_V^{-1}(k_1)$ *corresponds to the orbit of* $Q \in \mathrm{SH}_3(k_1)$, $G_x^\circ \cong \mathrm{GL}(1) \times \mathrm{SU}(k_1, Q)$. *Also* G_x/G_x° *is represented by an element of* $G_{x\,k}$ *of order two.*

We describe γ_V and the correspondence in Theorem 1.10(3) in details in Section 4 (see Theorem 4.11).

The case in the present paper is one of several irreducible prehomogeneous vector spaces where the interpretation of rational orbits is unknown. Rational orbits of prehomogeneous vector spaces sometimes have interesting arithmetic interpretations, especially when they are related to field extensions. One possible outcome arising from the interpretation of rational orbits of the present case is the expected density theorem if one can carry out necessary global and local zeta function theories. If $k = \mathbb{Q}$, one can expect to obtain the density of the "unnormalized Tamagawa numbers" of $\mathrm{SU}(F, 3)$, $\mathrm{SU}(F, 1, 2)$ of all quadratic fields F. The rank of the group Sp(6) is three and so the expected amount of labor necessary for the zeta function theories is fairly large, but probably not impossible.

We review the first Galois cohomology set and the irreducibility of the representation V in Section 2. In Section 3, we determine G_w for the element w in (1.2). We prove the main theorem in Section 4. In Section 5, we specialize to the case of number fields and describe the set of rational orbits more precisely. In particular, we describe representatives of rational orbits explicitly when $k = \mathbb{Q}$.

The author would like to thank Tamotsu Ikeda for helpful discussions. The author also would like to thank the referee for pointing out various mistakes in the manuscript and suggesting the title of the paper.

2. Preliminaries. In this section we review the definition of the first Galois cohomology set and describe $\mathrm{H}^1(k, \mathrm{SU}(k_1, 3))$ (see (1.7)). Also we review the irreducibility of the representation V.

If G is an algebraic group over k, a 1-cocycle with coefficients in G is a continuous map $h : \mathrm{Gal}(k^{\mathrm{sep}}/k) \ni \sigma \mapsto h_\sigma \in G_{k^{\mathrm{sep}}}$, where $G_{k^{\mathrm{sep}}}, \mathrm{Gal}(k^{\mathrm{sep}}/k)$ are equipped with the discrete topology and the Krull topology respectively, such that $h_{\sigma\tau} = h_\tau^\sigma h_\sigma$ for all $\sigma, \tau \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$. We use the notation $\{h_\sigma\}_{\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)}$ or $\{h_\sigma\}_\sigma$ for this 1-cocycle. Two 1-cocycles $\{h_\sigma\}_\sigma, \{h'_\sigma\}_\sigma$ are equivalent if there exists $g \in G_{k^{\mathrm{sep}}}$ such that $h'_\sigma = g^{-1} h_\sigma g^\sigma$ for all $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$. This is an equivalence relation and we denote the quotient set by $\mathrm{H}^1(k, G)$. This set $\mathrm{H}^1(k, G)$ is called the *first Galois cohomology set*. If L/k is a finite Galois extension, one can define 1-cocycles $h : \mathrm{Gal}(L/k) \ni \sigma \mapsto h_\sigma \in G_L$ similarly and define the first Galois cohomology set $\mathrm{H}^1(L/k, G)$. It is easy to see that

$$\mathrm{H}^1(k, G) = \varinjlim_L \mathrm{H}^1(L/k, G)$$

where the inductive limit on the right hand side is with respect to all finite Galois extensions L of k.

Note that if $\{h_\sigma\}$ is a 1-cocycle, $h_1 = 1_G$ ($1, 1_G$ are the unit elements of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, G_k respectively). If $h_\sigma = 1_G$ for all $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$, $\{h_\sigma\}_\sigma$ is a 1-cocycle. We call the cohomology class in $\mathrm{H}^1(k, G)$ determined by this 1-cocycle, the *trivial class* and use the notation 1. It is well-known (see [7, p.122, Lemma 1]) that

$$\mathrm{H}^1(k, \mathrm{GL}(n)),\ \mathrm{H}^1(k, \mathrm{SL}(n))$$

are trivial for all n (consider the exact sequence $1 \to \mathrm{SL}(n) \to \mathrm{GL}(n) \to \mathrm{GL}(1) \to 1$ and use the surjectivity of $\det : \mathrm{GL}(n)_k \to k^\times$ for $\mathrm{H}^1(k, \mathrm{SL}(n))$). We only need the case $n = 3$ in this paper.

It is well-known (see [7, p.123, Proposition 3]) that $\mathrm{H}^1(k, \mathrm{Sp}(2n))$ is trivial for all n. One can define $\mathrm{GSp}(2n)$ similarly as in the case $\mathrm{GSp}(6)$ and there is an exact sequence

$$1 \to \mathrm{Sp}(2n) \to \mathrm{GSp}(2n) \to \mathrm{GL}(1) \to 1\,.$$

So there is an exact sequence

$$\mathrm{H}^1(k, \mathrm{Sp}(2n)) \to \mathrm{H}^1(k, \mathrm{GSp}(2n)) \to \mathrm{H}^1(k, \mathrm{GL}(1))$$

(meaning that the inverse image of $1 \in \mathrm{H}^1(k, \mathrm{GL}(1))$ coincides with the image of $\mathrm{H}^1(k, \mathrm{Sp}(2n))$). So $\mathrm{H}^1(k, \mathrm{GSp}(2n))$ is also trivial. We only need the case $n = 3$ in this paper.

If k_1/k is a quadratic extension, and G is an algebraic group over k_1, the restriction of scalar $\mathrm{R}_{k_1/k}G$ is the algebraic group over k such that if L/k is a Galois extension containing k_1, $(\mathrm{R}_{k_1/k}G)_L = \{(g_1, g_2) \mid g_1, g_2 \in G_L\}$ and the action of $\sigma \in \mathrm{Gal}(L/k)$ on (g_1, g_2) is (g_1^σ, g_2^σ) (resp. (g_2^σ, g_1^σ)) if the restriction of σ to k_1 is trivial (resp. non-trivial). Let $\sigma(k_1)$ be the non-trivial element of $\mathrm{Gal}(k_1/k)$ as in Introduction.

If $Q \in \mathrm{H}_{3,\mathrm{ns}}(k_1)$, $\mathrm{U}(k_1, Q)$ (resp. $\mathrm{SU}(k_1, Q)$) is the algebraic subgroup of $\mathrm{R}_{k_1/k}\mathrm{GL}(3)$ (resp. $\mathrm{R}_{k_1/k}\mathrm{SL}(3)$) such that if L/k is a Galois extension containing k_1,

$$\begin{aligned} \mathrm{U}(k_1, Q)_L &= \{(g, {}^tQ\,{}^tg^{-1}\,{}^tQ^{-1}) \mid g \in \mathrm{GL}(3)_L\}, \\ \mathrm{SU}(k_1, Q)_L &= \{(g, {}^tQ\,{}^tg^{-1}\,{}^tQ^{-1}) \mid g \in \mathrm{SL}(3)_L\}. \end{aligned} \tag{2.1}$$

The set of k-rational points of $\mathrm{U}(k_1, Q)$ is

$$\{(g, g^{\sigma(k_1)}) \mid g \in \mathrm{GL}(3)_{k_1}, g^{\sigma(k_1)} = {}^tQ\,{}^tg^{-1}\,{}^tQ^{-1}\}$$

$$= \{(g, g^{\sigma(k_1)}) \mid g \in \mathrm{GL}(3)_{k_1}, gQg^* = Q\},$$

which coincides with (1.7). The set of rational points of $\mathrm{SU}(k_1, Q)$ is similar. It is well-known that $\mathrm{U}(k_1, Q), \mathrm{SU}(k_1, Q)$ are k-forms of $\mathrm{GL}(3), \mathrm{SL}(3)$ respectively. So $\mathrm{U}(k_1, Q), \mathrm{SU}(k_1, Q)$ are smooth reductive groups over k.

The following proposition is proved conceptually in [5, p.403]. However, we need an explicit description of cohomology classes and so we give a relatively computational proof here.

PROPOSITION 2.2. *There is a bijective correspondence between*

$$\mathrm{H}^1(k, \mathrm{SU}(k_1, 3)),\ \mathrm{H}^1(k_1/k, \mathrm{SU}(k_1, 3)),\ \mathrm{SL}(3)_{k_1}\backslash \mathrm{SH}_3(k_1).$$

Moreover, if $Q \in \mathrm{SH}_3(k_1)$*, the corresponding cohomology class in* $\mathrm{H}^1(k_1/k, \mathrm{SU}(\mathrm{k}_1, 3))$ *is determined by the* 1*-cocycle* $(h_{\sigma,1}, {}^t h_{\sigma,1}^{-1})$ *such that* $h_{1,1} = 1, h_{\sigma(k_1),1} = Q$.

PROOF. Suppose that $\{h_\sigma\}_\sigma$ is a 1-cocycle with coefficients in $\mathrm{SU}(k_1, 3)$. Then h_σ is in the form $(h_{\sigma,1}, {}^t h_{\sigma,1}^{-1})$ where $h_{\sigma,1} \in \mathrm{SL}(3)_{k^{\mathrm{sep}}}$ (see (2.1)). If $\sigma, \tau \in \mathrm{Gal}(k^{\mathrm{sep}}/k_1)$, then $h_{\sigma\tau,1} = h_{\tau,1}^\sigma h_{\sigma,1}$. So $\{h_{\sigma,1}\}_{\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k_1)}$ is a 1-cocycle with coefficients in $\mathrm{SL}(3)$. Therefore, there exists $g \in \mathrm{SL}(3)_{k^{\mathrm{sep}}}$ such that $h_{\sigma,1} = g^{-1} g^\sigma$ for all $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k_1)$. Replacing h_σ with $g h_\sigma (g^{-1})^\sigma$, we may assume that $h_\sigma = 1$ for all $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k_1)$.

We extend $\sigma(k_1)$ to k^{sep} (which is not canonical). If $\tau \in \mathrm{Gal}(k^{\mathrm{sep}}/k_1)$,

$$h_{\sigma(k_1)\tau,1} = h_{\tau,1}^{\sigma(k_1)} h_{\sigma(k_1),1} = h_{\sigma(k_1),1}.$$

So, if $\sigma \notin \mathrm{Gal}(k^{\mathrm{sep}}/k_1)$, $h_{\sigma,1} = h_{\sigma(k_1),1}$. Therefore, h is determined by $h_{\sigma(k_1),1}$. Then

$$h_{\sigma(k_1),1} = h_{\tau\sigma(k_1),1} = h_{\sigma(k_1),1}^\tau h_{\tau,1} = h_{\sigma(k_1),1}^\tau.$$

Therefore, $h_{\sigma(k_1),1} \in \mathrm{SL}(3)_{k_1}$. This implies that

$$\mathrm{H}^1(k, \mathrm{SU}(k_1, 3)) = \mathrm{H}^1(k_1/k, \mathrm{SU}(k_1, 3)).$$

Suppose that $\{(h_{\sigma,1}, {}^t h_{\sigma,1}^{-1})\}_{\sigma \in \mathrm{Gal}(k_1/k)}$ is a 1-cocycle with coefficients in $\mathrm{SU}(k_1, 3)$. Then

$$(h_{\sigma,1}, {}^t h_{\sigma,1}^{-1})^{\sigma(k_1)} = ((h_{\sigma,1}^*)^{-1}, h_{\sigma,1}^{\sigma(k_1)}).$$

So the cocycle condition becomes $h_{1,1} = 1$ and $(h_{\sigma(k_1),1}^*)^{-1} h_{\sigma(k_1),1} = 1$. This implies that $h_{\sigma(k_1),1} \in \mathrm{SH}_3(k_1)$. If $(g, {}^t g^{-1}) \in \mathrm{SU}(k_1, 3)_{k_1}$,

$$\begin{aligned}
&(g, {}^t g^{-1})^{-1} (h_{\sigma(k_1),1}, {}^t h_{\sigma(k_1),1}^{-1}) (g, {}^t g^{-1})^{\sigma(k_1)} \\
&= (g^{-1} h_{\sigma(k_1),1} (g^{-1})^*, {}^t g\, {}^t h_{\sigma(k_1),1}^{-1} g^{\sigma(k_1)}) \\
&= (g^{-1} h_{\sigma(k_1),1} (g^{-1})^*, {}^t (g^{-1} h_{\sigma(k_1),1} (g^{-1})^*)^{-1}).
\end{aligned}$$

So if we associate $Q = h_{\sigma(k_1),1} \in \mathrm{SH}_3(k_1)$ to the cohomology class in $\mathrm{H}^1(k_1/k, \mathrm{SU}(k_1, 3))$ determined by the 1-cocycle $\{(h_{\sigma,1}, {}^t h_{\sigma,1}^{-1})\}_{\sigma \in \mathrm{Gal}(k_1/k)}$, equivalent 1-cocyles correspond to elements of the orbit of Q by the action of $\mathrm{SL}(3)_{k_1}$. □

Note that Proposition 2.2 can easily be extended to a statement on $\mathrm{SU}(k_1, n)$ but we only need the case $n = 3$ in this paper.

We now briefly explain why V is an irreducible representation of G. It is enough to show that V is an irreducible representation of Sp(6).

If G is a group scheme, we denote the tangent space $\mathrm{T}(G)_e$ at the unit element e by $\mathrm{Lie}(G)$.

It is known that Sp(6) is a smooth simple group. Let X be the subspace of Lie(Sp(6)) spanned by matrices of the forms

$$\begin{pmatrix} 0 & u_1 & u_2 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u_1 & 0 & 0 \\ 0 & 0 & 0 & -u_2 & -u_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$$

where $u_1, u_2, u_3 \in k$ and S runs through 3×3 symmetric matrices. Then X is the Lie algebra of the unipotent radical of the standard Borel subgroup of Sp(6). Let Y be the subspace of Lie(Sp(6)) consisting of transposes of matrices in X.

It is easy to see that $Xe_{123} = \{0\}$ and so e_{123} is a highest weight vector (see [1, pp.190–193, 31.3,31.4] for the highest weight theory over an arbitrary field). Also straightforward computations show that Ye_{123} spans V. So if V is not irreducible, there is a highest weight vector other than e_{123} in V. The 14 standard weight vectors in V have distinct weights and so they are the only weight vectors. Therefore, it is enough to verify that $Xe_{i_1 i_2 i_3} \neq \{0\}$ unless $(i_1, i_2, i_3) = (1, 2, 3)$. Straightforward computations show that this is the case and V turns out to be an irreducible representation without the assumption $\mathrm{ch}\, k \neq 2, 3$. We do not carry out these computations here.

3. Stabilizer. In this section we determine the stabilizer of the element w in (1.2).

We put

$$H = \left\{ \left(t^{-3}, t \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \right) \middle| t \in \mathrm{GL}(1),\ A \in \mathrm{SL}(3) \right\} \cong \mathrm{GL}(1) \times \mathrm{SL}(3).$$

Then clearly, $H \subset G_w$ and H is connected. So $H \subset G_w^\circ$.

PROPOSITION 3.1. (1) $G_w^\circ = H \cong \mathrm{GL}(1) \times \mathrm{SL}(3)$.
(2) (G, V) *is an irreducible regular prehomogeneous vector space.*

PROOF. The statement (2) is known if $k = \mathbb{C}$.

We first prove that $\mathrm{Lie}(G_w) = \mathrm{Lie}(H)$. Our computation is essentially the same as that in [6, pp.107,108], except that our group is slightly different and k is arbitrary as long as $\mathrm{ch}\, k \neq 2, 3$. We identify $\mathrm{Lie}(G_w)$ with $k[\varepsilon]/(\varepsilon^2)$-valued points in G_w which reduce to 1_G.

Suppose that $(t, X) \in \mathrm{Lie}(G_w)$. Since $H \subset G_w$ and

$$\mathrm{Lie}(H) = \left\{ \left(-3a, \begin{pmatrix} a\mathrm{I}_3 + A & 0 \\ 0 & a\mathrm{I}_3 - {}^tA \end{pmatrix} \right) \middle| A \in \mathrm{M}(3), \mathrm{Tr}(A) = 0 \right\},$$

subtracting an element of $\mathrm{Lie}(H)$ from X, we may assume that X is in the form:

$$X = \begin{pmatrix} a & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & a & 0 & b_{12} & b_{22} & b_{23} \\ 0 & 0 & a & b_{13} & b_{23} & b_{33} \\ c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \end{pmatrix}.$$

We consider $((1, \mathrm{I}_3) + \varepsilon(t, X))w$ modulo U. Modulo U,

$$\begin{aligned} &e_{125} + e_{136} = 0,\ e_{124} - e_{236} = 0,\ e_{134} + e_{235} = 0, \\ &e_{245} + e_{346} = 0,\ e_{145} - e_{356} = 0,\ e_{146} + e_{256} = 0. \end{aligned} \tag{3.2}$$

By direct computation, $(t, X) \in \mathrm{Lie}(G_w)$ if and only if

$$\begin{aligned} &(ae_1 + c_{11}e_4 + c_{12}e_5 + c_{13}e_6) \wedge e_{23} - (ae_2 + c_{12}e_4 + c_{22}e_5 + c_{23}e_6) \wedge e_{13} \\ &+ (ae_3 + c_{13}e_4 + c_{23}e_5 + c_{33}e_6) \wedge e_{12} + (b_{11}e_1 + b_{12}e_2 + b_{13}e_3) \wedge e_{56} \\ &- (b_{12}e_1 + b_{22}e_2 + b_{23}e_3) \wedge e_{46} + (b_{13}e_1 + b_{23}e_2 + b_{33}e_3) \wedge e_{45} \\ &+ t(e_{123} + e_{456}) \in U. \end{aligned}$$

Expanding terms, this is equivalent to

$$\begin{aligned} &ae_{123} + c_{11}e_{234} + c_{12}e_{235} + c_{13}e_{236} + ae_{123} - c_{12}e_{134} - c_{22}e_{135} - c_{23}e_{136} \\ &+ ae_{123} + c_{13}e_{124} + c_{23}e_{125} + c_{33}e_{126} + b_{11}e_{156} + b_{12}e_{256} + b_{13}e_{356} \\ &- b_{12}e_{146} - b_{22}e_{246} - b_{23}e_{346} + b_{13}e_{145} + b_{23}e_{245} + b_{33}e_{345} \\ &+ t(e_{123} + e_{456}) \in U. \end{aligned}$$

Using the relations (3.2), this is equivalent to

$$\begin{aligned} &(3a + t)e_{123} + te_{456} + 2c_{13}e_{124} + 2c_{23}e_{125} + c_{33}e_{126} - 2c_{12}e_{134} - c_{22}e_{135} \\ &+ 2b_{13}e_{145} - 2b_{12}e_{146} + b_{11}e_{156} + c_{11}e_{234} + 2b_{23}e_{245} - b_{22}e_{246} + b_{33}e_{345} \in U. \end{aligned}$$

Since all terms are linearly independent modulo U,

$$t = a = b_{ij} = c_{ij} = 0.$$

Note that $2, 3 \neq 0$ by assumption. So $\mathrm{Lie}(G_w) = \mathrm{Lie}(H)$.

Since

$$\dim G_w \leq \dim \mathrm{Lie}(G_w) = \dim \mathrm{Lie}(H) = \dim H \leq \dim G_w.$$

$\dim G_w = \dim \mathrm{Lie}(G_w) = \dim H$. Therefore, G_w is smooth and $G_w^\circ = H$.

It is easy to see that $\dim \mathrm{GSp}(6) = 22, \dim G = 23, \dim H = 9$. So,

$$14 = \dim G - \dim G_w = \dim G \cdot w \leq \dim V = 14\,.$$

Therefore, $\dim G \cdot w = 14$. Since $G \cdot w$ is irreducible and is a constructible set, it is open in G. Since G_w is smooth and reductive, (G, V) is an irreducible regular prehomogeneous vector space (see Definition 1.4). □

By Proposition 3.1 $V_{k^{\mathrm{sep}}} = G_{k^{\mathrm{sep}}} \cdot w$.

Let ν be the element in (1.9). Then $(1, \nu) \in G_{w\,k}$ and it induces an outer automorphism on $\mathrm{SL}(3) \subset G_w^\circ$ by conjugation.

PROPOSITION 3.3. *$G_w/G_w^\circ \cong \mathbb{Z}/2\mathbb{Z}$ and the non-trivial element of G_w/G_w° is represented by $(1, \nu)$. Therefore, the action of* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ *on G_w/G_w° is trivial.*

PROOF. Since G_w is smooth and G_w/G_w° is finite, it is enough to prove $G_w/G_w^\circ \cong \mathbb{Z}/2\mathbb{Z}$ set-theoretically assuming $k = \overline{k}$.

Suppose that $(t, g) \in G_w$. There is an exact sequence

$$1 \to \mathrm{Inn}(\mathrm{SL}(3)) \to \mathrm{Aut}(\mathrm{SL}(3)) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

where $\mathrm{Inn}(\mathrm{SL}(3))$ is the inner automorphism group. Since the conjugation by ν (see (1.9)) induces an outer automorphism on $\mathrm{SL}(3)$, by multiplying $(1, \nu)$ if necessary, we may assume that g induces an inner automorphism on $\mathrm{SL}(3) \subset G_w^\circ$. Multiplying an element of $\mathrm{SL}(3)$, we may assume that g commutes with all elements of G_w°. Note that the $\mathrm{GL}(1)$-factor of G_w° is contained in the center of G.

Let $W = k^6$ (resp. $W_2 = k^3$) be the standard representation of $\mathrm{GL}(6)$ (resp. $\mathrm{SL}(3)$). Since $W \cong W_2 \oplus W_2^*$ (W_2^* is the dual space) and W_2, W_2^* are not equivalent as representations of $\mathrm{SL}(3)$, g leaves W_2, W_2^* stable. By Schur's lemma, g must be in the form

$$g = \begin{pmatrix} a\,\mathrm{I}_3 & 0 \\ 0 & b\,\mathrm{I}_3 \end{pmatrix}$$

where $a, b \in k^\times$. Multiplying an element of the $\mathrm{GL}(1)$-factor of G_w°, we may assume that $b = 1$. Then $(t, g)w = ta^3 e_{123} + te_{456} = e_{123} + e_{456}$. So $a^3 = t = 1$. This implies that

$$(t, g) = \left(1, \begin{pmatrix} a\mathrm{I}_3 & 0 \\ 0 & \mathrm{I}_3 \end{pmatrix}\right) = \left(a^3, a^{-1} \begin{pmatrix} a^2\mathrm{I}_3 & 0 \\ 0 & a\mathrm{I}_3 \end{pmatrix}\right).$$

Since $a = (a^2)^{-1}$, $g \in G_w^\circ$. So $G_w/G_w^\circ \cong \mathbb{Z}/2\mathbb{Z}$. □

4. Rational orbits. In this section, we prove the main result of this paper.

Since $\mathrm{H}^1(k, G) = \{1\}$,

$$G_k \backslash V_k^{\mathrm{ss}} \cong \mathrm{H}^1(k, G_w)\,.$$

If $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts trivially on $\mathbb{Z}/2\mathbb{Z}$ and $\{h_\sigma\}_\sigma$ is a 1-cocycle with coefficients in $\mathbb{Z}/2\mathbb{Z}$, $h_{\sigma\tau} = h_\tau h_\sigma = h_\sigma h_\tau$ ($\mathbb{Z}/2\mathbb{Z}$ is commutative). So $\mathrm{Gal}(k^{\mathrm{sep}}/k) \ni \sigma \mapsto h_\sigma$ is a homomorphism. The kernel of this homomorphism is an open normal subgroup of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ of index up to two, which corresponds to an extension of k of degree up to two. By associating this extension to

$\{h_\sigma\}_\sigma$, we obtain a bijection from $\mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z})$ to Ex(2). If $x \in V_k^{\mathrm{ss}}$, let $c_x \in \mathrm{H}^1(k, G_w)$ be the corresponding element. Since $G_w/G_w^\circ \cong \mathbb{Z}/2\mathbb{Z}$, the image of c_x in $\mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z})$ determines an element of Ex(2). We denote this element by $\gamma_V(x)$ or $\gamma_V(G_k \cdot x)$.

Obviously, $\gamma_V(w)$ is the trivial extension k of k. Since $\mathrm{ch}\, k \neq 2$, any quadratic extension is in the form $k(\sqrt{d})$ where $d \in k^\times \setminus (k^\times)^2$. We choose an element of $\gamma_V^{-1}(k(\sqrt{d}))$ for any such d in the following.

We put $k_1 = k(\sqrt{d})$. Let

$$g_{d,1} = \begin{pmatrix} \sqrt{d}\,\mathrm{I}_3 & \sqrt{d}\,\mathrm{I}_3 \\ (1+\sqrt{d})\,\mathrm{I}_3 & (-1+\sqrt{d})\,\mathrm{I}_3 \end{pmatrix}, \qquad g_d = \left(\frac{1}{2\sqrt{d}}, g_{d,1}\right). \tag{4.1}$$

Then $g_d \in G_{k_1}$ and $g_d^{\sigma(k_1)} = g_d(-1, -\nu)$. Note that $c(g_{d,1}) = -2\sqrt{d}$ (see (1.1)). It does not seem possible to choose an element of $\mathrm{Sp}(6)_{k_1}$ which satisfies a similar property as that of g_d and this is the reason why we chose $\mathrm{GL}(1)\times\mathrm{GSp}(6)$ rather than $\mathrm{GL}(1)\times\mathrm{Sp}(6)$ as the group. Since $(-1, -\nu) \in G_w$, if we put

$$w_d = g_d w\,, \tag{4.2}$$

$w_d \in V_k^{\mathrm{ss}}$. Explicitly,

$$w_d = de_{123} + (1+d)(e_{156} - e_{246} + e_{345}) + d(e_{126} - e_{135} + e_{234}) + (3+d)e_{456}\,. \tag{4.3}$$

PROPOSITION 4.4. *$\gamma_V(w_d) = k_1 = k(\sqrt{d})$.*

PROOF. The cohomology class corresponding to w_d is determined by the 1-cocycle $\{h_\sigma\}_\sigma$ such that

$$h_{\sigma(k_1)} = g_d^{-1} g_d^{\sigma(k_1)} = (-1, -\nu)\,.$$

Since $(-1, -\mathrm{I}_3)$ belongs to the GL(1)-part of G_w°, $(-1, -\nu)$ maps to the non-trivial element of G_w/G_w°. Therefore, $\gamma_V(w_d) = k_1$. □

Next we determine the stabilizer of w_d. Let $k_1 = k(\sqrt{d})$ as above.

PROPOSITION 4.5. (1) *$G_{w_d}^\circ \cong \mathrm{GL}(1) \times \mathrm{SU}(3, k_1)$.*

(2) *$G_{w_d}/G_{w_d}^\circ \cong \mathbb{Z}/2\mathbb{Z}$. Moreover, the non-trivial element of $G_{w_d}/G_{w_d}^\circ$ is represented by $g_d(1, \nu)g_d^{-1} \in G_{w_d k}$.*

PROOF. (1) Since $g_d \in G_{k_1}$,

$$G_{w_d k_1} = g_d G_{w k_1} g_d^{-1} = \left\{ g_d \left(t^{-3}, t\begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \right) g_d^{-1} \,\middle|\, t \in k_1^\times, A \in \mathrm{SL}(3)_{k_1} \right\}.$$

Suppose that $t \in k_1^\times, A \in \mathrm{GL}(3)_{k_1}$ and that

$$g_d \left(t^{-3}, t\begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \right) g_d^{-1} \in G_{w_d k}\,.$$

Since

$$\left(g_d\left(t^{-3}, t\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1}\end{pmatrix}\right)g_d^{-1}\right)^{\sigma(k_1)}$$
$$= g_d(-1,-\nu)\left((t^{\sigma(k_1)})^{-3}, t^{\sigma(k_1)}\begin{pmatrix} A^{\sigma(k_1)} & 0 \\ 0 & (A^*)^{-1}\end{pmatrix}\right)(-1,-\nu)g_d^{-1}$$
$$= g_d\left((t^{\sigma(k_1)})^{-3}, t^{\sigma(k_1)}\begin{pmatrix} (A^*)^{-1} & 0 \\ 0 & A^{\sigma(k_1)}\end{pmatrix}\right)g_d^{-1}\,,$$

we have

$$t^3 \in k^\times,\ tA = t^{\sigma(k_1)}(A^*)^{-1},\ t\,{}^tA^{-1} = t^{\sigma(k_1)}A^{\sigma(k_1)}\,.$$

Taking the product of the second equation and the transpose of the third equation, $t^2 \in k^\times$. Since, $t^3 \in k^\times$ also, we have $t \in k^\times$. This implies that $AA^* = \mathrm{I}_3$. Therefore,

$$G_{w_d\,k} = \left\{g_d\left(t^{-3}, t\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1}\end{pmatrix}\right)g_d^{-1}\,\middle|\, t \in k^\times, A \in \mathrm{SL}(3)_{k_1}, AA^* = \mathrm{I}_3\right\}$$
$$\cong k^\times \times \mathrm{SU}(3,k_1)\,.$$

We only considered k-rational points of G_{w_d}, but a similar consideration for any k-algebra R works and we obtain the statement (1) of the proposition.

(2) Since $G_w/G_w^\circ \cong \mathbb{Z}/2\mathbb{Z}$, $G_{w_d}/G_{w_d}^\circ \cong \mathbb{Z}/2\mathbb{Z}$. The only issue here is the action of $\mathrm{Gal}(k_1/k)$. It is easy to see that $G_{w_d}/G_{w_d}^\circ$ is represented by $g_d(1,\nu)g_d^{-1}$. Since

$$(g_d(1,\nu)g_d^{-1})^{\sigma(k_1)} = g_d(-1,-\nu)(1,\nu)(-1,-\nu)g_d^{-1} = g_d(1,\nu)g_d^{-1}\,,$$

$g_d(1,\nu)g_d^{-1} \in G_{w_d\,k} \setminus G_{w_d\,k}^\circ$. □

The following lemma is discussed in [9, p.120, LEMMA (1.8)]. Note that $\mathrm{H}^1(k,G)$ is trivial in our situation.

LEMMA 4.6. (1) *If* $x \in V_k^{\mathrm{ss}}$, $\gamma_V^{-1}(\gamma_V(G_kx)) \cong (G_x/G_x^\circ)_k\backslash \mathrm{H}^1(k,G_x^\circ)$.

(2) *By this correspondence, the cohomology class* $\{g^{-1}g^\sigma\} \in \mathrm{H}^1(k,G_x^\circ)$ *corresponds to the orbit of* G_kgx.

So to determine the set of rational orbits $G_k\backslash V_k^{\mathrm{ss}}$, it is enough to apply Lemma 4.6 to $x = w$ and $x = w_d$ for all d.

For

$$Q = \begin{pmatrix} q_1 & & \\ & q_2 & \\ & & q_3\end{pmatrix} \tag{4.7}$$

where $q_1, q_2, q_3 \in k^\times$, $q_1q_2q_3 = 1$ (which implies that $Q \in \mathrm{SH}_3(k_1)$), we put

$$
\begin{aligned}
A(Q) &= \begin{pmatrix} \frac{1+q_1}{2q_1} & & \\ & \frac{1+q_2}{2q_2} & \\ & & \frac{1+q_3}{2q_3} \end{pmatrix}, \quad B(Q) = \begin{pmatrix} \frac{1-q_1}{2} & & \\ & \frac{1-q_2}{2} & \\ & & \frac{1-q_3}{2} \end{pmatrix}, \\
C(Q) &= \begin{pmatrix} \frac{1-q_1}{2q_1} & & \\ & \frac{1-q_2}{2q_2} & \\ & & \frac{1-q_3}{2q_3} \end{pmatrix}, \quad D(Q) = \begin{pmatrix} \frac{1+q_1}{2} & & \\ & \frac{1+q_2}{2} & \\ & & \frac{1+q_3}{2} \end{pmatrix}, \\
m(Q) &= \begin{pmatrix} A(Q) & B(Q) \\ C(Q) & D(Q) \end{pmatrix}.
\end{aligned} \tag{4.8}
$$

LEMMA 4.9. (1) $g_{d,1} m(Q) g_{d,1}^{-1} \in \mathrm{Sp}(6)_{k_1}$.

(2) $(g_{d,1} m(Q) g_{d,1}^{-1})^{-1} (g_{d,1} m(Q) g_{d,1}^{-1})^{\sigma(k_1)} = g_{d,1} \begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix} g_{d,1}^{-1}$.

PROOF. (1) Since $A(Q)^tB(Q), C(Q)^tD(Q)$ are diagonal matrices, they are symmetric. By direct computation,

$$A(Q)^tD(Q) - B(Q)^tC(Q) = \mathrm{I}_3 .$$

Therefore,

$$\begin{pmatrix} A(Q) & B(Q) \\ C(Q) & D(Q) \end{pmatrix} \in \mathrm{Sp}(6)_k .$$

Since $g_{d,1} \in \mathrm{GSp}(6)_{k_1}$, $g_{d,1} m(Q) g_{d,1}^{-1} \in \mathrm{GSp}(6)_{k_1}$. Since $c : \mathrm{GSp}(6) \to \mathrm{GL}(1)$ is a character, $c(g_{d,1} m(Q) g_{d,1}^{-1}) = c(g_{d,1}) c(m(Q)) c(g_{d,1})^{-1} = 1$. So $g_{d,1} m(Q) g_{d,1}^{-1} \in \mathrm{Sp}(6)_{k_1}$.

(2) We show that

$$m(Q)^{-1} \nu m(Q) \nu = \begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix} . \tag{4.10}$$

This is equivalent to

$$
\begin{aligned}
& \nu m(Q) \nu = m(Q) \begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix} \\
\iff & \begin{pmatrix} D(Q) & C(Q) \\ B(Q) & A(Q) \end{pmatrix} = \begin{pmatrix} A(Q)Q & B(Q)Q^{-1} \\ C(Q)Q & D(Q)Q^{-1} \end{pmatrix} \\
\iff & A(Q)Q = D(Q), \ C(Q)Q = B(Q) .
\end{aligned}
$$

The last condition is clearly satisfied and so (4.10) is satisfied.

This implies that

$$
\begin{aligned}
& (g_{d,1} m(Q) g_{d,1}^{-1})^{-1} (g_{d,1} m(Q) g_{d,1}^{-1})^{\sigma(k_1)} \\
& = g_{d,1} m(Q)^{-1} g_{d,1}^{-1} g_{d,1} \nu m(Q)^{-1} \nu g_{d,1}^{-1} \\
& = g_{d,1} m(Q)^{-1} \nu m(Q) \nu g_{d,1}^{-1}
\end{aligned}
$$

$$= g_{d,1}\begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix} g_{d,1}^{-1}.$$

□

Now we are ready to state the main theorem.

THEOREM 4.11. *Let $\gamma_V : G_k\backslash V_k^{\mathrm{ss}} \to \mathrm{Ex}(2)$ be the map defined at the beginning of this section. Then the following* (1)–(5) *hold.*

(1) $\gamma_V^{-1}(k) = G_k \cdot w$.

(2) $G_w^\circ \cong \mathrm{GL}(1)\times \mathrm{SL}(3)$ *and* $G_w/G_w^\circ \cong \mathbb{Z}/2\mathbb{Z}$. *Also* G_w/G_w° *is represented by* $(1,\nu)$.

(3) *If* k_1 *is a quadratic extension of* k, $\gamma_V^{-1}(k_1)$ *is in bijective correspondence with* $(\mathrm{SL}(3)_{k_1} \rtimes (\mathbb{Z}/2\mathbb{Z}))\backslash \mathrm{SH}_3(k_1)$, *where the action of the non-trivial element of* $\mathbb{Z}/2\mathbb{Z}$ *on* $\mathrm{SH}_3(k_1)$ *is given by* $Q \mapsto {}^tQ^{-1}$.

(4) *If* $G\cdot x \in \gamma_V^{-1}(k_1)$ *corresponds to the orbit of* $Q \in \mathrm{SH}_3(k_1)$, $G_x^\circ \cong \mathrm{GL}(1)\times \mathrm{SU}(k_1, Q)$. *Also* G_x/G_x° *is represented by an element of* $G_{x\,k}$ *of order two.*

(5) *If* k_1 *is a quadratic extension of* k *and* $Q \in \mathrm{SH}_3(k_1)$ *is diagonal as in* (4.7), *the corresponding orbit in* (3) *is* $G_k g_d(1, m(Q))w$.

PROOF. (1) Since $G_w^\circ \cong \mathrm{GL}(1)\times\mathrm{SL}(3)$ and $\mathrm{H}^1(k, \mathrm{GL}(1)\times\mathrm{SL}(3))$ is trivial, (1) follows. (2) is proved in Propositions 3.1, 3.3.

(3) By Propositions 2.2, 4.5 and Lemma 4.6, we obtain (3) except for the action of $\mathbb{Z}/2\mathbb{Z}$. If $Q \in \mathrm{SH}_3(k_1)$, the corresponding cohomology class in

$$\mathrm{H}^1(k_1/k, G_{w_d}^\circ) \cong \mathrm{H}^1(k_1/k, \mathrm{GL}(1)\times \mathrm{SU}(k_1, 3)) \cong \mathrm{H}^1(k_1/k, \mathrm{SU}(k_1, 3))$$

is determined by the 1-cocycle $\{h_\sigma\}_{\sigma\in \mathrm{Gal}(k_1/k)}$ such that

$$h_{\sigma(k_1)} = g_d\left(1, \begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix}\right) g_d^{-1}.$$

Since $(G_{w_d}/G_{w_d}^\circ)_k$ is represented by $g_d(1,\nu)g_d^{-1}$, the conjugation by this element is

$$\begin{aligned}
&(g_d(1,\nu)g_d^{-1})g_d\left(1, \begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix}\right) g_d^{-1}(g_d(1,\nu)g_d^{-1})^{-1} \\
&= g_d\left(1, \begin{pmatrix} 0 & \mathrm{I}_3 \\ \mathrm{I}_3 & 0 \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix}\begin{pmatrix} 0 & \mathrm{I}_3 \\ \mathrm{I}_3 & 0 \end{pmatrix}\right) g_d^{-1} \\
&= g_d\left(1, \begin{pmatrix} {}^tQ^{-1} & 0 \\ 0 & Q \end{pmatrix}\right) g_d^{-1}.
\end{aligned}$$

Therefore, the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{SL}(3)_{k_1}\backslash \mathrm{SH}_3(k_1)$ can be regarded as $Q \mapsto {}^tQ^{-1}$.

(4) Since $\mathrm{H}^1(k, G)$ is trivial, there exists $(r,h) \in k_1^\times \times \mathrm{GSp}(6)_{k_1}$ such that

$$\begin{aligned}
g_d\left(1, \begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix}\right) g_d^{-1} &= (g_d(r,h)g_d^{-1})^{-1}(g_d(r,h)g_d^{-1})^{\sigma(k_1)} \\
&= g_d(r^{-1}, h^{-1})(-1,-\nu)(r^{\sigma(k_1)}, h^{\sigma(k_1)})(-1,-\nu)g_d^{-1} \\
&= g_d(r^{-1}r^{\sigma(k_1)}, h^{-1}\nu h^{\sigma(k_1)}\nu)g_d^{-1}.
\end{aligned}$$

Then Q corresponds to the orbit of $x = g_d(r,h)g_d^{-1}w_d = g_d(r,h)w$. The above condition is equivalent to

$$r \in k^\times, \quad \begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix} = h^{-1}\nu h^{\sigma(k_1)}\nu\,. \tag{4.12}$$

This condition is satisfied even if r is replaced by 1. So we assume that $r = 1$.

Obviously,

$$G_{x\,k_1} = \left\{ g_d(1,h)\left(t^{-3}, t\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\right)(g_d(1,h))^{-1} \,\middle|\, t \in k_1^\times, A \in \mathrm{SL}(3)_{k_1} \right\}.$$

We use the notation such as $\overline{h}$ for $h^{\sigma(k_1)}$ here. Since

$$\begin{aligned} &\left(g_d(1,h)\left(t^{-3}, t\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\right)(g_d(1,h))^{-1}\right)^{\sigma(k_1)} \\ &= g_d(-1,-\nu)(1,\overline{h})\left(\overline{t}^{-3}, \overline{t}\begin{pmatrix} \overline{A} & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}\right)(1,\overline{h}^{-1}))(-1,-\nu)g_d^{-1} \\ &= g_d\left(\overline{t}^{-3}, \nu\overline{h}\,\overline{t}\begin{pmatrix} \overline{A} & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}\overline{h}^{-1}\nu\right)g_d^{-1}, \end{aligned}$$

the condition

$$g_d(1,h)\left(t^{-3}, t\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\right)(g_d(1,h))^{-1} \in G_{x\,k}$$

is satisfied if and only if $t^3 \in k^\times$ and

$$\nu\overline{h}\,\overline{t}\begin{pmatrix} \overline{A} & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}\overline{h}^{-1}\nu = ht\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}h^{-1}.$$

By (4.12), this is equivalent to

$$h\overline{t}\begin{pmatrix} Q & 0 \\ 0 & {}^tQ^{-1} \end{pmatrix}\begin{pmatrix} (A^*)^{-1} & 0 \\ 0 & \overline{A} \end{pmatrix}\begin{pmatrix} Q^{-1} & 0 \\ 0 & {}^tQ \end{pmatrix}h^{-1} = ht\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}h^{-1}.$$

Simplifying, we obtain the condition,

$$\overline{t}Q(A^*)^{-1}Q^{-1} = tA,\ \overline{t}\,{}^tQ^{-1}\overline{A}\,{}^tQ = t\,{}^tA^{-1}.$$

Taking the product of the first equation and the transpose of the second equation, $t^2 \in k^\times$. Since $t^3 \in k^\times$ also, this implies that $t \in k^\times$ and that

$$Q(A^*)^{-1}Q^{-1} = A\,,$$

which is equivalent to

$$AQA^* = Q\,.$$

So $G_{x\,k} = k^\times \times \mathrm{SU}(k_1, Q)_k$. We only considered k-rational points, but by considering k-algebras R, we obtain an isomorphism of algebraic groups $G_x^\circ \cong \mathrm{GL}(1) \times \mathrm{SU}(k_1, Q)$.

(5) By Lemma 4.9, we can choose $(1, m(Q))$ as (r, h) in the proof of (4). Therefore, the corresponding orbit is $G_k g_d(1, m(Q))w$. □

Note that any element of $\mathrm{SH}_3(k_1)$ can be diagonalized and so Theorem 4.11 (5) makes it possible in principle to find the orbit corresponding to any element of $\mathrm{SH}_3(k_1)$.

Let W be the standard representation of GL(6) as in Introduction. We now describe the map γ_V by constructing an equivariant map from V to $\mathrm{Hom}(W, W)$.

We define a map $D : \wedge^3 W \to W \otimes \wedge^2 W$ by

$$D(v_1 \wedge v_2 \wedge v_3) = v_1 \otimes (v_2 \wedge v_3) - v_2 \otimes (v_1 \wedge v_3) + v_3 \otimes (v_1 \wedge v_2)\,.$$

This is well-defined and GL(6)-equivariant. Let

$$\phi(x) = x \wedge D(x) \in W \otimes \wedge^5 W \cong W \otimes W^* \cong \mathrm{Hom}(W, W)$$

for $x \in \wedge^3 W$.

Let $w, U, U^\perp$ be as in Introduction. It is easy to see that $w \wedge \omega = 0$ and so $w \wedge x = 0$ for all $x \in U$. Therefore, $w \in U^\perp$. The composition $V \cong U^\perp \to \wedge^3 W \to \mathrm{Hom}(W, W)$ is G-equivariant. We denote this map by Φ. It is known (see [6, pp.79–81]) that

$$\Phi(w) = \sum_{i=1}^{3} e_i \otimes e_i^* - \sum_{i=4}^{6} e_i \otimes e_i^*\,.$$

Therefore, eigenvalues of $\Phi(w)$ are ± 1 and $\Phi(w) \circ \Phi(w) = \mathrm{I}_6$.

It is easy to see that if $t \in \mathrm{GL}(1), g \in \mathrm{GSp}(6)$ and $x \in V$,

$$\Phi((t, g)x) = t^2 (\det g) g \Phi(x) \tag{4.13}$$

where $g\Phi(x)(v) = g(\Phi(x)(g^{-1}v))$ for $v \in W$. So

$$\begin{aligned}\Phi((t, g)x) \circ \Phi((t, g)x)\mathrm{GL}(6)) &= t^4 (\det g)^2 (g\Phi(x)) \circ (g\Phi(x)) \\ &= t^4 (\det g)^2 g(\Phi(x) \circ \Phi(x)).\end{aligned}$$

Since $Gw \subset V$ is Zariski open, and $\Phi(w) \circ \Phi(w) = \mathrm{I}_6$, there is a polynomial $\Delta(x)$ of $x \in V$ such that

$$\Phi(x) \circ \Phi(x) = \Delta(x)\,\mathrm{I}_6\,.$$

Moreover, by the above consideration,

$$\Delta((t, g)x) = t^4 (\det g)^2 \Delta(x)\,,$$

i.e., $\Delta(x)$ is a relative invariant polynomial.

If $x = (t, g)w$, eigenvalues of $\Phi(x)$ are $\pm t^2 (\det g)$ by (4.13). Since

$$\Delta(x) = t^4 (\det g)^2 \Delta(w) = t^4 (\det g)^2\,,$$

eigenvalues of $\Phi(x)$ are $\pm\sqrt{\Delta(x)}$. Therefore, we obtain the following proposition.

PROPOSITION 4.14. *If $x \in V_k^{\mathrm{ss}}$, $\gamma_V(x)$ is the quadratic field generated over k by eigenvalues of $\Phi(x)$ (which are $\pm\sqrt{\Delta(x)}$).*

5. The case of number fields. In this section we consider the case where k is a number field. Throughout this section, we assume that k is a number field.

Suppose that k_1/k is a quadratic extension. Then $\gamma_V^{-1}(k_1)$ is in bijective correspondence with $\mathrm{H}^1(k_1/k, \mathrm{SU}(k_1, 3)) \cong (\mathrm{SL}(3)_{k_1} \rtimes \mathbb{Z}/2\mathbb{Z})\backslash \mathrm{SH}_3(k_1)$. Let $\mathfrak{M}$, $\mathfrak{M}_\infty$, $\mathfrak{M}_\mathrm{f}$, $\mathfrak{M}_\mathbb{R}$, $\mathfrak{M}_\mathbb{C}$ be the set of all places, all infinite places and all finite places, all real places and all imaginary places respectively. If $v \in \mathfrak{M}$, we denote the completion of k at v by k_v.

The following proposition is well-known (see [4]). Note that $\mathrm{SU}(k_1, 3)$ is simply connected.

PROPOSITION 5.1. (1) *If* $g \in \mathfrak{M}_\mathrm{f}$, $\mathrm{H}^1(k_v, \mathrm{SU}(k_1, 3)) = \{1\}$.
(2) $\mathrm{H}^1(k, \mathrm{SU}(k_1, 3)) \cong \prod_{v \in \mathfrak{M}_\infty} \mathrm{H}^1(k_v, \mathrm{SU}(k_1, 3))$.

Let $\mathfrak{M}(k_1)$ be the set of $v \in \mathfrak{M}_\mathbb{R}$ such that $k_1 \not\subset k_v$. If $k_1 = k(\sqrt{d})$ and $v \in \mathfrak{M}$, $v \in \mathfrak{M}(k_1)$ if and only if $v \in \mathfrak{M}_\mathbb{R}$ and the image of d in k_v is negative.

If $v \in \mathfrak{M}_\mathbb{C}$, $\mathrm{H}^1(k_v, \mathrm{SU}(k_1, 3)) = \{1\}$ of course. If $k_1 \subset k_v$, $\mathrm{SU}(k_1, 3) \cong \mathrm{SL}(3)$ over k_v. Therefore, $\mathrm{H}^1(k_v, \mathrm{SU}(k_1, 3)) = \{1\}$ also. Let $v \in \mathfrak{M}(k_1)$. Then $k_1 \cdot k_v = \mathbb{C}$. So

$$\mathrm{H}^1(k_v, \mathrm{SU}(k_1, 3)) = \mathrm{H}^1(\mathbb{C}/\mathbb{R}, \mathrm{SU}(\mathbb{C}, 3))\,.$$

Let Q_1, Q_2 be the matrices in (1.8).

LEMMA 5.2. *The set* $(\mathrm{SL}(3)_\mathbb{C} \rtimes \mathbb{Z}/2\mathbb{Z})\backslash \mathrm{SH}_3(\mathbb{C})$ *consists of two elements and one can choose* Q_1, Q_2 *as their representatives.*

PROOF. Any Hermitian matrix can be diagonalized by elements of $\mathrm{SL}(3)_\mathbb{C}$. If $Q \in \mathrm{SH}_3(\mathbb{C})$ is diagonal (consequently diagonal entries are real), applying elements of the forms

$$\begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & t & \\ & & t^{-1} \end{pmatrix},$$

we may assume that Q is in the form

$$Q = \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & * \end{pmatrix}.$$

Since $\det Q = 1$, Q is in the form

$$Q = \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}.$$

Applying permutation matrices, Q becomes Q_1 or Q_2. Since the signature of a Hermitian matrix does not change by the action of $\mathrm{SL}(3)_\mathbb{C}$, Q_1, Q_2 are not equivalent. Moreover, both Q_1, Q_2 are invariant by the action $Q \mapsto {}^tQ^{-1}$, Q_1, Q_2 are not equivalent by the action of $\mathbb{Z}/2\mathbb{Z}$. □

These considerations show the following proposition

PROPOSITION 5.3. $|\mathrm{H}^1(k, \mathrm{SU}(k_1, 3))| = 2^{|\mathfrak{M}(k_1)|}$.

Suppose that $k = \mathbb{Q}$. Quadratic extensions are in bijective correspondence with square-free integers $d \neq 1$. Let $d \neq 1$ be a square-free integer and $k_1 = \mathbb{Q}(\sqrt{d})$. Then if $d > 0$, $\mathfrak{M}(k_1) = \emptyset$. If $d < 0$, $\mathfrak{M}(k_1)$ consists of the infinite place of $\mathbb{Q}$.

Since entries of Q_1, Q_2 belong to $\mathbb{Q}$, We can choose $\{Q_1, Q_2\}$ as a set of representatives of $(\mathrm{SL}(3)_{\mathbb{Q}} \rtimes \mathbb{Z}/2\mathbb{Z})\backslash \mathrm{SH}_3(\mathrm{k}_1)$. By computations,

$$m(Q_2) = \begin{pmatrix} 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \\ 0 & & & 1 & & \\ & -1 & & & 0 & \\ & & -1 & & & 0 \end{pmatrix}.$$

LEMMA 5.4. *$g_d m(Q_2) w$ is the following element.*

$$d(e_{123} + e_{126} - e_{135} + e_{234}) + (d-1)(-e_{246} + e_{345} + e_{456}) + (d+1)e_{156}\,.$$

PROOF. By $m(Q_2)$, $\mathbb{e}_1, \dots, \mathbb{e}_6$ map to

$$\mathbb{e}_1,\ -\mathbb{e}_5,\ -\mathbb{e}_6,\ \mathbb{e}_4,\ \mathbb{e}_2,\ \mathbb{e}_3$$

respectively. So $m_d(Q_2)w = e_{156} + e_{234}$. This implies that $g_d m(Q_2) w$ is equal to

$$\frac{1}{2\sqrt{d}}(\sqrt{d}\mathbb{e}_1 + (1+\sqrt{d})\mathbb{e}_4) \wedge (\sqrt{d}\mathbb{e}_2 + (-1+\sqrt{d})\mathbb{e}_5) \wedge (\sqrt{d}\mathbb{e}_3 + (-1+\sqrt{d})\mathbb{e}_6)$$
$$+ \frac{1}{2\sqrt{d}}(\sqrt{d}\mathbb{e}_2 + (1+\sqrt{d})\mathbb{e}_5) \wedge (\sqrt{d}\mathbb{e}_3 + (1+\sqrt{d})\mathbb{e}_6) \wedge (\sqrt{d}\mathbb{e}_1 + (-1+\sqrt{d})\mathbb{e}_4).$$

Straightforward computations show that this is equal to the element in the statement of the lemma. □

By these considerations, we obtain the following theorem in the case $k = \mathbb{Q}$.

THEOREM 5.5. (1) *$G_{\mathbb{Q}} \backslash V^{\mathrm{ss}}_{\mathbb{Q}}$ has the following representatives x.*

(i) *$x = w$.*

(ii) *x is the element given in* (4.3) *where d in* (4.3) *runs through all square-free integers not equal to* 1.

(iii) *x is the element given in Lemma* 5.4 *where $d < 0$ runs through all square-free integers.*

(2) *The stabilizer for elements in* (1)(*i*)–(*iii*) *are as follows.*

(i) $G^{\circ}_x \cong \mathrm{GL}(1) \times \mathrm{SL}(3)$.

(ii) $G^{\circ}_x \cong \mathrm{GL}(1) \times \mathrm{SU}(\mathbb{Q}(\sqrt{d}), 3)$.

(iii) $G^{\circ}_x \cong \mathrm{GL}(1) \times \mathrm{SU}(\mathbb{Q}(\sqrt{d}), 1, 2)$.

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DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
KYOTO UNIVERSITY
KYOTO 606–8502
JAPAN

E-mail address: yukie@math.kyoto-u.ac.jp