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STOCHASTIC CALCULUS FOR MARKOV PROCESSES ASSOCIATED WITH SEMI-DIRICHLET FORMS

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Abstract. We present a new Fukushima type decomposition in the framework of semi-Dirichlet forms. This generalizes the result of Ma, Sun and Wang [17, Theorem 1.4] by removing the condition (S). We also extend Nakao's integral to semi-Dirichlet forms and derive Itô's formula related to it.

Introduction. Let E be a metrizable Lusin space, i.e., E is topologically isomorphic to a Borel subset of a complete separable metric space, and m be a σ-finite positive measure on its Borel σ-algebra $\mathcal{B}(E)$. We consider a quasi-regular semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E; m)$ with associated Markov process $\mathbf{M} = ((X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$, where Δ (the cemetery) is an extra point adjoined to E and $E_\Delta = E \cup \{\Delta\}$. Throughout this paper, any function u on E is considered as a function on E_Δ by putting $u(\Delta) = 0$. For $u \in D(\mathcal{E})_{loc}$ (see (5) below for the precise definition), we define the additive functional (AF in short) $A^{[u]}$ by

$$A_t^{[u]} := \tilde{u}(X_t) - \tilde{u}(X_0)\,,$$

where $\tilde{u}$ is an $\mathcal{E}$-quasi-continuous m-version of u. The aim of this paper is to establish a Fukushima type decomposition for $A^{[u]}$ and study the stochastic integral $\int_0^t \tilde{v}(X_{s-})dA_s^{[u]}$ for $v \in D(\mathcal{E})_{loc}$.

We refer the reader to [14, 15, 20] for notations and terminologies related to semi-Dirichlet forms. In particular, we refer the reader to the new monograph [20] for the potential theory of semi-Dirichlet forms including the correspondence between positive continuous additive functionals and smooth measures.

Let us start with a brief introduction to the development of Fukushima's decomposition. Fukushima's celebrated decomposition theorem was originally established for regular symmetric Dirichlet forms (see [6] and [7, Theorem 5.2.2]) and then extended to the non-symmetric and quasi-regular ones (cf. [19, Theorem 5.1.3] and [15, Theorem VI.2.5]). If $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form and $u \in D(\mathcal{E})$, Fukushima's decomposition tells us that there exist a unique martingale AF (MAF in short) $M^{[u]}$ of finite energy and a unique continuous AF (CAF in short) $N^{[u]}$ of zero energy such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}\,. \tag{1}$$

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If $(\mathcal{E}, D(\mathcal{E}))$ is a strongly local symmetric Dirichlet form, Fukushima's decomposition (1) holds also for $u \in D(\mathcal{E})_{loc}$ with $M^{[u]}$ being a MAF locally of finite energy and $N^{[u]}$ being a CAF locally of zero energy (cf. [7, Theorem 5.5.1]). For a general symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, Kuwae showed that the Fukushima type decomposition holds for a subclass of $D(\mathcal{E})_{loc}$ (see [12, Theorem 4.2]). If $(\mathcal{E}, D(\mathcal{E}))$ is a (not necessarily symmetric) Dirichlet form, Walsh showed in [26, 27] that for $u \in D(\mathcal{E})_{loc}$ there exist a MAF $W^{[u]}$ locally of finite energy and a CAF $C^{[u]}$ locally of zero energy such that

$$A_t^{[u]} = W_t^{[u]} + C_t^{[u]} + V_t^{[u]}, \tag{2}$$

where

$$V_t^{[u]} := \sum_{0<s\le t} (\tilde{u}(X_s) - \tilde{u}(X_{s-}))1_{\{|\tilde{u}(X_s)-\tilde{u}(X_{s-})|>1\}}1_{\{t<\zeta\}} - u(X_{\zeta-})1_{\{t\ge\zeta\}}.$$

Hereafter ζ denotes the lifetime of $\mathbf{M}$.

If $(\mathcal{E}, D(\mathcal{E}))$ is only a semi-Dirichlet form, the situation becomes more complicated. Note that the assumption of the existence of dual Markov process plays a crucial role in Fukushima's decomposition. In fact, without that assumption, the usual definition of energy of AFs is questionable. If $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form, Ma et al. showed in [13] that Fukushima's decomposition holds for $u \in D(\mathcal{E})_{loc}$. For a general regular semi-Dirichlet form, Oshima showed in [20] that Fukushima's decomposition holds for $u \in D(\mathcal{E})_b$.

Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form. We define $I(\zeta) := [\![0, \zeta[\![\cup [\![\zeta_i]\!]$, with ζ_i being the totally inaccessible part of ζ. We refer the reader to [9, 3.14] for the definition of stochastic interval. Denote by J the jumping measure of $(\mathcal{E}, D(\mathcal{E}))$. For $u \in D(\mathcal{E})_{loc}$, Z. M. Ma et al. showed in [17, Theorem 1.4] (cf. also [24]) that the following two assertions are equivalent.

(i) u admits a Fukushima type decomposition. That is, there exist a locally square integrable MAF $M^{[u]}$ on $I(\zeta)$ and a local CAF $N^{[u]}$ on $I(\zeta)$ which has zero quadratic variation such that (1) holds.

(ii) u satisfies

$$(S): \quad \mu_u(dx) := \int_E (\tilde{u}(x) - \tilde{u}(y))^2 J(dy, dx) \text{ is a smooth measure}.$$

Moreover, if u satisfies Condition (S), then the decomposition (1) is unique up to the equivalence of local AFs. We refer the reader to [7, page 271] for the notion of local AFs.

In the first part of this paper, we will establish a new Fukushima type decomposition for $u \in D(\mathcal{E})_{loc}$ without Condition (S). Define

$$F_t^{[u]} := \sum_{0<s\le t} (\tilde{u}(X_s) - \tilde{u}(X_{s-}))1_{\{|\tilde{u}(X_s)-\tilde{u}(X_{s-})|>1\}}. \tag{3}$$

In Section 1 (see Theorem 1.2 below), we will show that, for any $u \in D(\mathcal{E})_{loc}$, there exist a unique locally square integrable MAF $Y^{[u]}$ on $I(\zeta)$ and a unique continuous local AF $Z^{[u]}$ of zero quadratic variation such that

$$A_t^{[u]} = Y_t^{[u]} + Z_t^{[u]} + F_t^{[u]}. \tag{4}$$

The decomposition (4) gives the most general form of the Fukushima type decomposition in the framework of semi-Dirichlet forms. It implies in particular that $A^{[u]}$ is a Dirichlet process (cf. [4, 5]), i.e., is the summation of a semi-martingale and a zero quadratic variation process.

In the second part of this paper, we will define the stochastic integral $\int_0^t \tilde{v}(X_{s-})dA_s^{[u]}$ for $u, v \in D(\mathcal{E})_{loc}$ and derive the related Itô's formula.

Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular symmetric Dirichlet form. For $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b$, Nakao studied in [18] the stochastic integral $\int_0^t \tilde{v}(X_{s-})dA_s^{[u]}$ by introducing so-called Nakao's integral $\int_0^t \tilde{v}(X_{s-})dN_s^{[u]}$. Later, Z. Q. Chen et al. and Kuwae (see [3] and [12]) extended Nakao's integral to a larger class of integrators as well as integrands. By using different methods, Walsh ([25]) and C. Z. Chen et al. ([2]) independently extended Nakao's integral from the setting of symmetric Dirichlet forms to that of non-symmetric Dirichlet forms. By virtue of the decomposition (2), Walsh also defined Nakao's integral for more general integrators as well as integrands in the setting of non-symmetric Dirichlet forms (see [27]). In all of these references, the related Itô's formulas have been derived for the stochastic integral $\int_0^t \tilde{v}(X_{s-})dA_s^{[u]}$.

In Section 2, we will define the stochastic integral $\int_0^t \tilde{v}(X_{s-})dA_s^{[u]}$ for $u, v \in D(\mathcal{E})_{loc}$ and derive the related Itô's formula in the setting of semi-Dirichlet forms. Owing to the non-Markovian property of the dual form, all the previous known methods in defining Nakao's integral ceased to work. Note that if $(\mathcal{E}, D(\mathcal{E}))$ is only a semi-Dirichlet form, its symmetric part is not a symmetric Dirichlet form in general but a symmetric positivity preserving form and the dual killing measure might not exist. These cause extra difficulties in defining Nakao's integral. In this paper, we will combine the method of [2] with the localization technique of [13] and [17] to define the stochastic integral $\int_0^t \tilde{v}(X_{s-})dA_s^{[u]}$ and derive the related Itô's formula.

In Section 3, we will give concrete examples of semi-Dirichlet forms for which our results can be applied.

1. Decomposition of $\tilde{u}(X_t) - \tilde{u}(X_0)$ without Condition (S). The basic setting of this paper is the same as that in [17], to which we refer the reader for more details. Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E; m)$ with E being a metrizable Lusin space and m being a σ-finite positive measure on $\mathcal{B}(E)$. Denote by $(T_t)_{t\geq 0}$ and $(G_\alpha)_{\alpha\geq 0}$ (resp. $(\hat{T}_t)_{t\geq 0}$ and $(\hat{G}_\alpha)_{\alpha\geq 0}$) the semigroup and resolvent (resp. co-semigroup and co-resolvent) associated with $(\mathcal{E}, D(\mathcal{E}))$. Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be an m-tight special standard process which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$.

Throughout this paper, we fix a function $\phi \in L^1(E; m)$ with $0 < \phi \leq 1$ m-a.e. and set $h = G_1\phi$, $\hat{h} = \hat{G}_1\phi$. Denote $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$ for $B \subset E$. Let V be a quasi-open subset of E. We denote by $\mathbf{M}^V = (X_t^V)_{t\geq 0}$ the part process of $\mathbf{M}$ on V and denote by $(\mathcal{E}^V, D(\mathcal{E}^V))$ the part form of $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(V;m)$. It is known that $\mathbf{M}^V$ is a standard process, $D(\mathcal{E}^V) = D(\mathcal{E})_V = \{u \in D(\mathcal{E}) \mid \tilde{u} = 0,\ \mathcal{E}\text{-q.e. on } V^c\}$, and $(\mathcal{E}^V, D(\mathcal{E})_V)$ is a quasi-regular semi-Dirichlet form (cf. [11]). Denote by $(T_t^V)_{t\geq 0}$, $(\hat{T}_t^V)_{t\geq 0}$, $(G_\alpha^V)_{\alpha\geq 0}$ and $(\hat{G}_\alpha^V)_{\alpha\geq 0}$ the semigroup, co-semigroup, resolvent and co-resolvent associated with $(\mathcal{E}^V, D(\mathcal{E})_V)$, respectively. Define $\bar{h}^V := \hat{G}_1^V\phi$ and $\bar{h}^{V,*} := e^{-2}\hat{T}_1^V(\hat{G}_2^V\phi)$. Then $\bar{h}^V, \bar{h}^{V,*} \in D(\mathcal{E})_V$ and $\bar{h}^{V,*} \leq \bar{h}^V$. Denote $D(\mathcal{E})_{V,b} := \mathcal{B}_b(E) \cap D(\mathcal{E})_V$.

For an AF $A = (A_t)_{t\geq 0}$ of $\mathbf{M}^V$, we define

$$e^V(A) := \lim_{t\downarrow 0} \frac{1}{2t} E_{\bar{h}^V\cdot m}(A_t^2)$$

whenever the limit exists in $[0,\infty]$. For a local AF $B = (B_t)_{t\geq 0}$ of $\mathbf{M}$, we define

$$e^{V,*}(B) := \lim_{t\downarrow 0} \frac{1}{2t} E_{\bar{h}^{V,*}\cdot m}(B_{t\wedge\tau_V}^2)$$

whenever the limit exists in $[0,\infty]$.

Define

$$\begin{aligned}\dot{\mathcal{M}}^V := \{M \mid\, & M \text{ is an AF of } \mathbf{M}^V,\ E_x(M_t^2) < \infty,\ E_x(M_t) = 0\\ & \text{for all } t \geq 0 \text{ and } \mathcal{E}\text{-q.e. } x \in V,\ e^V(M) < \infty\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_c^V := \{N \mid\, & N \text{ is a CAF of } \mathbf{M}^V,\ E_x(|N_t|) < \infty \text{ for all } t \geq 0\\ & \text{and } \mathcal{E}\text{-q.e. } x \in V,\ e^V(N) = 0\},\end{aligned}$$

$$\begin{aligned}\Theta := \{\{V_n\} \mid\, & V_n \text{ is } \mathcal{E}\text{-quasi-open},\ V_n \subset V_{n+1}\ \mathcal{E}\text{-q.e.}\\ & \forall\, n \in \mathbb{N},\ \text{and } E = \cup_{n=1}^\infty V_n\ \mathcal{E}\text{-q.e.}\},\end{aligned}$$

(5)
$$\begin{aligned}D(\mathcal{E})_{loc} := \{u \mid\, & \exists\, \{V_n\} \in \Theta \text{ and } \{u_n\} \subset D(\mathcal{E})\\ & \text{such that } u = u_n\ m\text{-a.e. on } V_n,\ \forall\, n \in \mathbb{N}\},\end{aligned}$$

$$\begin{aligned}\dot{\mathcal{M}}_{loc} := \{M \mid\, & M \text{ is a local AF of } \mathbf{M},\ \exists\, \{V_n\}, \{E_n\} \in \Theta \text{ and } \{M^n \mid M^n \in \dot{\mathcal{M}}^{V_n}\}\\ & \text{such that } E_n \subset V_n,\ M_{t\wedge\tau_{E_n}} = M^n_{t\wedge\tau_{E_n}},\ t \geq 0,\ n \in \mathbb{N}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_c := \{N \mid\, & N \text{ is a local AF of } \mathbf{M},\ \exists\, \{E_n\} \in \Theta \text{ such that } t \mapsto N_{t\wedge\tau_{E_n}}\\ & \text{is continuous and of zero quadratic variation},\ n \in \mathbb{N}\}.\end{aligned}$$

In the above definition, $\{N_{t\wedge\tau_{E_n}}\}$ is said to be of zero quadratic variation if its quadratic variation vanishes in P_m-measure, more precisely, if it satisfies

$$\sum_{k=0}^{[T/\varepsilon_l]} (N_{\{(k+1)\varepsilon_l\}\wedge\tau_{E_n}} - N_{\{k\varepsilon_l\}\wedge\tau_{E_n}})^2 \to 0 \text{ as } l \to \infty \text{ in } P_m\text{-measure},$$

for any $T > 0$ and any sequence $\{\varepsilon_l\}_{l\in\mathbb{N}}$ converging to 0.

We use ζ_i to denote the totally inaccessible part of ζ, by which we mean that ζ_i is an $\{\mathcal{F}_t\}$-stopping time and is the totally inaccessible part of ζ with respect to P_x for $\mathcal{E}$-q.e. $x \in E$. By [17, Proposition 2.4], such ζ_i exists and is unique in the sense of P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. We write $I(\zeta) := [\![0, \zeta[\![\cup [\![\zeta_i]\!]$. By [17, Proposition 2.4], there exists a $\{V_n\} \in \Theta$ such that for any $\{U_n\} \in \Theta$, $I(\zeta) = \cup_n [\![0, \tau_{V_n \cap U_n}]\!]$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Therefore $I(\zeta)$ is a predictable set of interval type (cf. [9, Theorem 8.18]). By the local compactification method (see [15, Theorem VI.1.6] and [10, Theorem 3.5]) in the semi-Dirichlet forms setting, we may assume without loss of generality that $\mathbf{M}$ is a Hunt process and E is a locally compact separable metric space whenever necessary.

In this paper a local AF M is called a locally square integrable MAF on $I(\zeta)$, denoted by $M \in \mathcal{M}_{loc}^{I(\zeta)}$, if $M \in (\mathcal{M}_{loc}^2)^{I(\zeta)}$ in the sense of [9, Definition 8.19]. For $u \in D(\mathcal{E})_{loc}$, we define the bounded variation process $F^{[u]}$ as in (3). Denote by $J(dx, dy)$ and $K(dx)$ the jumping and killing measures of $(\mathcal{E}, D(\mathcal{E}))$, respectively (cf. [10]). Let $(N(x, dy), H_s)$ be a Lévy system of $\mathbf{M}$ and μ_H be the Revuz measure of the positive CAF (PCAF in short) H. Then, we have

$$J(dy, dx) = \frac{1}{2} N(x, dy)\mu_H(dx), \quad K(dx) = N(x, \{\Delta\})\mu_H(dx)\,. \tag{6}$$

Define (cf. [13, Theorem 5.3])

$$\hat{S}_{00}^* := \{\mu \in S_0 \mid \hat{U}_1\mu \le c\hat{G}_1\phi \text{ for some constant } c > 0\}\,,$$

where S_0 denotes the family of positive measures of finite energy integral and $\hat{U}_1\mu$ is the 1-co-potential.

We put the following assumption:

ASSUMPTION 1.1. There exist $\{V_n\} \in \Theta$ and a sequence of locally bounded functions $\{C_n\}$ on $\mathbb{R}$ such that for each $n \in \mathbb{N}$, if $u, v \in D(\mathcal{E})_{V_n, b}$ then $uv \in D(\mathcal{E})$ and

$$\mathcal{E}(uv, uv) \le C_n(\|u\|_\infty + \|v\|_\infty)(\mathcal{E}_1(u, u) + \mathcal{E}_1(v, v))\,.$$

Now we can state the main result of this section.

THEOREM 1.2. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption* 1.1. *Suppose $u \in D(\mathcal{E})_{loc}$. Then,*

(i) *There exist $Y^{[u]} \in \mathcal{M}_{loc}^{I(\zeta)}$ and $Z^{[u]} \in \mathcal{L}_c$ such that*

$$\tilde{u}(X_t) - \tilde{u}(X_0) = Y_t^{[u]} + Z_t^{[u]} + F_t^{[u]}, \quad t \ge 0, \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in E\,. \tag{7}$$

The decomposition (7) *is unique up to the equivalence of local AFs, and the continuous part of $Y^{[u]}$ belongs to $\dot{\mathcal{M}}_{loc}$.*

(ii) *There exists an $\{E_n\} \in \Theta$ such that for $n \in \mathbb{N}$, $\{Y_{t\wedge\tau_{E_n}}^{[u]}\}$ is a P_x-square-integrable martingale for $\mathcal{E}$-q.e. $x \in E$, $e^{E_n,*}(Y^{[u]}) < \infty$; $E_x[(Z_{t\wedge\tau_{E_n}}^{[u]})^2] < \infty$ for $t \ge 0$, $\mathcal{E}$-q.e. $x \in E$, $e^{E_n,*}(Z^{[u]}) = 0$.*

A Fukushima type decomposition for $A^{[u]}$ has been established in [17] under Condition (S). Below we will follow the argument of [17] to establish the decomposition for $A^{[u]} - F^{[u]}$ without assuming Condition (S). Before proving Theorem 1.2, we prepare some lemmas.

We fix a $\{V_n\} \in \Theta$ satisfying Assumption 1.1. Without loss of generality, we assume that $\widetilde{\hat{h}}$ is bounded on each V_n, otherwise we may replace V_n by $V_n \cap \{\widetilde{\hat{h}} < n\}$. Since $\bar{h}^{V_n} = \hat{G}_1^{V_n}\phi \leq \hat{G}_1\phi = \hat{h}$, $\bar{h}^{V_n}$ is bounded on V_n. To simplify notations, we write

$$\bar{h}_n := \bar{h}^{V_n}.$$

LEMMA 1.3 ([17, Lemma 1.12]). *Let $u \in D(\mathcal{E})_{V_n,b}$. Then there exist unique $M^{n,[u]} \in \dot{\mathcal{M}}^{V_n}$ and $N^{n,[u]} \in \mathcal{N}_c^{V_n}$ such that* for *$\mathcal{E}$-q.e. $x \in V_n$,*

$$\tilde{u}(X_t^{V_n}) - \tilde{u}(X_0^{V_n}) = M_t^{n,[u]} + N_t^{n,[u]}, \quad t \geq 0, \quad P_x\text{-}a.s. \tag{8}$$

We now fix a $u \in D(\mathcal{E})_{loc}$. Then, there exist $\{V_n^1\} \in \Theta$ and $\{u_n\} \subset D(\mathcal{E})$ such that $u = u_n$ m-a.e. on V_n^1. By [16, Proposition 3.6], we may assume without loss of generality that each u_n is $\mathcal{E}$-quasi-continuous. By [16, Proposition 2.16], there exists an $\mathcal{E}$-nest $\{F_n^2\}$ of compact subsets of E such that $\{u_n\} \subset C(\{F_k^2\})$. Denote by V_n^2 the fine interior of F_n^2. Then $\{V_n^2\} \in \Theta$. Denote $V_n^3 = V_n \cap V_n^1 \cap V_n^2$. Then $\{V_n^3\} \in \Theta$ and each u_n is bounded on V_n^3.

For $n \in \mathbb{N}$, we define $E_n = \{x \in E \mid \widetilde{h_n}(x) > \frac{1}{n}\}$, where $h_n := G_1^{V_n}\phi$. Then $\{E_n\} \in \Theta$ satisfying $\overline{E_n}^{\mathcal{E}} \subset E_{n+1}$ $\mathcal{E}$-q.e. and $E_n \subset V_n$ $\mathcal{E}$-q.e. for each $n \in \mathbb{N}$ (cf. [11, Lemma 3.8]). Hereafter, for $B \subset E$, we use $\overline{B}^{\mathcal{E}}$ to denote its $\mathcal{E}$-quasi-closure. Define $f_n = n\widetilde{h_n} \wedge 1$. Then $f_n \in D(\mathcal{E})_{V_n,b}$, $f_n = 1$ on E_n and $f_n = 0$ on V_n^c. Denote by Q_n the bound of $|u_n|$ on V_n^3. By [11, (2.1)] and Assumption 1.1, we find that $[(-Q_n f_n) \vee u_n \wedge (Q_n f_n)]f_n \in D(\mathcal{E})_{V_n,b}$. To simplify notations, below we still use u_n to denote $[(-Q_n f_n) \vee u_n \wedge (Q_n f_n)]$. Then we have $u_n, u_n f_n \in D(\mathcal{E})_{V_n,b}$, and $u = u_n = u_n f_n$ on $E_n \cap V_n^3$.

Denote by $J^n(dx, dy)$ and K^n the jumping and killing measures of $(\mathcal{E}^{V_n}, D(\mathcal{E}^{V_n}))$, respectively. Let $(N^n(x, dy), H_s^n)$ be a Lévy system of $\mathbf{M}^{V_n}$ and μ_{H^n} be the Revuz measure of H^n. Then $J^n(dy, dx) = \frac{1}{2}N^n(x, dy)\mu_{H^n}(dx)$ and $K^n(dx) = N^n(x, \{\Delta\})\mu_{H^n}(dx)$. For $n \in \mathbb{N}$, since $f_n, u_n f_n \in D(\mathcal{E})_{V_n,b}$, we obtain by [17, Proposition 1.8] that $f_n, u_n f_n$ satisfy Condition (S). That is, $\mu_{f_n}^n(dx) := \int_{V_n}(f_n(x) - f_n(y))^2 J^n(dy, dx)$ and $\mu_{u_n f_n}^n(dx) := \int_{V_n}((u_n f_n)(x) - (u_n f_n)(y))^2 J^n(dy, dx)$ are smooth measures with respect to $\mathbf{M}^{V_n}$. Let V be an $\mathcal{E}$-quasi-open set of E. We define

$$\Theta_V := \{\{R_k\} \mid R_k \text{ is } \mathcal{E}\text{-quasi-open}, \ R_k \subset R_{k+1} \ \mathcal{E}\text{-q.e.}$$
$$\forall\, k \in \mathbb{N}, \text{ and } V = \bigcup_{k=1}^{\infty} R_k \ \mathcal{E}\text{-q.e.}\}.$$

Then, for each $n \in \mathbb{N}$, there exists a $\{R_k^n\}_{k\in\mathbb{N}} \in \Theta_{V_n}$ such that for each $k \in \mathbb{N}$,

$$K^n(R_k^n) < \infty, \quad \int_{R_k^n}\int_{V_n}(f_n(x) - f_n(y))^2 J^n(dy, dx) < \infty, \tag{9}$$

$$\int_{R_k^n}\int_{V_n}((u_n f_n)(x)-(u_n f_n)(y))^2 J^n(dy,dx)<\infty .$$

By [11, Lemma 3.8], we may assume without loss of generality that $\overline{R_k^n}^{\mathcal{E}} \subset R_{k+1}^n$ $\mathcal{E}$-q.e.

Since $\{V_n\} \in \Theta$, by [11, Lemma 3.6] and the separability of $D(\mathcal{E})$ with respect to the $\mathcal{E}_1^{1/2}$-norm, we know that there exists a sequence $\{\xi_n\}$ satisfying $\xi_n \in D(\mathcal{E})_{V_n}$ for $n \in \mathbb{N}$ and $\{\xi_n \mid n \in \mathbb{N}\}$ is $\mathcal{E}_1^{1/2}$-dense in $D(\mathcal{E})$. For each $n \in \mathbb{N}$, we select an $a_n \in \mathbb{N}$ such that $\inf_{\xi\in D(\mathcal{E})_{R_{a_n}^n}} \mathcal{E}_1^{1/2}(\xi_n-\xi,\xi_n-\xi)<\frac{1}{n}$. Then $\bigcup_{n=1}^{\infty} D(\mathcal{E})_{R_{a_n}^n}$ is $\mathcal{E}_1^{1/2}$-dense in $D(\mathcal{E})$ and thus $\lim_{n\to\infty} \mathrm{cap}_\phi(E\backslash R_{a_n}^n)=0$ by [14]. We select a subsequence $\{n_l\}$ such that $\mathrm{cap}_\phi(E\backslash R_{a_{n_l}}^{n_l}) < \frac{1}{2^l}$ for each $l \in \mathbb{N}$. Define $F_l := \bigcap_{k=l}^{\infty} \overline{R_{a_{n_k}}^{n_k}}^{\mathcal{E}}$ for $l \in \mathbb{N}$. Then, $\{F_l\}$ is an $\mathcal{E}$-q.e. increasing sequence of $\mathcal{E}$-quasi-closed sets satisfying $\lim_{l\to\infty}\mathrm{cap}_\phi(E\backslash F_l)=0$. For $l \in \mathbb{N}$, we define by $V_{n_l}^4$ the fine interior of F_l. Therefore, we obtain by [11, Lemma 3.7] and (9) that $\{V_{n_l}^4\}_{l=1}^{\infty} \in \Theta$ and for each $l \in \mathbb{N}$, $V_{n_l}^4 \subset V_{n_l}$,

$$K^{n_l}(V_{n_l}^4)<\infty,\quad \int_{V_{n_l}^4}\int_{V_{n_l}}(f_{n_l}(x)-f_{n_l}(y))^2 J^{n_l}(dy,dx)<\infty ,$$
$$\int_{V_{n_l}^4}\int_{V_{n_l}}((u_{n_l} f_{n_l})(x)-(u_{n_l} f_{n_l})(y))^2 J^{n_l}(dy,dx)<\infty .$$

To simplify notations, we still use $\{n\}$ to denote $\{n_l\}$ and use E_n to denote $E_{n_l}\cap V_{n_l}^3\cap V_{n_l}^4$. Then we have $\{E_n\}\in\Theta$ and for each $n \in \mathbb{N}$, $E_n \subset V_n$, $u_n f_n \in D(\mathcal{E})_{V_n,b}$, $u = u_n f_n$ on E_n,

$$K^n(E_n)<\infty,\quad \int_{E_n}\int_{V_n}(f_n(x)-f_n(y))^2 J^n(dy,dx)<\infty , \tag{10}$$
$$\int_{E_n}\int_{V_n}((u_n f_n)(x)-(u_n f_n)(y))^2 J^n(dy,dx)<\infty .$$

LEMMA 1.4. *Let $u \in D(\mathcal{E})_{loc}$. Denote*

$$F_t^{[u],*} := \sum_{0<s\le t}(\tilde{u}(X_s)-\tilde{u}(X_{s-}))^2 1_{\{|\tilde{u}(X_s)-\tilde{u}(X_{s-})|\le 1\}} .$$

Then, $F_{t\wedge\tau_{E_n}}^{[u],*}$ is integrable with respect to $P_\nu := \int P_x\nu(dx)$ for any $\nu \in \hat{S}_{00}^*$ satisfying $\nu(E) < \infty$.

PROOF. Let $\nu \in \hat{S}_{00}^*$ with $\nu(E)<\infty$. By [13, Lemma A.9], there exists a constant $C_\nu > 0$ such that for any PCAF A with Revuz measure μ_A, we have

$$E_\nu(A_t)\le C_\nu(1+t)\int_E \tilde{\tilde{h}}\,d\mu_A,\quad t>0 . \tag{11}$$

Note that $u(X_s)=u_n(X_s)$ for any $s<\tau_{E_n}$. By [7, (A.3.23)], (6) and (11), we get

(12) $E_\nu[F_{t\wedge\tau_{E_n}}^{[u],*}]$

$$
\begin{aligned}
&\le E_\nu\left[\sum_{0<s\le t\wedge\tau_{E_n}}(u_n(X_s)-u_n(X_{s-}))^2 1_{\{|u_n(X_s)-u_n(X_{s-})|\le 1\}}\right]+\nu(E)\\
&=E_\nu\left[\int_0^{t\wedge\tau_{E_n}}\int_{E_\Delta}[u_n(y)-u_n(X_s)]^2 1_{\{|u_n(y)-u_n(X_s)|\le 1\}}N(X_s,dy)dH_s\right]+\nu(E)\\
&\le C_\nu(1+t)\int_{E_n}\tilde{\hat{h}}(x)\int_{E_\Delta}(u_n(y)-u_n(x))^2 1_{\{|u_n(y)-u_n(x)|\le 1\}}N(x,dy)\mu_H(dx)+\nu(E)\\
&=C_\nu(1+t)\left\{2\int_{E_n}\tilde{\hat{h}}(x)\int_E(u_n(y)-u_n(x))^2 1_{\{|u_n(y)-u_n(x)|\le 1\}}J(dy,dx)\right.\\
&\quad\left.+\int_{E_n}\tilde{\hat{h}}(x)u_n^2(x)1_{\{|u_n(x)|\le 1\}}K(dx)\right\}+\nu(E)\\
&=C_\nu(1+t)\left\{2\int_{E_n}\tilde{\hat{h}}(x)\int_{V_n}(u_n(y)-u_n(x))^2 1_{\{|u_n(y)-u_n(x)|\le 1\}}J^n(dy,dx)\right.\\
&\quad\left.+\int_{E_n}\tilde{\hat{h}}(x)u_n^2(x)1_{\{|u_n(x)|\le 1\}}K^n(dx)\right\}+\nu(E)\,.
\end{aligned}
$$

Note here that $K^n(dx)=K(dx)+2J(V_n^c,dx)$ on V_n and $J^n=J$ on $V_n\times V_n$.

Further, we obtain by $f_n=1$ on E_n, (10) and (12) that

$$
\begin{aligned}
&E_\nu[F^{[u],*}_{t\wedge\tau_{E_n}}]\\
&\le C_\nu(1+t)\|\tilde{\hat{h}}|_{E_n}\|_\infty\left\{2\int_{E_n}f_n^2(x)\int_{V_n}(u_n(y)-u_n(x))^2 1_{\{|u_n(y)-u_n(x)|\le 1\}}J^n(dy,dx)+K^n(E_n)\right\}\\
&\quad+\nu(E)\\
&\le C_\nu(1+t)\|\tilde{\hat{h}}|_{E_n}\|_\infty\left\{4\int_{E_n}\int_{V_n}(f_n(x)-f_n(y))^2J^n(dy,dx)\right.\\
&\quad\left.+4\int_{E_n}\int_{V_n}f_n^2(y)(u_n(y)-u_n(x))^2J^n(dy,dx)+K^n(E_n)\right\}+\nu(E)\\
&\le C_\nu(1+t)\|\tilde{\hat{h}}|_{E_n}\|_\infty\left\{4\int_{E_n}\int_{V_n}(f_n(x)-f_n(y))^2J^n(dy,dx)\right.\\
&\quad+8\int_{E_n}\int_{V_n}((u_nf_n)(x)-(u_nf_n)(y))^2J^n(dy,dx)\\
&\quad\left.+8\int_{E_n}u_n^2(x)\int_{V_n}(f_n(x)-f_n(y))^2J^n(dy,dx)+K^n(E_n)\right\}+\nu(E)\\
&<\infty\,.
\end{aligned}
$$

□

Proof of Theorem 1.2 (i). Let $\{V_n\}$, $\{E_n\}$ and $\{u_nf_n\}$ be given as before. By Lemma 1.3, for $n\in\mathbb{N}$, there exist unique $M^{n,[u_nf_n]}\in\dot{\mathcal{M}}^{V_n}$ and $N^{n,[u_nf_n]}\in\mathcal{N}_c^{V_n}$ such that for $\mathcal{E}$-q.e. $x\in V_n$,

$$u_nf_n(X_t^{V_n})-u_nf_n(X_0^{V_n})=M_t^{n,[u_nf_n]}+N_t^{n,[u_nf_n]},\quad t\ge 0,\quad P_x\text{-a.s.}$$

Hereafter, for a martingale M, we denote by M^c and M^d its continuous part and purely discontinuous part, respectively. By [17, Lemma 1.14], for $n < l$, we have $M^{n,[u_n f_n],c}_{t\wedge\tau_{E_n}} = M^{l,[u_l f_l],c}_{t\wedge\tau_{E_n}}$, $t \geq 0$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_n$. Therefore, we can define $\{M^{[u],c}_t \mid 0 \leq t < \infty\}$ by $M^{[u],c}_t := \lim_{l\to\infty} M^{l,[u_l f_l],c}_t$ for $0 \leq t \leq \tau_{E_n}$ and $n \in \mathbb{N}$; $M^{[u],c}_t := 0$ for $t > \zeta$ if there exists some $n \in \mathbb{N}$ such that $\tau_{E_n} = \zeta$ and $\zeta < \infty$, or $M^{[u],c}_t := 0$ for $t \geq \zeta$ if $\tau_{E_n} < \zeta$ for any $n \in \mathbb{N}$. Following the argument of the proof of [17, Theorem 1.4], we can show that $M^{[u],c}$ is well defined, $M^{[u],c} \in \dot{\mathcal{M}}_{loc}$ and $M^{[u],c} \in \mathcal{M}^{I(\zeta)}_{loc}$.

Denote $\Delta u(X_s) := \tilde{u}(X_s) - \tilde{u}(X_{s-})$. By Lemma 1.4,

$$\begin{aligned} Y^l_t &:= \sum_{0<s\leq t} \Delta u(X_s) 1_{\{\frac{1}{l}\leq|\Delta u(X_s)|\leq 1\}} - \left(\sum_{0<s\leq t} \Delta u(X_s) 1_{\{\frac{1}{l}\leq|\Delta u(X_s)|\leq 1\}} \right)^p \\ &= \sum_{0<s\leq t} \Delta u(X_s) 1_{\{\frac{1}{l}\leq|\Delta u(X_s)|\leq 1\}} \\ &\quad - \int_0^t \int_{\{\frac{1}{l}\leq|\tilde{u}(y)-\tilde{u}(X_s)|\leq 1\}} (\tilde{u}(y) - \tilde{u}(X_s)) N(X_s, dy) dH_s \end{aligned}$$

is well-defined. Hereafter ^{p} denotes the dual predictable projection. Further, by Lemma 1.4 and following the argument of the proof of [17, Theorem 1.4] (with M^l therein replaced with Y^l of this paper), we can show that for $\mathcal{E}$-q.e. $x \in E$, $Y^{l_k}_{t\wedge\tau_{E_n}}$ converges uniformly in t on each finite interval for a subsequence $\{l_k \to \infty\}$ and for each k,

$$Y^{l_k}_{(t+s)\wedge\tau_{E_n}} = Y^{l_k}_{t\wedge\tau_{E_n}} + Y^{l_k}_{s\wedge\tau_{E_n}} \circ \theta_{t\wedge\tau_{E_n}}, \text{ if } 0 \leq t, s < \infty .$$

Thus, L^n, the limit of $\{Y^{l_k}_{s\wedge\tau_{E_n}}\}_{k=1}^{\infty}$, is a P_x-square integrable purely discontinuous martingale for $\mathcal{E}$-q.e. $x \in E$ and satisfies:

$$L^n_{(t+s)\wedge\tau_{E_n}} = L^n_{t\wedge\tau_{E_n}} + L^n_{s\wedge\tau_{E_n}} \circ \theta_{t\wedge\tau_{E_n}}, \text{ if } 0 \leq t, s < \infty .$$

By the above construction, we find that $L^{n_1}_{t\wedge\tau_{E_{n_1}}} = L^{n_2}_{t\wedge\tau_{E_{n_1}}}$ for $n_1 \leq n_2$. We define $\{Y^{[u],d}_t \mid 0 \leq t < \infty\}$ by $Y^{[u],d}_t := L^n_t$ for $0 \leq t \leq \tau_{E_n}$ and $n \in \mathbb{N}$; $Y^{[u],d}_t := 0$ for $t > \zeta$ if there exists some $n \in \mathbb{N}$ such that $\tau_{E_n} = \zeta$ and $\zeta < \infty$, or $Y^{[u],d}_t := 0$ for $t \geq \zeta$ if $\tau_{E_n} < \zeta$ for any $n \in \mathbb{N}$. Then $Y^{[u],d} \in \mathcal{M}^{I(\zeta)}_{loc}$, which gives all the jumps of $\tilde{u}(X_t) - \tilde{u}(X_0)$ on $I(\zeta)$ with jump size less than or equal to 1. Since $\{Y^l_t\}$ is an MAF for each l, we find that $\{Y^{[u],d}_t\}$ is a local MAF by the locally uniform convergence on $I(\zeta)$.

We define $Y^{[u]} := M^{[u],c} + Y^{[u],d}$ and $Z^{[u]}_{t\wedge\tau_{E_n}} := \tilde{u}(X_{t\wedge\tau_{E_n}}) - \tilde{u}(X_0) - Y^{[u]}_{t\wedge\tau_{E_n}} - F^{[u]}_{t\wedge\tau_{E_n}}$ for each $n \in \mathbb{N}$. Then $Z^{[u]}$ is a local AF of $\mathbf{M}$. Note that

$$\begin{aligned} \Delta Z^{[u]}_{t\wedge\tau_{E_n}} &= \Delta\tilde{u}(X_{t\wedge\tau_{E_n}}) - \Delta Y^{[u]}_{t\wedge\tau_{E_n}} - \Delta F^{[u]}_{t\wedge\tau_{E_n}} \\ &= \Delta\tilde{u}(X_{t\wedge\tau_{E_n}}) - \Delta\tilde{u}(X_{t\wedge\tau_{E_n}}) 1_{\{|\Delta\tilde{u}(X_{t\wedge\tau_{E_n}})|\leq 1\}} \\ &\quad - \Delta\tilde{u}(X_{t\wedge\tau_{E_n}}) 1_{\{|\Delta\tilde{u}(X_{t\wedge\tau_{E_n}})|> 1\}} \\ &= 0 . \end{aligned}$$

Hence $t \mapsto Z^{[u]}_{t\wedge\tau_{E_n}}$ is continuous. Now we show that $\{Z^{[u]}_{t\wedge\tau_{E_n}}\}$ has zero quadratic variation and thus $Z^{[u]} \in \mathcal{L}_c$. Note that $f_n = 0$ on V_n^c. By Fukushima's decomposition for part processes, we have that

$$
\begin{aligned}
(13)\qquad & u_n f_n(X_{t\wedge\tau_{E_n}}) - u_n f_n(X_0)\\
&= u_n f_n(X^{V_n}_{t\wedge\tau_{E_n}}) - u_n f_n(X^{V_n}_0)\\
&= M^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} + N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}}\\
&= M^{n,[u_n f_n],c}_{t\wedge\tau_{E_n}} + M^{n,[u_n f_n],d}_{t\wedge\tau_{E_n}} + N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}}\\
&= M^{n,[u_n f_n],c}_{t\wedge\tau_{E_n}} + M^{n,[u_n f_n],sd}_{t\wedge\tau_{E_n}} + M^{n,[u_n f_n],bd}_{t\wedge\tau_{E_n}} + N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}},
\end{aligned}
$$

where

$$
\begin{aligned}
M^{n,[u_n f_n],sd}_t = \lim_{l\to\infty}\Bigg\{ & \sum_{0<s\le t} (u_n f_n(X^{V_n}_s) - u_n f_n(X^{V_n}_{s-}))1_{\{\frac{1}{l}\le |u_n f_n(X^{V_n}_s) - u_n f_n(X^{V_n}_{s-})|\le 1\}}\\
& - \int_0^t \int_{\{\frac{1}{l}\le |u_n f_n(y) - u_n f_n(X^{V_n}_s)|\le 1\}} (u_n f_n(y) - u_n f_n(X^{V_n}_s)) N^n(X^{V_n}_s, dy)\, dH^n_s \Bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
M^{n,[u_n f_n],bd}_t = & \sum_{0<s\le t} (u_n f_n(X^{V_n}_s) - u_n f_n(X^{V_n}_{s-}))1_{\{|u_n f_n(X^{V_n}_s) - u_n f_n(X^{V_n}_{s-})|>1\}}\\
& - \int_0^t \int_{\{|u_n f_n(y) - u_n f_n(X^{V_n}_s)|>1\}} (u_n f_n(y) - u_n f_n(X^{V_n}_s)) N^n(X^{V_n}_s, dy)\, dH^n_s.
\end{aligned}
$$

We define

$$
\begin{aligned}
B_t := \Big\{ & (\tilde u(X_{\tau_{E_n}}) - \tilde u(X_{\tau_{E_n}-}))1_{\{|\tilde u(X_{\tau_{E_n}}) - \tilde u(X_{\tau_{E_n}-})|\le 1\}}\\
& - (u_n f_n(X_{\tau_{E_n}}) - u_n f_n(X_{\tau_{E_n}-}))1_{\{|u_n f_n(X_{\tau_{E_n}}) - u_n f_n(X_{\tau_{E_n}-})|\le 1\}} \Big\} 1_{\{\tau_{E_n}\le t\}}.
\end{aligned}
$$

$\{B_t\}$ is an adapted quasi-left continuous bounded variation process and hence its dual predictable projection $\{B^p_t\}$ is an adapted continuous bounded variation process (cf. [7, Theorem A.3.5]). By comparing (13) to

$$
\tilde u(X_{t\wedge\tau_{E_n}}) - \tilde u(X_0) = M^{[u],c}_{t\wedge\tau_{E_n}} + Y^{[u],d}_{t\wedge\tau_{E_n}} + Z^{[u]}_{t\wedge\tau_{E_n}} + F^{[u]}_{t\wedge\tau_{E_n}},
$$

we get

$$
\begin{aligned}
(14)\qquad Z^{[u]}_{t\wedge\tau_{E_n}} &= N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} + M^{n,[u_n f_n],sd}_{t\wedge\tau_{E_n}} - Y^{[u],d}_{t\wedge\tau_{E_n}} + M^{n,[u_n f_n],bd}_{t\wedge\tau_{E_n}} - F^{[u]}_{t\wedge\tau_{E_n}}\\
&\quad + \tilde u(X_{t\wedge\tau_{E_n}}) - u_n f_n(X_{t\wedge\tau_{E_n}})\\
&= N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} + (M^{n,[u_n f_n],sd}_{t\wedge\tau_{E_n}} - Y^{[u],d}_{t\wedge\tau_{E_n}} + B_t - B^p_t) + B^p_t\\
&\quad - \int_0^{t\wedge\tau_{E_n}} \int_{\{|u_n f_n(y) - u_n f_n(X^{V_n}_s)|>1\}} (u_n f_n(y) - u_n f_n(X^{V_n}_s))\\
&\qquad \cdot N^n(X^{V_n}_s, dy)\, dH^n_s.
\end{aligned}
$$

Hence $\{M^{n,[u_nf_n],sd}_{t\wedge\tau_{E_n}} - Y^{[u],d}_{t\wedge\tau_{E_n}} + B_t - B^p_t\}$ is a purely discontinuous martingale with zero jump, which must be equal to zero. The quadratic variation of $\{N^{n,[u_nf_n]}_{t\wedge\tau_{E_n}}\}$ vanishes in $P_{\bar{h}_n\cdot m}$-measure (see the proof of [17, Lemma 1.14]) and the quadratic variation of $\{B^p_t\}$ vanishes in $P_{\phi\cdot m}$-measure since $\{B^p_t\}$ is a continuous bounded variation process. Denote by C^n_t the last term of (14). By (10), one finds that $\{C^n_t\}$ is a P_ν-square-integrable continuous bounded variation process for any $\nu \in \hat{S}^*_{00}$ satisfying $\nu(E) < \infty$. Hence its quadratic variation vanishes in $P_{\phi\cdot m}$-measure. Therefore, the quadratic variation of $\{Z^{[u]}_{t\wedge\tau_{E_n}}\}$ vanishes in P_m-measure since $m(E_n) < \infty$, i.e., $\{Z^{[u]}_{t\wedge\tau_{E_n}}\}$ has zero quadratic variation.

Finally, we prove the uniqueness of decomposition (7). Suppose that $Y' \in \mathcal{M}^{I(\zeta)}_{loc}$ and $Z' \in \mathcal{L}_c$ such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = Y'_t + Z'_t + F^{[u]}_t,\ \ t \geq 0,\ \ P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in E\,.$$

By [17, Proposition 2.4], we can choose an $\{E_n\} \in \Theta$ such that $I(\zeta) = \cup_n [\![0, \tau_{E_n}]\!]$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Then, for each $n \in \mathbb{N}$, $\{(Y^{[u]} - Y')^{\tau_{E_n}}\}$ is a locally square integrable martingale and a zero quadratic variation process. This implies that $P_m(\langle (Y^{[u]} - Y')^{\tau_{E_n}}\rangle_t = 0, \forall t \in [0,\infty)) = 0$. By [13, Theorem A.8], following the proof of [7, Lemma 5.1.10(iii)], we have that $P_x(\langle (Y^{[u]} - Y')^{\tau_{E_n}}\rangle_t = 0, \forall t \in [0,\infty)) = 0$ for $\mathcal{E}$-q.e. $x \in E$. Therefore $Y^{[u]}_t = Y'_t$, $0 \leq t \leq \tau_{E_n}$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Since n is arbitrary, we obtain the uniqueness of decomposition (7) up to the equivalence of local AFs.

Proof of Theorem 1.2 (ii). By (i), $Y^{[u]} \in \mathcal{M}^{I(\zeta)}_{loc}$. Hence $\langle Y^{[u],d}\rangle_t = (\int_0^t \int_{E_\Delta} (\tilde{u}(X_s) - \tilde{u}(y))^2 1_{\{|\tilde{u}(X_s)-\tilde{u}(y)|\leq 1\}} N(X_s, dy) dH_s) 1_{I(\zeta)}$ is a PCAF on $I(\zeta)$ and can be extended to a PCAF by [3, Remark 2.2]. The Revuz measure of $\langle Y^{[u],d}\rangle$ is given by

$$\begin{aligned}\mu^d_{\langle u\rangle}(dx) =& 2\int_E (\tilde{u}(x) - \tilde{u}(y))^2 1_{\{|\tilde{u}(x)-\tilde{u}(y)|\leq 1\}} J(dy, dx)\\ &+ \tilde{u}^2(x) 1_{\{|\tilde{u}(x)|\leq 1\}} K(dx)\,.\end{aligned}$$

By [17, Lemma 1.1], $\mu^d_{\langle u\rangle}$ is a smooth measure. Therefore, there exists an $\{E'_n\} \in \Theta$ such that $\mu^d_{\langle u\rangle}(E'_n) < \infty$ for each $n \in \mathbb{N}$. To simplify notations, we still use E_n to denote $E_n \cap E'_n$. The remaining part of the proof is similar to that of [17, Theorem 1.15]. We omit the details here. □

REMARK 1.5. (i) As in [17, Theorem 1.4], if we use $\mathcal{M}^{[\![0,\zeta[\![}_{loc}$ instead of $\mathcal{M}^{I(\zeta)}_{loc}$, then the uniqueness of the decomposition (7) may fail to be true.

(ii) For $u \in D(\mathcal{E})_{loc}$, if Condition (S) holds, i.e., $\mu_u \in S$, then by [17, Theorem 1.4], there exist unique $M^{[u]} \in \mathcal{M}^{I(\zeta)}_{loc}$ and $N^{[u]} \in \mathcal{L}_c$ such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M^{[u]}_t + N^{[u]}_t,\ \ t \geq 0,\ \ P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in E\,, \tag{15}$$

with

$$M^{[u]}_t = M^{[u],c}_t + M^{[u],d}_t\,, \tag{16}$$

and

$$M_t^{[u],d} = \lim_{l\to\infty}\left\{\sum_{0<s\le t}(\tilde u(X_s)-\tilde u(X_{s-}))1_{\{\frac{1}{l}\le|\tilde u(X_s)-\tilde u(X_{s-})|\}} - \int_0^t\int_{\{\frac{1}{l}\le|\tilde u(y)-\tilde u(X_s)|\}}(\tilde u(y)-\tilde u(X_s))N(X_s,dy)dH_s\right\}. \tag{17}$$

By comparing (15)-(17) with

$$\begin{aligned}\tilde u(X_t)-\tilde u(X_0) =& Y_t^{[u]}+Z_t^{[u]}+F_t^{[u]}\\ =& M_t^{[u],c}+Y_t^{[u],d}+Z_t^{[u]}+F_t^{[u]},\end{aligned}$$

$$Y_t^{[u],d} = \lim_{l\to\infty}\left\{\sum_{0<s\le t}(\tilde u(X_s)-\tilde u(X_{s-}))1_{\{\frac{1}{l}\le|\tilde u(X_s)-\tilde u(X_{s-})|\le 1\}} - \int_0^t\int_{\{\frac{1}{l}\le|\tilde u(y)-\tilde u(X_s)|\le 1\}}(\tilde u(y)-\tilde u(X_s))N(X_s,dy)dH_s\right\},$$

we get

$$\begin{aligned}M_t^{[u]} =& Y^{[u]}+\sum_{0<s\le t}(\tilde u(X_s)-\tilde u(X_{s-}))1_{\{|\tilde u(X_s)-\tilde u(X_{s-})|>1\}}\\ &-\int_0^t\int_{\{|\tilde u(y)-\tilde u(X_s)|>1\}}(\tilde u(y)-\tilde u(X_s))N(X_s,dy)dH_s,\end{aligned}$$

and

$$N_t^{[u]} = Z^{[u]}+\int_0^t\int_{\{|\tilde u(y)-\tilde u(X_s)|>1\}}(\tilde u(y)-\tilde u(X_s))N(X_s,dy)dH_s.$$

2. Stochastic integral and Itô's formula.

Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E;m)$ with associated Markov process $\mathbf{M} = ((X_t)_{t\ge0},(P_x)_{x\in E_\Delta})$. Throughout this section, we put the following assumption.

ASSUMPTION 2.1. There exist $\{V_n\}\in\Theta$, Dirichlet forms $(\eta^{(n)}, D(\eta^{(n)}))$ on $L^2(V_n;m)$, and constants $\{C_n>1\}$ such that for each $n\in\mathbb{N}$, $D(\eta^{(n)}) = D(\mathcal{E})_{V_n}$ and

$$\frac{1}{C_n}\eta_1^{(n)}(u,u)\le\mathcal{E}_1(u,u)\le C_n\eta_1^{(n)}(u,u),\quad \forall u\in D(\mathcal{E})_{V_n}.$$

By [15, Corollary 4.15], Assumption 2.1 implies Assumption 1.1. In this section, we will first define stochastic integrals for part forms $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ and then extend them to $(\mathcal{E}, D(\mathcal{E}))$.

2.1. Stochastic integral for part process. We fix a $\{V_n\} \in \Theta$ satisfying Assumption 2.1. Without loss of generality, we assume that $\widetilde{\hat{h}}$ is bounded on each V_n, otherwise we may replace V_n by $V_n \cap \{\widetilde{\hat{h}} < n\}$. For $n \in \mathbb{N}$, let $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ be the part form of $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(V_n; m)$. Then, $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ is a quasi regular semi-Dirichlet form with associated Markov process $\mathbf{M}^{V_n} = ((X_t^{V_n})_{t\geq 0}, (P_x^{V_n})_{x\in (V_n)_\Delta})$ (cf. [11]).

Let $u \in D(\mathcal{E})_{V_n}$ and denote $A_t^{n,[u]} = \tilde{u}(X_t^{V_n}) - \tilde{u}(X_0^{V_n})$. By Lemma 1.3, we have the decomposition (8). For $v \in D(\mathcal{E})_{V_n,b}$, we will follow [2] to define the stochastic integral $\int_0^t \tilde{v}(X_{s-}^{V_n}) dA_s^{n,[u]}$ and derive the related Itô's formula. Note that since $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ is only a semi-Dirichlet form, its symmetric part $(\tilde{\mathcal{E}}^{V_n}, D(\mathcal{E})_{V_n})$ might not be a Dirichlet form. However, we can use $(\tilde{\eta}^{(n)}, D(\eta^{(n)}))$, the symmetric part of $(\eta^{(n)}, D(\eta^{(n)}))$, to substitute $(\tilde{\mathcal{E}}^{V_n}, D(\mathcal{E})_{V_n})$ and then follow the argument of [2] to define Nakao's integral $\int_0^t \tilde{v}(X_{s-}^{V_n}) dN_s^{n,[u]}$ and prove its related properties. Below we will mainly state the results and point out only the necessary modifications in proofs. For more details we refer the reader to [2].

We use $A_c^{n,+}$ to denote the family of all PCAFs of $\mathbf{M}^{V_n}$. Define

$$A_c^{n,+,f} := \{A \in A_c^{n,+} \mid \text{the smooth measure, } \mu_A, \text{ corresponding to } A \text{ is finite}\}$$

and

$$\mathcal{N}_c^{n,*} := \left\{ N_t^{[u]} + \int_0^t g(X_s)ds + A_t^{(1)} - A_t^{(2)} \,\middle|\, u \in D(\mathcal{E})_{V_n}, g \in L^2(V_n; m) \text{ and } A^{(1)}, A^{(2)} \in A_c^{n,+,f} \right\}.$$

Note that any $C \in \mathcal{N}_c^{n,*}$ is finite and continuous on $[0, \infty)$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Similar to [18, Theorem 2.2], we can prove the following lemma.

LEMMA 2.2. *Let Υ be a finely open set such that $\Upsilon \subset V_n$. If $C^{(1)}, C^{(2)} \in \mathcal{N}_c^{n,*}$ satisfy*

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}^{V_n}[C_t^{(1)}] = \lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}^{V_n}[C_t^{(2)}], \quad \forall h \in D(\mathcal{E})_{\Upsilon,b},$$

then $C^{(1)} = C^{(2)}$ for $t < \tau_\Upsilon$ $P_x^{V_n}$-a.s. for $\mathcal{E}$-q.e. $x \in V_n$.

For $u \in D(\mathcal{E})_{V_n}$ and $v \in D(\mathcal{E})_{V_n,b}$, we will define $\int_0^t \tilde{v}(X_{s-}^{V_n}) dN_s^{n,[u]}$ to be the unique AF $(C_t)_{t\geq 0}$ in $\mathcal{N}_c^{n,*}$ that satisfies $\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}^{V_n}[C_t] = -\mathcal{E}^{V_n}(u, hv)$ for any $h \in D(\mathcal{E})_{V_n,b}$ (see Definition 2.5 and Remark 2.6 below). Denote by $(L^{V_n}, D(L^{V_n}))$ the generator of $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$. Note that if $u \in D(L^{V_n})$ then $dN_s^{n,[u]} = L^{V_n} u(X_s^{V_n}) ds$. In this case, it is easy to see that for any $v, h \in D(\mathcal{E})_{V_n,b}$,

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}^{V_n}\left[\int_0^t v(X_s^{V_n}) L^{V_n} u(X_s^{V_n}) ds\right] = \int_{V_n} hvL^{V_n} u\, dm = -\mathcal{E}^{V_n}(u, hv)$$

(cf. [13, Theorem A.8(vi)]). Hence our definition of the stochastic integral $\int_0^t \tilde{v}(X_s^{V_n}) dN_s^{n,[u]}$ for $u \in D(\mathcal{E})_{V_n}$ is an extension of the ordinary Lebesgue integral $\int_0^t \tilde{v}(X_s^{V_n}) L^{V_n} u(X_s^{V_n}) ds$ for $u \in D(L^{V_n})$.

Similar to [2, Lemma 2.1], we can prove the following lemma.

LEMMA 2.3. *Let $f \in D(\mathcal{E})_{V_n}$. Then there exist unique $f^* \in D(\mathcal{E})_{V_n}$ and $f^\triangle \in D(\mathcal{E})_{V_n}$ such that for any $g \in D(\mathcal{E})_{V_n}$,*

$$\mathcal{E}_1^{V_n}(f, g) = \tilde{\eta}_1^{(n)}(f^*, g) \tag{18}$$

and

$$\tilde{\eta}_1^{(n)}(f, g) = \mathcal{E}_1^{V_n}(f^\triangle, g)\,. \tag{19}$$

Let $f, g \in D(\mathcal{E})_{V_n}$. We use $\tilde{\mu}^{(n)}_{\langle f,g\rangle}$ to denote the mutual energy measure of f and g with respect to the symmetric Dirichlet form $(\tilde{\eta}^{(n)}, D(\mathcal{E})_{V_n})$. Suppose that $u \in D(\mathcal{E})_{V_n}$ and $v \in D(\mathcal{E})_{V_n,b}$. By [7, Theorem 5.2.3 and Lemma 5.6.1], we get

$$\begin{aligned}\left|\int_{V_n} \tilde{v}\, d\tilde{\mu}^{(n)}_{\langle h,u^*\rangle}\right| \le &\left(\int_{V_n} \tilde{v}^2 d\tilde{\mu}^{(n)}_{\langle h,h\rangle}\right)^{\frac{1}{2}} \left(\int_{V_n} d\tilde{\mu}^{(n)}_{\langle u^*,u^*\rangle}\right)^{\frac{1}{2}}\\ \le &2\|\tilde{v}\|_\infty (\tilde{\eta}_1^{(n)}(h,h))^{\frac{1}{2}} (\tilde{\eta}_1^{(n)}(u^*,u^*))^{\frac{1}{2}}\,.\end{aligned}$$

Hence $h \mapsto \frac{1}{2}\int_{V_n} \tilde{v}\, d\tilde{\mu}^{(n)}_{\langle h,u^*\rangle}$ is a bounded linear function on $D(\mathcal{E})_{V_n}$. By the Riesz representation theorem, there exists a unique element in $D(\mathcal{E})_{V_n}$, which is denoted by $\lambda(u, v)$, such that

$$\frac{1}{2}\int_{V_n} \tilde{v}\, d\tilde{\mu}^{(n)}_{\langle h,u^*\rangle} = \tilde{\eta}_1^{(n)}(\lambda(u, v), h), \quad \forall h \in D(\mathcal{E})_{V_n}\,.$$

Let u^* and $\lambda(u, v)^\triangle$ be the unique elements in $D(\mathcal{E})_{V_n}$ as defined by (18) and (19) relative to u and $\lambda(u, v)$, respectively. Similar to [2, Theorem 2.2], we can prove the following result.

THEOREM 2.4. *Let $u \in D(\mathcal{E})_{V_n}$ and $v \in D(\mathcal{E})_{V_n,b}$. Then, for any $h \in D(\mathcal{E})_{V_n,b}$,*

$$\mathcal{E}^{V_n}(u, hv) = \mathcal{E}_1^{V_n}(\lambda(u, v)^\triangle, h) + \frac{1}{2}\int_{V_n} \tilde{h}\, d\tilde{\mu}^{(n)}_{\langle v,u^*\rangle} + \int_{V_n} (u^* - u)hv\, dm\,. \tag{20}$$

Note that $\tilde{\mu}^{(n)}_{\langle v,u^*\rangle}$ is a signed smooth measure with respect to $(\tilde{\eta}^{(n)}, D(\eta^{(n)}))$ and hence $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ by Assumption 2.1. We use $G(u, v)$ to denote the unique element in $A_c^{n,+} - A_c^{n,+}$ that is corresponding to $\tilde{\mu}^{(n)}_{\langle v,u^*\rangle}$ under the Revuz correspondence between smooth measures of $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ and PCAFs of $\mathbf{M}^{V_n}$ (cf. [13, Theorem A.8]). To simplify notations, we define

$$\Gamma(u, v)_t := N_t^{[\lambda(u,v)^\triangle]} - \int_0^t \lambda(u, v)^\triangle(X_s^{V_n})\,ds, \quad t \ge 0\,.$$

DEFINITION 2.5. Let $u \in D(\mathcal{E})_{V_n}$ and $v \in D(\mathcal{E})_{V_n,b}$. We define for $t \ge 0$,

$$\begin{aligned}\int_0^t \tilde{v}(X_{s-}^{V_n})\,dN_s^{n,[u]} :=& \int_0^t \tilde{v}(X_s^{V_n})\,dN_s^{n,[u]}\\ :=&\Gamma(u, v)_t - \frac{1}{2}G(u, v)_t - \int_0^t (u^* - u)v(X_s^{V_n})\,ds\,.\end{aligned}$$

REMARK 2.6. Let $u \in D(\mathcal{E})_{V_n}$ and $v \in D(\mathcal{E})_{V_n,b}$. Then one can check that $\int_0^t \tilde{v}(X_s^{V_n}) dN_s^{n,[u]} \in \mathcal{N}_c^{n,*}$. By Definition 2.5, (8), [1, Theorem 3.4], [13, Theorem A.8(iii)] and (20), we obtain that

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}^{V_n}\left[\int_0^t \tilde{v}(X_s^{V_n})dN_s^{[u],n}\right]\\
&=\lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}^{V_n}\left[N_t^{[\lambda(u,v)^{\triangle}]}-\int_0^t \lambda(u,v)^{\triangle}(X_s^{V_n})ds-\frac{1}{2}G(u,v)_t-\int_0^t (u^*-u)v(X_s^{V_n})ds\right]\\
&=-\mathcal{E}_1^{V_n}(\lambda(u,v)^{\triangle},h)-\frac{1}{2}\int_{V_n}\tilde{h}d\tilde{\mu}_{\langle v,u^*\rangle}^{(n)}-\int_{V_n}(u^*-u)hvdm\\
&=-\mathcal{E}^{V_n}(u,hv),\quad \forall h\in D(\mathcal{E})_{V_n,b}\,.
\end{aligned}$$

Therefore, by Lemma 2.2, $\int_0^t \tilde{v}(X_s^{V_n})dN_s^{n,[u]}$ is the unique AF $(C_t)_{t\geq 0}$ in $\mathcal{N}_c^{n,*}$ that satisfies $\lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}^{V_n}[C_t]=-\mathcal{E}^{V_n}(u,hv)$ for any $h\in D(\mathcal{E})_{V_n,b}$.

Similar to [2, Proposition 2.6], we can prove the following proposition.

PROPOSITION 2.7. *Let $u \in D(\mathcal{E})_{V_n}$, $v \in D(\mathcal{E})_{V_n,b}$ and Υ be a finely open set such that $\Upsilon \subset V_n$. Suppose that there exist $A^{(1)}, A^{(2)} \in A_c^{n,+}$ such that $N_t^{n,[u]} = A_t^{(1)} - A_t^{(2)}$ for $t < \tau_\Upsilon$. Then*

$$\int_0^t \tilde{v}(X_s^{V_n})dN_s^{n,[u]}=\int_0^t \tilde{v}(X_s^{V_n})d(A_s^{(1)}-A_s^{(2)}) \text{ for } t<\tau_\Upsilon$$

$P_x^{V_n}$-a.s. for $\mathcal{E}$-q.e. $x \in V_n$.

THEOREM 2.8. *Let $v \in D(\mathcal{E})_{V_n,b}$ and $\{u_k\}_{k=0}^\infty \subset D(\mathcal{E})_{V_n}$ such that u_k converges to u_0 with respect to the $\tilde{\mathcal{E}}_1^{1/2}$-norm as $k \to \infty$. Then there exists a subsequence $\{k'\}$ such that for $\mathcal{E}$-q.e. $x \in V_n$,*

$$\begin{aligned}
&P_x^{V_n}\Bigg(\lim_{k'\to\infty}\int_0^t \tilde{v}(X_s^{V_n})dN_s^{n,[u_{k'}]}\\
&\qquad=\int_0^t \tilde{v}(X_s^{V_n})dN_s^{n,[u_0]} \text{ uniformly on any finite interval of } t\Bigg)=1\,.
\end{aligned}$$

PROOF. By Definition 2.5, we have

$$\begin{aligned}
\int_0^t \tilde{v}(X_s^{V_n})dN_s^{n,[u_k]}=&N_t^{n,[\lambda(u_k,v)^{\triangle}]}-\int_0^t \lambda(u_k,v)^{\triangle}(X_s^{V_n})ds\\
&-\frac{1}{2}G(u_k,v)_t-\int_0^t (u_k^*-u_k)v(X_s^{V_n})ds\,.
\end{aligned}$$

For each term of the right hand side of the above equation, we can prove that there exists a subsequence which converges uniformly on any finite interval of t. Below we will only give the proof for the convergence of the third term. The convergence of the other three terms can be proved similar to [2, Theorem 2.7] by virtue of [13, Lemmas 2.5 and A.6].

We use S_0^n and $\hat{U}_1^{V_n}\mu$ to denote respectively the family of positive measures of finite energy integral and 1-co-potential relative to $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$. Define

$$\hat{S}_{00}^{n,*} := \{\mu \in S_0^n \mid \hat{U}_1^{V_n}\mu \leq c\hat{G}_1^{V_n}\phi \text{ for some constant } c > 0\}.$$

Let $A \in \mathcal{B}(E)$. By [13, Theorem A.3], if $\nu(A) = 0$ for every $\nu \in \hat{S}_{00}^{n,*}$ then $\mathrm{cap}_\phi(A) = 0$, where the capacity cap_ϕ is defined as in [14].

Let $\nu \in \hat{S}_{00}^{n,*}$. Recall that for $u \in D(\mathcal{E})_{V_n}$, $G(u, v)$ denotes the unique element in $A_c^{n,+} - A_c^{n,+}$ that is corresponding to $\tilde{\mu}^{(n)}_{\langle v,u^*\rangle}$ under the Revuz correspondence between smooth measures of $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ and PCAFs of $\mathbf{M}^{V_n}$. Hence $G(u_k, v) - G(u_0, v) = G(u_k - u_0, v)$ for $k \geq 1$. We use $G^+(u_k - u_0, v)$ and $G^-(u_k - u_0, v)$ to denote the PCAFs corresponding to $\tilde{\mu}^{(n),+}_{\langle v,(u_k-u_0)^*\rangle}$ and $\tilde{\mu}^{(n),-}_{\langle v,(u_k-u_0)^*\rangle}$, respectively. Then,

$$\begin{aligned}
&E_\nu^{V_n}\left[\sup_{0\leq s\leq t} |G(u_k, v)_s - G(u_0, v)_s|\right] \\
&\quad = E_\nu^{V_n}\left[\sup_{0\leq s\leq t} |G(u_k - u_0, v)_s|\right] \\
&\quad \leq E_\nu^{V_n}\left[\sup_{0\leq s\leq t} G^+(u_k - u_0, v)_s\right] + E_\nu^{V_n}\left[\sup_{0\leq s\leq t} G^-(u_k - u_0, v)_s\right] \\
&\quad = E_\nu^{V_n}[G^+(u_k - u_0, v)_t] + E_\nu^{V_n}[G^-(u_k - u_0, v)_t].
\end{aligned}$$

Therefore, by [13, Lemma A.9], we find that there exists a constant $C_\nu > 0$ (independent of k) such that

$$\begin{aligned}
&E_\nu^{V_n}\left[\sup_{0\leq s\leq t} |G(u_k, v)_s - G(u_0, v)_s|\right] \\
&\quad \leq C_\nu(1+t)\int_{V_n} \tilde{\bar{h}}_n d|\tilde{\mu}^{(n)}_{\langle v,(u_k-u_0)^*\rangle}| \\
&\quad \leq C_\nu(1+t)\left(\int_{V_n} \tilde{\bar{h}}_n^{\,2} d\tilde{\mu}^{(n)}_{\langle v\rangle}\right)^{\frac{1}{2}}\left(\int_{V_n} d\tilde{\mu}^{(n)}_{\langle (u_k-u_0)^*\rangle}\right)^{\frac{1}{2}} \\
&\quad \leq 2C_\nu(1+t)\|\tilde{\bar{h}}_n\|_\infty(\eta^{(n)}(v, v))^{\frac{1}{2}}(\eta^{(n)}((u_k - u_0)^*, (u_k - u_0)^*))^{\frac{1}{2}},
\end{aligned}$$

which converges to 0 as $k \to \infty$. The proof is completed by the same method used in the proof of [7, Lemma 5.1.2] (cf. [19, Theorem 2.3.8]). □

Similar to [2, Proposition 2.6 and Corollary 3.2], we can prove the following two propositions.

PROPOSITION 2.9. *Let $u, v \in D(\mathcal{E})_{V_n,b}$. Then*

$$\int_0^t \tilde{v}(X_s^{V_n})dN_s^{n,[u]} + \int_0^t \tilde{u}(X_s^{V_n})dN_s^{n,[v]} = N_t^{n,[uv]} - \langle M^{n,[u]}, M^{n,[v]}\rangle_t, \quad t \geq 0,$$

$P_x^{V_n}$*-a.s. for* $\mathcal{E}$*-q.e.* $x \in V_n$.

PROPOSITION 2.10. *Let* $u \in D(\mathcal{E})_{V_n,b}$ *and* $\{v_k\}_{k=0}^{\infty} \subset D(\mathcal{E})_{V_n,b}$ *such that* v_k *converges to* v_0 *with respect to the* $\|\cdot\|_\infty$*-norm and the* $\tilde{\mathcal{E}}_1^{1/2}$*-norm as* $k \to \infty$. *Then there exists a subsequence* $\{k'\}$ *such that for* $\mathcal{E}$*-q.e.* $x \in V_n$,

$$P_x^{V_n}\left(\lim_{k'\to\infty}\int_0^t \widetilde{v_{k'}}(X_s^{V_n})dN_s^{n,[u]} = \int_0^t \widetilde{v}_0(X_s^{V_n})dN_s^{n,[u]} \text{ uniformly on any finite interval of } t\right) = 1\,.$$

DEFINITION 2.11. Let $u \in D(\mathcal{E})_{V_n}$ and $v \in D(\mathcal{E})_{V_n,b}$. We define for $0 \le t < \zeta$,

$$\int_0^t \tilde{v}(X_{s-}^{V_n})dA_s^{n,[u]} := \int_0^t \tilde{v}(X_{s-}^{V_n})dM_s^{n,[u]} + \int_0^t \tilde{v}(X_{s-}^{V_n})dN_s^{n,[u]}\,.$$

Finally, by virtue of [17, Theorem 3.1] and similar to [2, Theorem 3.4], we can prove the following result.

THEOREM 2.12. (i) *Let* $u, v \in D(\mathcal{E})_{V_n,b}$. *Then,*

$$\begin{aligned}\tilde{u}\tilde{v}(X_t^{V_n}) - \tilde{u}\tilde{v}(X_0^{V_n}) &= \int_0^t \tilde{v}(X_{s-}^{V_n})dA^{n,[u]}(X_s^{V_n}) + \int_0^t \tilde{u}(X_{s-}^{V_n})dA^{n,[v]}(X_s^{V_n}) \\ &\quad + \langle M^{n,[u],c}, M^{n,[v],c}\rangle_t \\ &\quad + \sum_{0<s\le t}[\Delta(uv)(X_s^{V_n}) - \tilde{v}(X_{s-}^{V_n})\Delta u(X_s^{V_n}) - \tilde{u}(X_{s-}^{V_n})\Delta v(X_s^{V_n})]\end{aligned} \tag{21}$$

on $[0,\zeta)$ $P_x^{V_n}$*-a.s. for* $\mathcal{E}$*-q.e.* $x \in V_n$.

(ii) *Let* $\Phi \in C^2(\mathbb{R}^n)$ *and* $u_1, \ldots, u_n \in D(\mathcal{E})_{V_n,b}$. *Then,*

$$\begin{aligned}\Phi(\tilde{u})(X_t^{V_n}) - \Phi(\tilde{u})(X_0^{V_n}) &= \sum_{i=1}^n \int_0^t \Phi_i(\tilde{u}(X_{s-}^{V_n}))dA_s^{n,[u_i]} \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \Phi_{ij}(\tilde{u}(X_s^{V_n}))d\langle M^{n,[u_i],c}, M^{n,[u_j],c}\rangle_s \\ &\quad + \sum_{0<s\le t}\left[\Delta\Phi(\tilde{u}(X_s^{V_n})) - \sum_{i=1}^n \Phi_i(\tilde{u}(X_{s-}^{V_n}))\Delta u_i(X_s^{V_n})\right]\end{aligned}$$

on $[0,\zeta)$ $P_x^{V_n}$*-a.s. for* $\mathcal{E}$*-q.e.* $x \in V_n$, *where*

$$\Phi_i(x) = \frac{\partial\Phi}{\partial x_i}(x), \quad \Phi_{ij}(x) = \frac{\partial^2\Phi}{\partial x_i \partial x_j}(x), \quad i, j = 1, \ldots, n\,,$$

and $u = (u_1, \ldots, u_n)$.

2.2. Stochastic integral for M. In this subsection, for $u, v \in D(\mathcal{E})_{loc}$, we will define the stochastic integral $\int_0^t \tilde{v}(X_{s-})dA_s^{[u]}$. To this end, we first choose a $\{V_n\} \in \Theta$ such that Assumption 2.1 is satisfied and $\widetilde{\hat{h}}$ is bounded on each V_n. Then, we choose $\{E_n\} \in \Theta$ and $\{u_n, v_n\}$ such that $E_n \subset V_n$, $u_n, v_n \in D(\mathcal{E})_{V_n,b}$, $u = u_n$ and $v = v_n$ on E_n for each $n \in \mathbb{N}$. The existence of $\{E_n\}$ and $\{u_n, v_n\}$ is justified by the argument before Lemma 1.4. Now we define $\int_0^t \tilde{v}(X_{s-})dA_s^{[u]}$ by

$$\int_0^t \tilde{v}(X_{s-})dA_s^{[u]} := \lim_{n\to\infty} \int_0^t \widetilde{v}_n(X_{s-}^{V_n})dA_s^{n,[u_n]}, \quad 0 \le t < \zeta\,, \tag{22}$$

where the stochastic integral $\int_0^t \widetilde{v}_n(X_{s-}^{V_n})dA_s^{n,[u_n]}$ is defined as in Definition 2.11.

THEOREM 2.13. *For $u, v \in D(\mathcal{E})_{loc}$, the stochastic integral in* (22) *is well-defined. Moreover, if $u, u', v, v' \in D(\mathcal{E})_{loc}$ satisfy $u = u'$ and $v = v'$ on U for some finely open set U, then*

$$\int_0^t \tilde{v}(X_{s-})dA_s^{[u]} = \int_0^t \widetilde{v}'(X_{s-})dA_s^{[u']}\,, \tag{23}$$

for $0 \le t < \tau_U$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$.

PROOF. First, we fix a $\{V_n\} \in \Theta$ such that Assumption 2.1 is satisfied and $\widetilde{\hat{h}}$ is bounded on each V_n. Suppose that there are two finely open sets F_k, F_l satisfying $F_k \subset V_k$, $F_l \subset V_l$, $k < l$; $f_k, g_k \in D(\mathcal{E})_{V_k,b}$, $u = f_k$, $v = g_k$ on F_k; $f_l, g_l \in D(\mathcal{E})_{V_l,b}$, $u = f_l$, $v = g_l$ on F_l. Below we will show that

$$\int_0^t \widetilde{g}_k(X_{s-}^{V_k})dA_s^{k,[f_k]} = \int_0^t \widetilde{g}_l(X_{s-}^{V_l})dA_s^{l,[f_l]}\,, \tag{24}$$

for $0 \le t < \tau_{F_k\cap F_l}$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_k$.

In fact, by approximating f_l by a sequence of functions $\{f_l^r\}$ in $D(L^{V_l})$, we obtain by Proposition 2.7 and Theorem 2.8 that

$$\begin{aligned}
\int_0^t \widetilde{g}_k(X_{s-}^{V_l})dA_s^{l,[f_l]} &= \int_0^t \widetilde{g}_k(X_{s-}^{V_l})dM_s^{l,[f_l]} + \int_0^t \widetilde{g}_k(X_{s-}^{V_l})dN_s^{l,[f_l]} \\
&= \int_0^t \widetilde{g}_l(X_{s-}^{V_l})dM_s^{l,[f_l]} + \lim_{r\to\infty} \int_0^t \widetilde{g}_k(X_{s-}^{V_l})dN_s^{l,[f_l^r]} \\
&= \int_0^t \widetilde{g}_l(X_{s-}^{V_l})dM_s^{l,[f_l]} + \lim_{r\to\infty} \int_0^t \widetilde{g}_l(X_{s-}^{V_l})dN_s^{l,[f_l^r]} \\
&= \int_0^t \widetilde{g}_l(X_{s-}^{V_l})dA_s^{l,[f_l]}, \quad 0 \le t < \tau_{F_k\cap F_l}\,,
\end{aligned} \tag{25}$$

$P_x^{V_l}$-a.s. for $\mathcal{E}$-q.e. $x \in V_l$. Since $A_{t\wedge\tau_{V_l}}^{l,[f_l]} \in \mathcal{F}_{t\wedge\tau_{V_l}-}^{V_l}$, (24) holds P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_l$.

Further, we obtain by the integration by parts (21) that

$$\int_0^t \widetilde{g}_k(X_{s-}^{V_l})dA_s^{l,[f_k]} = \int_0^t \widetilde{g}_k(X_{s-}^{V_l})dA_s^{l,[f_l]}\,, \tag{26}$$

for $0 \le t < \tau_{F_k \cap F_l}$, $P_x^{V_l}$-a.s. and hence P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_l$. Note that $M_{t\wedge\tau_{F_k}}^{k,[f_k]} = M_{t\wedge\tau_{F_k}}^{l,[f_k]}$ and $N_{t\wedge\tau_{F_k}}^{k,[f_k]} = N_{t\wedge\tau_{F_k}}^{l,[f_k]}$ $P_x^{V_k}$-a.s. for $\mathcal{E}$-q.e. $x \in V_k$ (cf. the proof of [17, Lemma 1.14]). By approximating f_k by a sequence of functions in $D(L^{V_k})$, Proposition 2.7 and Theorem 2.8, we get

$$\int_0^t \widetilde{g}_k(X_{s-}^{V_k})dA_s^{k,[f_k]} = \int_0^t \widetilde{g}_k(X_{s-}^{V_l})dA_s^{l,[f_k]}, \quad 0 \le t < \tau_{F_k}\,, \tag{27}$$

$P_x^{V_k}$-a.s. and hence P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_k$. Therefore, (24) holds for $0 \le t < \tau_{F_k \cap F_l}$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_k$ by (25)-(27).

Now we suppose that (22) is defined by a different $\{V_n\} \in \Theta$, say $\{V_n'\} \in \Theta$. By considering $\{V_n \cap V_n'\}$, [17, Proposition 2.4] and the above argument, we find that the two limits in (22) are equal on $[0, \zeta)$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. Therefore, (22) is well-defined.

From (24) and its proof, we find that if $u, u', v, v' \in D(\mathcal{E})_{loc}$ satisfy $u = u'$ and $v = v'$ on U for some finely open set U, then there exists an $\{E_n\} \in \Theta$ such that (23) holds on $\bigcup_n [0, \tau_{E_n \cap U})$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. By [17, Proposition 2.4], this implies that (23) holds for $0 \le t < \tau_U$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$. The proof is complete. □

From the proof of Theorem 1.2, we find that $M^{[u],c}$ is well defined whenever $u \in D(\mathcal{E})_{loc}$. Therefore, we obtain by Theorem 2.12 and (23) the following theorem.

THEOREM 2.14. *Let $\Phi \in C^2(\mathbb{R}^n)$ and $u_1, \dots, u_n \in D(\mathcal{E})_{loc}$. Then,*

$$\begin{aligned} A_t^{[\Phi(u)]} = &\sum_{i=1}^n \int_0^t \Phi_i(\tilde{u}(X_{s-}))dA_s^{[u_i]} + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \Phi_{ij}(\tilde{u}(X_s))d\langle M^{[u_i],c}, M^{[u_j],c}\rangle_s \\ &+ \sum_{0<s\le t}\left[\Delta\Phi(\tilde{u}(X_s)) - \sum_{i=1}^n \Phi_i(\tilde{u}(X_{s-}))\Delta u_i(X_s)\right] \end{aligned} \tag{28}$$

on $[0, \zeta)$ P_x-a.s. for $\mathcal{E}$-q.e. $x \in E$, where

$$\Phi_i(x) = \frac{\partial \Phi}{\partial x_i}(x), \quad \Phi_{ij}(x) = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x), \quad i, j = 1, \dots, n\,,$$

and $u = (u_1, \dots, u_n)$.

3. Some Examples. In this section, we give concrete examples for which all results of the previous two sections can be applied.

First, we consider a local semi-Dirichlet form.

EXAMPLE 3.1 (see [21]). Let $d \ge 3$, U be an open subset of $\mathbb{R}^d$, $\sigma, \rho \in L^1_{loc}(U; dx)$, $\sigma, \rho > 0$ dx-*a.e.* For $u, v \in C_0^\infty(U)$, we define

$$\mathcal{E}_\rho(u, v) = \sum_{i,j=1}^d \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\rho dx\,.$$

Assume that

$$(\mathcal{E}_\rho, C_0^\infty(U)) \text{ is closable on } L^2(U; \sigma dx).$$

Let $a_{ij}, b_i, d_i, c \in L^1_{loc}(U; dx)$, $1 \le i, j \le d$. For $u, v \in C_0^\infty(U)$, we define

$$\mathcal{E}(u,v) = \sum_{i,j=1}^d \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij} dx + \sum_{i=1}^d \int_U \frac{\partial u}{\partial x_i} v b_i dx + \sum_{i=1}^d \int_U u \frac{\partial v}{\partial x_i} d_i dx + \int_U uvc dx .$$

Set $\tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$, $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$, $\underline{b} := (b_1, \ldots, b_d)$, and $\underline{d} := (d_1, \ldots, d_d)$. Define F to be the set of all functions $g \in L^1_{loc}(U; dx)$ such that the distributional derivatives $\frac{\partial g}{\partial x_i}$, $1 \le i \le d$, are in $L^1_{loc}(U; dx)$ such that $\|\nabla g\|(g\sigma)^{-\frac{1}{2}} \in L^\infty(U; dx)$ or $\|\nabla g\|^p (g^{p+1}\sigma^{p/q})^{-\frac{1}{2}} \in L^d(U; dx)$ for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p < \infty$, where $\|\cdot\|$ denotes Euclidean distance in $\mathbb{R}^d$. We say that a $\mathcal{B}(U)$-measurable function f has property $(A_{\rho,\sigma})$ if one of the following conditions holds:

(i) $f(\rho\sigma)^{-\frac{1}{2}} \in L^\infty(U; dx)$.

(ii) $f^p(\rho^{p+1}\sigma^{p/q})^{-\frac{1}{2}} \in L^d(U, dx)$ for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p < \infty$, and $\rho \in F$.

Suppose that

(A.I) There exists $\eta > 0$ such that $\sum_{i,j=1}^d \tilde{a}_{ij}\xi_i\xi_j \ge \eta|\underline{\xi}|^2$, $\forall \underline{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.

(A.II) $\check{a}_{ij}\rho^{-1} \in L^\infty(U; dx)$ for $1 \le i, j \le d$.

(A.III) For all $K \subset U$, K compact, $1_K\|\underline{b} + \underline{d}\|$ and $1_K c^{1/2}$ have property $(A_{\rho,\sigma})$, and $(c + \alpha_0\sigma)dx - \sum_{i=1}^d \frac{\partial d_i}{\partial x_i}$ is a positive measure on $\mathcal{B}(U)$ for some $\alpha_0 \in (0, \infty)$.

(A.IV) $||\underline{b} - \underline{d}||$ has property $(A_{\rho,\sigma})$.

(A.V) $\underline{b} = \underline{\beta} + \underline{\gamma}$ such that $\|\underline{\beta}\|, \|\underline{\gamma}\| \in L^1_{loc}(U, dx)$, $(\alpha_0\sigma + c)dx - \sum_{i=1}^d \frac{\partial \gamma_i}{\partial x_i}$ is a positive measure on $\mathcal{B}(U)$ and $\|\underline{\beta}\|$ has property $(A_{\rho,\sigma})$.

Then, by [21, Theorem 1.2], there exists $\alpha > 0$ such that $(\mathcal{E}_\alpha, C_0^\infty(U))$ is closable on $L^2(U; dx)$ and its closure $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a regular local semi-Dirichlet form on $L^2(U; dx)$. Define $\eta_\alpha(u,u) := \mathcal{E}_\alpha(u,u) - \int \langle \nabla u, \underline{\beta} \rangle u dx$ for $u \in D(\mathcal{E}_\alpha)$. By [21, Theorem 1.2 (ii) and (1.28)], we know $(\eta_\alpha, D(\mathcal{E})_\alpha)$ is a Dirichlet form and there exists $C > 1$ such that for any $u \in D(\mathcal{E}_\alpha)$,

$$\frac{1}{C}\eta_\alpha(u,u) \le \mathcal{E}_\alpha(u,u) \le C\eta_\alpha(u,u).$$

Let $\mathbf{M}$ be the diffusion process associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$. For $u \in D(\mathcal{E}_\alpha)_{loc}$, we have the decomposition (15) and Itô's formula (28).

Next we consider a semi-Dirichlet form of pure jump type.

EXAMPLE 3.2 (See [8] and cf. also [22]). Let (E, d) be a locally compact separable metric space, m be a positive Radon Measure on E with full topological support, and $k(x, y)$ be a nonnegative Borel measurable function on $\{(x, y) \in E \times E \mid x \neq y\}$. Set $k_s(x, y) = \frac{1}{2}(k(x, y) + k(y, x))$ and $k_a = \frac{1}{2}(k(x, y) - k(y, x))$. Denote by $C_0^{lip}(E)$ the family of all uniformly Lipschitz continuous functions on E with compact support. Suppose that the following conditions hold:

(B.I) $x \mapsto \int_{y \neq x}(1 \wedge d(x, y)^2)k_s(x, y)m(dy) \in L^1_{loc}(E; dx)$.

(B.II) $\sup_{x \in E} \int_{\{y:\, k_s(x,y) \neq 0\}} \frac{k_a^2(x,y)}{k_s(x,y)} m(dy) < \infty$.

Define for $u, v \in C_0^{lip}(E)$,

$$\eta(u, v) = \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y))k_s(x, y)m(dx)m(dy)\,,$$

and

$$\mathcal{E}(u, v) = \frac{1}{2}\eta(u, v) + \iint_{x \neq y} (u(x) - u(y))v(y)k_a(x, y)m(dx)m(dy)\,.$$

Then, there exists $\alpha > 0$ such that $(\mathcal{E}_\alpha, C_0^{lip}(E))$ is closable on $L^2(E; dx)$ and its closure $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a regular semi-Dirichlet form on $L^2(E, dx)$. Moreover, there exists $C > 1$ such that for any $u \in D(\mathcal{E}_\alpha)$,

$$\frac{1}{C}\eta_\alpha(u, u) \leq \mathcal{E}_\alpha(u, u) \leq C\eta_\alpha(u, u)\,.$$

Let $\mathbf{M}$ be the pure jump process associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$. For $u \in D(\mathcal{E}_\alpha)_{loc}$, we have the decomposition (7) and Itô's formula (28).

Finally, we consider a general semi-Dirichlet form with diffusion, jumping and killing terms.

EXAMPLE 3.3 (See [23]). Let G be an open set of $\mathbb{R}^d$. Suppose that the following conditions hold:

(C.I) There exist $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \ \text{ for } x \in G,\ \xi \in \mathbb{R}^d\,.$$

(C.II) $b_i \in L^d(G; dx)$, $i = 1, \ldots, d$.

(C.III) $c \in L_+^{d/2}(G; dx)$.

(C.IV) $x \mapsto \int_{y \neq x}(1 \wedge |x - y|^2)k_s(x, y)dy \in L^1_{loc}(G; dx)$.

(C.V) $\sup_{x \in G} \int_{\{|x-y| \geq 1, y \in G\}} |k_a(x, y)|dy < \infty$, $\sup_{x \in G} \int_{\{|x-y| < 1, y \in G\}} |k_a(x, y)|^\gamma dy < \infty$ for some $0 < \gamma \leq 1$, and $|k_a(x, y)|^{2-\gamma} \leq C_1 k_s(x, y)$, $x, y \in G$ with $0 < |x - y| < 1$ for some constant $C_1 > 0$.

Define for $u, v \in C_0^1(G)$,

$$\eta(u,v) = \frac{1}{2}\sum_{i=1}^{d}\int_G \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)dx + \frac{1}{2}\iint_{x\neq y}(u(x)-u(y))(v(x)-v(y))k_s(x,y)dxdy$$

and

$$\begin{aligned}\mathcal{E}(u,v) =& \frac{1}{2}\sum_{i,j=1}^{d}\int_G a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)dx + \sum_{i=1}^{d}\int_G b_i(x)u(x)\frac{\partial v}{\partial x_i}(x)dx \\ &+ \int_G u(x)v(x)c(x)dx \\ &+ \frac{1}{2}\iint_{x\neq y}(u(x)-u(y))(v(x)-v(y))k_s(x,y)dxdy \\ &+ \iint_{x\neq y}(u(x)-u(y))v(x)k_a(x,y)dxdy\,.\end{aligned}$$

Then, when λ is sufficiently large, there exists $\alpha > 0$ such that $(\mathcal{E}_\alpha, C_0^1(G))$ is closable on $L^2(G;dx)$ and its closure $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a regular semi-Dirichlet form on $L^2(G;dx)$. Moreover, there exists $C' > 1$ such that for any $u \in D(\mathcal{E}_\alpha)$,

$$\frac{1}{C'}\eta_\alpha(u,u) \le \mathcal{E}_\alpha(u,u) \le C'\eta_\alpha(u,u)\,.$$

Let $\mathbf{M}$ be the Markov process associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$. For $u \in D(\mathcal{E}_\alpha)_{loc}$, we have the decomposition (7) and Itô's formula (28).

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