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BOUNDEDNESS OF THE MAXIMAL OPERATOR ON MUSIELAK-ORLICZ-MORREY SPACES

Dedicated to Professor Yoshihiro Mizuta on the occasion of his seventieth birthday

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Abstract. We give the boundedness of the maximal operator on Musielak-Orlicz-Morrey spaces, which is an improvement of [7, Theorem 4.1]. We also discuss the sharpness of our conditions.

1. Introduction. For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, its maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy\,,$$

where $B(x,r)$ is the ball in $\mathbf{R}^N$ with center x and of radius $r > 0$ and $|B(x,r)|$ denotes its Lebesgue measure. The mapping $f \mapsto Mf$ is called the maximal operator. When f is a function on an open set Ω in $\mathbf{R}^N$, we define Mf by extending f to be zero outside Ω.

The classical result that M is a bounded operator on $L^p(\mathbf{R}^N)$ for $p > 1$ has been extended to various function spaces. Boundedness of the maximal operator on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was investigated in [3] and [4]. Variable exponent Lebesgue spaces are special cases of the Musielak-Orlicz spaces, which were first considered by H. Nakano as modulared function spaces in [12] and then developed by J. Musielak as generalized Orlicz spaces in [9]. The boundedness of the maximal operator was also studied for variable exponent Morrey spaces (see [1, 8]). All the above spaces are special cases of the so-called Musielak-Orlicz-Morrey spaces

In [7, Theorem 4.1], we established the boundedness of the maximal operator M on Musielak-Orlicz-Morrey spaces $L^{\Phi,\kappa}(\mathbf{R}^N)$ defined by general functions $\Phi(x,t)$ and $\kappa(x,r)$ satisfying certain conditions. Our aim in this paper is to give its improvement by relaxing assumptions on $\Phi(x,t)$ (Theorem 7). In fact, we shall show our result by assuming ($\Phi5$; ν) and ($\Phi6$; ω) below instead of ($\Phi5$) and ($\Phi6$) in [7]. Further, the result is proved without the doubling condition on $\Phi(x,\cdot)$ which is ($\Phi4$) in [7]. As a result, we can include a variety of examples of $\Phi(x,t)$ to which our theory applies; in particular, non-doubling functions $\Phi(x,t)$ as in Examples 2–5. See also Hästö [6]. His conditions for the boundedness of the maximal operator on Musielak-Orlicz spaces are different from ours.

In the final section, we discuss the sharpness of the conditions ($\Phi5$; ν) and ($\Phi6$; ω).

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2. Preliminaries. Let Ω be an open set in $\mathbf{R}^N$ and consider a function

$$\Phi(x,t) : \Omega \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions ($\Phi 1$)–($\Phi 4$):

($\Phi 1$) $\Phi(\,\cdot\,,t)$ is measurable on Ω for each $t \ge 0$ and $\Phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in \Omega$;

($\Phi 2$) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \Phi(x,1) \le A_1 \quad \text{for all } x \in \Omega;$$

($\Phi 3$) $t \mapsto \Phi(x,t)/t$ is uniformly almost increasing on $(0,\infty)$, namely there exists a constant $A_2 \ge 1$ such that

$$\Phi(x,t_1)/t_1 \le A_2 \Phi(x,t_2)/t_2 \quad \text{for all } x \in \Omega \text{ whenever } 0 < t_1 < t_2 .$$

Note that ($\Phi 2$) and ($\Phi 3$) imply

$$\Phi(x,t) \le A_1 A_2 t \quad \text{for } 0 \le t \le 1 \quad \text{and} \quad \Phi(x,t) \ge (A_1 A_2)^{-1} t \quad \text{for } t \ge 1. \tag{1}$$

Let $\bar{\phi}(x,t) = \sup_{0<s\le t} \Phi(x,s)/s$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in \Omega$ and $t \ge 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\Phi(x,t/2) \le \overline{\Phi}(x,t) \le A_2 \Phi(x,t) \tag{2}$$

for all $x \in \Omega$ and $t \ge 0$.

We also consider a function $\kappa(x,r) : \Omega \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

($\kappa 1$) there is a constant $Q_1 \ge 1$ such that

$$\kappa(x,2r) \le Q_1 \kappa(x,r)$$

for all $x \in \Omega$ and $r > 0$;

($\kappa 2$) $r \mapsto r^{-\varepsilon}\kappa(x,r)$ is uniformly almost increasing on $(0,\infty)$ for some $\varepsilon > 0$, namely there exists a constant $Q_2 \ge 1$ such that

$$r^{-\varepsilon}\kappa(x,r) \le Q_2 s^{-\varepsilon}\kappa(x,s)$$

for all $x \in \Omega$ whenever $0 < r < s$;

($\kappa 3$) there is a constant $Q_3 \ge 1$ such that

$$Q_3^{-1}\min(1,\ r^N) \le \kappa(x,r) \le Q_3 \max(1,\ r^N)$$

for all $x \in \Omega$ and $r > 0$.

Given $\Phi(x,t)$ and $\kappa(x,r)$ as above, the Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(\Omega)$ is defined by

$$L^{\Phi,\kappa}(\Omega) = \left\{ f \in L^1_{\mathrm{loc}}(\Omega) \colon \sup_{x\in\Omega, r>0} \frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)\cap\Omega} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) dy < \infty \text{ for some } \lambda > 0 \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\Phi,\kappa;\Omega} = \|f\|_{L^{\Phi,\kappa}(\Omega)} = \inf\left\{\lambda > 0 : \sup_{x\in\Omega, r>0} \frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)\cap\Omega} \overline{\Phi}\left(y, \frac{|f(y)|}{\lambda}\right) dy \le 1\right\}$$

(cf. [11]).

In case $\kappa(x,r) = r^N$, $L^{\Phi,\kappa}(\Omega)$ is the Musielak-Orlicz space $L^{\Phi}(\Omega)$ (cf. [9]).

We shall also consider the following conditions for $\Phi(x,t)$: Let $p \ge 1$, $q \ge 1$, $\nu > 0$ and $\omega > 0$.

($\Phi3; 0; p$) $t \mapsto t^{-p}\Phi(x,t)$ is uniformly almost increasing on $(0,1]$, namely there exists a constant $A_{2,0,p} \ge 1$ such that

$$t_1^{-p}\Phi(x,t_1) \le A_{2,0,p}\, t_2^{-p}\, \Phi(x,t_2) \quad \text{for all } x \in \Omega \text{ whenever } 0 < t_1 < t_2 \le 1;$$

($\Phi3; \infty; q$) $t \mapsto t^{-q}\Phi(x,t)$ is uniformly almost increasing on $[1,\infty)$, namely there exists a constant $A_{2,\infty,q} \ge 1$ such that

$$t_1^{-q}\Phi(x,t_1) \le A_{2,\infty,q}\, t_2^{-q}\Phi(x,t_2) \quad \text{for all } x \in \Omega \text{ whenever } 1 \le t_1 < t_2;$$

($\Phi5; \nu$) for every $\gamma > 0$, there exists a constant $B_{\gamma,\nu} \ge 1$ such that

$$\Phi(x,t) \le B_{\gamma,\nu}\Phi(y,t)$$

whenever $x, y \in \Omega$, $|x-y| \le \gamma t^{-\nu}$ and $t \ge 1$;

($\Phi6; \omega$) there exist a function g on Ω and a constant $B_\infty \ge 1$ such that $0 \le g(x) \le 1$ for all $x \in \Omega$, $g \in L^{\omega}(\Omega)$ and

$$B_\infty^{-1}\Phi(x,t) \le \Phi(x',t) \le B_\infty\Phi(x,t)$$

whenever $x, x' \in \Omega$, $|x'| \ge |x|$ and $g(x) \le t \le 1$.

Note that ($\Phi3; 0; 1$) + ($\Phi3; \infty; 1$) = ($\Phi3$). If $\Phi(x,t)$ satisfies ($\Phi3; 0; p$), then it satisfies ($\Phi3; 0; p'$) for $1 \le p' \le p$; if $\Phi(x,t)$ satisfies ($\Phi3; \infty; q$), then it satisfies ($\Phi3; \infty; q'$) for $1 \le q' \le q$.

If $\Phi(x,t)$ satisfies ($\Phi3; 0; p$), then

$$\Phi(x,t) \le A_1 A_{2,0,p}\, t^p \quad \text{for } 0 \le t \le 1; \tag{3}$$

if $\Phi(x,t)$ satisfies ($\Phi3; \infty; q$), then

$$\Phi(x,t) \ge (A_1 A_{2,\infty,q})^{-1}\, t^q \quad \text{for } t \ge 1.$$

If $\Phi(x,t)$ satisfies ($\Phi5; \nu$), then it satisfies ($\Phi5; \nu'$) for all $\nu' \ge \nu$; if $\Phi(x,t)$ satisfies ($\Phi6; \omega$), then it satisfies ($\Phi6; \omega'$) for all $\omega' \ge \omega$.

REMARK 1. In view of (Φ2), if $|\Omega| < \infty$, then ($\Phi 6$; ω) is automatically satisfied for every $\omega > 0$ with $g(x) \equiv 1$.

In the following examples, let

$$f^- := \inf_{x \in \Omega} f(x) \quad \text{and} \quad f^+ := \sup_{x \in \Omega} f(x)$$

for a measurable function f on Ω.

EXAMPLE 2. Let $p_i(\cdot)$, $i = 1, 2$ and $q_{i,j}(\cdot)$, $j = 1, \ldots, k_i$, be real valued measurable functions on Ω such that $p_i^- > 1$ and $q_{i,j}^- > -\infty$, $i = 1, 2$, $j = 1, \ldots, k_i$.

Set $L_c(t) = \log(c + t)$ for $c > 1$ and $t \geq 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$. Let

$$\Phi(x, t) = \begin{cases} t^{p_1(x)} \prod_{j=1}^{k_1} (L_{e-1}^{(j)}(1/t))^{-q_{1,j}(x)} & \text{if } 0 \leq t \leq 1\,; \\ t^{p_2(x)} \prod_{j=1}^{k_2} (L_{e-1}^{(j)}(t))^{q_{2,j}(x)} & \text{if } t \geq 1\,. \end{cases}$$

Then, $\Phi(x, t)$ satisfies ($\Phi 1$), ($\Phi 2$) and ($\Phi 3$). It satisfies ($\Phi 3$; 0; p) for $1 \leq p < p_1^-$ in general and for $1 \leq p \leq p_1^-$ in case $q_{1,j}^- \geq 0$ for all $j = 1, \ldots, k_1$; it satisfies ($\Phi 3$; ∞; q) for $1 \leq q < p_2^-$ in general and for $1 \leq q \leq p_2^-$ in case $q_{2,j}^- \geq 0$ for all $j = 1, \ldots, k_2$.

Moreover, we see that $\Phi(x, t)$ satisfies ($\Phi 5$; ν) for every $\nu > 0$ if $p_2(\cdot)$ is log-Hölder continuous, namely

$$|p_2(x) - p_2(y)| \leq \frac{C_p}{L_e(1/|x - y|)} \quad (x,\ y \in \Omega)$$

with a constant $C_p \geq 0$ and $q_{2,j}(\cdot)$ is $(j + 1)$-log-Hölder continuous, namely

$$|q_{2,j}(x) - q_{2,j}(y)| \leq \frac{C_j}{L_e^{(j+1)}(1/|x - y|)} \quad (x,\ y \in \Omega)$$

with constants $C_j \geq 0$ for each $j = 1, \ldots, k_2$.

Finally, we see that $\Phi(x, t)$ satisfies ($\Phi 6$; ω) for every $\omega > 0$ with $g(x) = (1 + |x|)^{-(N+1)/\omega}$ if $p_1(\cdot)$ is log-Hölder continuous at ∞, namely

$$|p_1(x) - p_1(x')| \leq \frac{C_{p,\infty}}{L_e(|x|)}$$

whenever $|x'| \geq |x|$ $(x,\ x' \in \Omega)$ with a constant $C_{p,\infty} \geq 0$, and $q_{1,j}(\cdot)$ is $(j + 1)$-log-Hölder continuous at ∞, namely

$$|q_{1,j}(x) - q_{1,j}(x')| \leq \frac{C'_j}{L_e^{(j+1)}(|x|)}$$

whenever $|x'| \geq |x|$ $(x,\ x' \in \Omega)$ with a constant $C'_j \geq 0$, for each $j = 1, \ldots, k_1$. In fact, if $(1 + |x|)^{-(N+1)/\omega} < t \leq 1$, then $t^{-|p_1(x) - p_1(x')|} \leq e^{(N+1)C_{p,\infty}/\omega}$ for $|x'| \geq |x|$ and $(L_{e-1}^{(j)}(1/t))^{|q_{1,j}(x) - q_{1,j}(x')|} \leq C(N, C'_j)$ for $|x'| \geq |x|$.

EXAMPLE 3. Let $p(\cdot)$ be a measurable function on Ω such that $0 < p^- \le p^+ < \infty$. Then,

$$\Phi(x,t) = e^{p(x)t} - p(x)t - 1$$

satisfies (Φ1), (Φ2) and (Φ3). It satisfies (Φ3; 0; p) and (Φ3; ∞; q) for $1 \le p \le 2$ and $q \ge 1$.

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5; ν) for every $\nu > 0$ if $p(\cdot) = \text{const.}$ and for $\nu \ge 1/\alpha$ if $p(\cdot)$ is α-Hölder continuous , namely

$$|p(x) - p(y)| \le C_\alpha |x-y|^\alpha$$

with a constant $C_\alpha \ge 0$ ($0 < \alpha \le 1$). In fact,

$$(1 - (1+p^-)e^{-p^-})e^{p(x)t} \le \Phi(x,t) \le e^{p(x)t}$$

for all $x \in \Omega$ and $t \ge 1$ and

$$|p(x)t - p(y)t| \le C_\alpha \gamma^\alpha$$

whenever $1 \le t \le (\gamma/|x-y|)^\alpha$.

Finally, we see that $\Phi(x,t)$ satisfies (Φ6; ω) for every $\omega > 0$ with $g(x) \equiv 0$, since

$$\frac{1}{2}(p(x)t)^2 \le \Phi(x,t) \le \frac{e^{p^+}}{2}(p(x)t)^2$$

for all $x \in \Omega$ and $0 \le t \le 1$.

EXAMPLE 4. Let $p(\cdot)$ be a real valued measurable function on Ω such that $p^- \ge 1$. Then,

$$\Phi(x,t) = e^t t^{p(x)}$$

satisfies (Φ1), (Φ2) and (Φ3). It satisfies (Φ3; 0; p) and (Φ3; ∞; q) for $1 \le p \le p^-$ and $q \ge 1$.

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5; ν) for $\nu > 0$ if $p(\cdot)$ is log-Hölder continuous, and $\Phi(x,t)$ satisfies (Φ6; ω) with $g(x) = (1+|x|)^{-(N+1)/\omega}$ for every $\omega > 0$ if $p(\cdot)$ is log-Hölder continuous at ∞.

EXAMPLE 5. Let $p(\cdot)$ be a measurable function on Ω such that $p^- \ge 1$ and $p^+ < \infty$. Then,

$$\Phi(x,t) = e^{t^{p(x)}} - 1$$

satisfies (Φ1), (Φ2) and (Φ3). It satisfies (Φ3; 0; p) and (Φ3; ∞; q) for $1 \le p \le p^-$ and $q \ge 1$.

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5; ν) for every $\nu > 0$ if $p(\cdot) = \text{const.}$ and for $\nu > p^+/\alpha$ if $p(\cdot)$ is α-Hölder continuous ($0 < \alpha \le 1$). In fact, there exists a constant $C > 1$ such that

$$C^{-1} e^{t^{p(x)}} \le \Phi(x,t) \le e^{t^{p(x)}}$$

for all $x \in \Omega$ and $t \geq 1$ and

$$|t^{p(x)} - t^{p(y)}| \leq |p(x) - p(y)|(\log t)t^{p^+} \leq C$$

whenever $1 \leq t \leq (\gamma/|x-y|)^{1/\nu}$ for $\nu > p^+/\alpha$.

Finally, since

$$t^{p(x)} \leq \Phi(x,t) \leq et^{p(x)}$$

for all $x \in \Omega$ and $0 \leq t \leq 1$, we see that $\Phi(x,t)$ satisfies $(\Phi 6; \omega)$ with $g(x) = (1 + |x|)^{-(N+1)/\omega}$ for every $\omega > 0$ if $p(\cdot)$ is log-Hölder continuous at ∞.

EXAMPLE 6. Let $\nu(\cdot)$ and $\beta(\cdot)$ be functions on Ω such that $\inf_{x \in \Omega} \nu(x) > 0$, $\sup_{x \in \Omega} \nu(x) \leq N$ and $-c(N - \nu(x)) \leq \beta(x) \leq c(N - \nu(x))$ for all $x \in \Omega$ and some constant $c > 0$. Then $\kappa(x,r) = r^{\nu(x)}(\log(e + r + 1/r))^{\beta(x)}$ satisfies $(\kappa 1)$, $(\kappa 2)$ and $(\kappa 3)$.

3. Boundedness of the maximal operator. Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \ldots)$ be a constant that depends on $a, b, \ldots$.

For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, its maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy\,.$$

When f is a function on Ω, we define Mf by extending f to be zero outside Ω. As the boundedness of the maximal operator M on $L^{\Phi,\kappa}(\Omega)$, we give the following theorem, which is an improvement of [7, Theorem 4.1]:

THEOREM 7. *Suppose that* $\Phi(x,t)$ *satisfies* $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \nu)$ *and* $(\Phi 6; \omega)$ *for* $p > 1$, $q > 1$, $\nu > 0$ *and* $\omega > 0$ *satisfying*

$$\nu < q/N \quad \text{and} \quad \omega \leq p\,. \tag{4}$$

Then the maximal operator M *is a bounded operator from* $L^{\Phi,\kappa}(\Omega)$ *into itself, namely* $Mf|_\Omega \in L^{\Phi,\kappa}(\Omega)$ *for all* $f \in L^{\Phi,\kappa}(\Omega)$ *and*

$$\|Mf\|_{\Phi,\kappa;\Omega} \leq C\|f\|_{\Phi,\kappa;\Omega}$$

with a constant $C > 0$ *depending only on* N *and constants appearing in conditions for* Φ.

In case $\kappa(x,r) = r^N$, we have the following corollary.

COROLLARY 8. *Suppose that* $\Phi(x,t)$ *satisfies* $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \nu)$ *and* $(\Phi 6; \omega)$ *for* $p > 1$, $q > 1$, $\nu > 0$ *and* $\omega > 0$ *satisfying* $\nu < q/N$ *and* $\omega \leq p$. *Then the maximal operator* M *is a bounded operator from* $L^{\Phi}(\Omega)$ *into itself.*

We prove this theorem by modifying the proof of [7, Theorem 4.1].

For a nonnegative measurable function f on Ω, $x \in \Omega$ and $r > 0$, let

$$I(f;x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} f(y)\, dy$$

and

$$J(f;x,r) = \frac{1}{|B(x,r)|}\int_{B(x,r)\cap\Omega} \Phi\big(y, f(y)\big)\,dy\,.$$

LEMMA 9. *Suppose $\Phi(x,t)$ satisfies ($\Phi 3; \infty; q_1$) and ($\Phi 5; \nu$) for $q_1 \geq 1$ and $\nu > 0$ satisfying $\nu \leq q_1/N$. Then, given $L \geq 1$, there exist constants $C_1 = C(L) \geq 2$ and $C_2 > 0$ such that*

$$\Phi\big(x, I(f;x,r)/C_1\big) \leq C_2 J(f;x,r)$$

for all $x \in \Omega$, $r > 0$ and for all nonnegative measurable function f on Ω such that $f(y) \geq 1$ or $f(y) = 0$ for each $y \in \Omega$ and

$$\sup_{x\in\Omega,\, r>0} \frac{\kappa(x,r)}{|B(x,r)|}\int_{B(x,r)\cap\Omega} \Phi(y, f(y))dy \leq L\,. \tag{5}$$

PROOF. Given f as in the statement of the lemma, $x \in \Omega$ and $r > 0$, set $I = I(f;x,r)$ and $J = J(f;x,r)$. Note that (5) implies

$$J \leq \kappa(x,r)^{-1} L\,. \tag{6}$$

By (1), $\Phi\big(y, f(y)\big) \geq (A_1A_2)^{-1} f(y)$ for all $y \in \Omega$. Hence $I \leq A_1A_2 J$. Thus, if $J \leq 1$, then by ($\Phi 3$)

$$\Phi(x, I/C_1) \leq A_2 J \Phi(x,1) \leq A_1A_2 J$$

whenever $C_1 \geq A_1A_2$.

Next, suppose $J > 1$. Since $\Phi(x,t) \to \infty$ as $t \to \infty$ by (1), there exists $K > 1$ such that

$$\Phi(x,K) = \Phi(x,1)J\,.$$

With this K, we have by ($\Phi 3$)

$$\begin{aligned}\int_{B(x,r)\cap\Omega} f(y)\,dy &\leq \int_{B(x,r)\cap\Omega\cap\{y: f(y)\leq K\}} f(y)\,dy + \int_{B(x,r)\cap\Omega\cap\{y: f(y)>K\}} f(y)\,dy\\ &\leq K|B(x,r)| + A_2 K \int_{B(x,r)\cap\Omega} \frac{\Phi\big(y, f(y)\big)}{\Phi(y,K)}\,dy\,.\end{aligned}$$

Since $K > 1$, by ($\Phi 3; \infty; q_1$) we have

$$\Phi(x,1)J = \Phi(x,K) \geq A_{2,\infty,q_1}^{-1} K^{q_1} \Phi(x,1)\,,$$

so that, in view of (6) and ($\kappa 3$),

$$\begin{aligned}K^{q_1} &\leq A_{2,\infty,q_1} J \leq A_{2,\infty,q_1}\kappa(x,r)^{-1} L\\ &\leq A_{2,\infty,q_1} Q_3 L \max(1, r^{-N})\,.\end{aligned}$$

Since $J > 1$, $\kappa(x,r) < L$ by (6). By ($\kappa 2$) and ($\kappa 3$), $\kappa(x,\rho) \geq (Q_2Q_3)^{-1}\rho^{\varepsilon}$ for $\rho \geq 1$, so that $\kappa(x,\rho) \geq L$ for $x \in \Omega$ if $\rho \geq R := (Q_2Q_3L)^{1/\varepsilon}$. Thus $r < R$. Since $R \geq 1$, it follows that $\max(1, r^{-N}) \le R^N r^{-N}$. Hence

$$K^{q_1} \le A_{2,\infty,q_1} Q_3 L R^N r^{-N}$$

or $r \le \gamma K^{-q_1/N}$ with $\gamma = (A_{2,\infty,q_1} Q_3 L)^{1/N} R$. Thus, if $|y-x| \le r$, then $|y-x| \le \gamma K^{-q_1/N} \le \gamma K^{-\nu}$. Hence, by ($\Phi5$; ν) there is $\beta \ge 1$, independent of f, x, r, such that

$$\Phi(x, K) \le \beta \Phi(y, K) \quad \text{for all } y \in B(x, r) \cap \Omega\,.$$

Thus, we have by ($\Phi2$) and ($\Phi3$)

$$\begin{aligned}\int_{B(x,r)\cap\Omega} f(y)dy &\le K|B(x,r)| + \frac{A_2 \beta K}{\Phi(x,K)} \int_{B(x,r)\cap\Omega} \Phi\big(y, f(y)\big)dy \\ &= K|B(x,r)| + A_2\beta K|B(x,r)| \frac{J}{\Phi(x,K)} \\ &= K|B(x,r)| \left(1 + \frac{A_2\beta}{\Phi(x,1)}\right) \le K|B(x,r)|\,(1 + A_1A_2\beta)\ .\end{aligned}$$

Therefore

$$I \le (1 + A_1 A_2 \beta) K\,,$$

so that by ($\Phi2$) and ($\Phi3$)

$$\Phi(x, I/C_1) \le A_2 \Phi(x, K) \le A_1 A_2 J$$

whenever $C_1 \ge 1 + A_1 A_2 \beta$. □

The next lemma can be shown in the same way as [7, Lemma 3.2]; note that the value of ω is irrelevant in this lemma.

LEMMA 10. *Suppose $\Phi(x,t)$ satisfies ($\Phi6$; ω) for some $\omega > 0$. Then there exists a constant $C_3 > 0$ such that*

$$\Phi\big(x, I(f; x, r)/2\big) \le C_3 \{J(f; x, r) + \Phi(x, g(x))\}$$

for all $x \in \Omega$, $r > 0$ and for all nonnegative measurable function f on Ω such that $g(y) \le f(y) \le 1$ or $f(y) = 0$ for each $y \in \Omega$, where g is the function appearing in ($\Phi6$; ω).

We use the following lemma which is the special case of the theorem when $\Phi(x,t) = t^{p_0}$ ($p_0 > 1$); this lemma can be proved in a way similar to the proof of [10, Theorem 1].

LEMMA 11. *Let $p_0 > 1$. Then there exists a constant $C > 0$ for which the following holds: If f is a measurable function such that*

$$\int_{B(x,r)\cap\Omega} |f(y)|^{p_0}\, dy \le |B(x,r)| \kappa(x,r)^{-1}$$

for all $x \in \Omega$ and $r > 0$, then

$$\int_{B(x,r)\cap\Omega} [Mf(y)]^{p_0}\, dy \le C|B(x,r)| \kappa(x,r)^{-1}$$

for all $x \in \Omega$ and $r > 0$.

PROOF OF THEOREM 7. Set $p_0 = \min(p, q, q/(N\nu))$. Then $p_0 > 1$. Consider the function

$$\Phi_0(x, t) = \Phi(x, t)^{1/p_0} .$$

Then $\Phi_0(x, t)$ satisfies the conditions (Φj), $j = 1, 2$, $(\Phi 5; \nu)$ and $(\Phi 6; \omega)$ with the same g.

Condition $(\Phi 3; 0; p)$ implies that $\Phi_0(x, t)$ satisfies $(\Phi 3; 0; p/p_0)$ and condition $(\Phi 3; \infty; q)$ implies that $\Phi_0(x, t)$ satisfies $(\Phi 3; \infty; q/p_0)$.

Let $f \geq 0$ and $\|f\|_{\Phi,\kappa;\Omega} \leq 1/2$. Let $f_1 = f\chi_{\{x: f(x) \geq 1\}}$, $f_2 = f\chi_{\{x: g(x) \leq f(x) < 1\}}$ with g in $(\Phi 6; \omega)$ and $f_3 = f - f_1 - f_2$, where χ_E is the characteristic function of E.

Since $\Phi(x, t) \geq (A_1 A_2)^{-1}$ for $t \geq 1$ by (1), in view of (2) we have

$$\Phi_0(x, t) \leq (A_1 A_2)^{1-1/p_0} \Phi(x, t) \leq (A_1 A_2)^{1-1/p_0} \overline{\Phi}(x, 2t)$$

if $t \geq 1$. Hence

$$\sup_{x \in \Omega,\, r > 0} \frac{\kappa(x, r)}{|B(x, r)|} \int_{B(x,r) \cap \Omega} \Phi_0(y, f_1(y))\, dy \leq (A_1 A_2)^{1-1/p_0} .$$

If we set $q_1 = q/p_0$, then $q_1 \geq 1$ and $\nu \leq q_1/N$. Hence applying Lemma 9 to Φ_0, f_1 and $L = (A_1 A_2)^{1-1/p_0}$, there exist constants $C_1 \geq 2$ and $C_2 > 0$ such that

$$\Phi_0\big(x, Mf_1(x)/C_1\big) \leq C_2 M[\Phi_0\big(\cdot, f_1(\cdot)\big)](x) ,$$

so that

$$\Phi\big(x, Mf_1(x)/C_1\big) \leq C_2^{p_0} \Big[M[\Phi_0\big(\cdot, f(\cdot)\big)](x)\Big]^{p_0} \tag{7}$$

for all $x \in \Omega$.

Next, applying Lemma 10 to Φ_0 and f_2, we have

$$\Phi_0(x, Mf_2(x)/C_1) \leq C \left[M[\Phi_0\big(\cdot, f_2(\cdot)\big)](x) + \Phi_0\big(x, g(x)\big)\right] .$$

Noting that $\Phi_0(x, g(x)) \leq C g(x)^{p/p_0}$ by (3), we have

$$\Phi\big(x, Mf_2(x)/C_1\big) \leq C \left\{\Big[M[\Phi_0\big(\cdot, f(\cdot)\big)](x)\Big]^{p_0} + g(x)^p\right\} \tag{8}$$

for all $x \in \Omega$ with a constant $C > 0$ independent of f.

Since $0 \leq f_3 \leq g \leq 1$, $0 \leq Mf_3 \leq Mg \leq 1$. Hence by (3) we have

$$\Phi\big(x, Mf_3(x)/C_1\big) \leq C[Mg(x)]^p \tag{9}$$

for all $x \in \Omega$ with a constant $C > 0$ independent of f.

Combining (7), (8) and (9), and noting that $g(x) \leq Mg(x)$ for a.e. $x \in \Omega$, we obtain

$$\Phi\big(x, Mf(x)/(3C_1)\big) \leq C \left\{\Big[M[\Phi_0\big(\cdot, f(\cdot)\big)](x)\Big]^{p_0} + [Mg(x)]^p\right\} \tag{10}$$

for a.e. $x \in \Omega$ with a constant $C > 0$ independent of f.

Since $\|f\|_{\Phi,\kappa;\Omega} \leq 1/2$, in view of (2), we have

$$\int_{B(x,r) \cap \Omega} \Phi_0(y, f(y))^{p_0}\, dy = \int_{B(x,r) \cap \Omega} \Phi(y, f(y))\, dy \leq |B(x, r)| \kappa(x, r)^{-1}$$

for all $x \in \Omega$ and $r > 0$. Hence, applying Lemma 11 to $\Phi_0(y, f(y))$, we have

$$\int_{B(x,r)\cap\Omega} \Big[M[\Phi_0\big(\cdot, f(\cdot)\big)](y)\Big]^{p_0} dy \le C|B(x,r)|\kappa(x,r)^{-1}$$

with a constant $C > 0$ independent of x, r and f. Here note from $g \in L^\infty(\Omega) \cap L^\omega(\Omega)$ for $\omega \le p$ that $g \in L^p(\Omega)$. Therefore, by ($\kappa 3$) we see that

$$\int_{B(x,r)\cap\Omega} g(y)^p\, dy \le \min(\|g\|^p_{L^p(\Omega)}, |B(x,r)|) \le C|B(x,r)|\kappa(x,r)^{-1}$$

with a constant $C = C(\|g\|^p_{L^p(\Omega)}) > 0$ independent of x and r. Hence, by Lemma 11 again,

$$\int_{B(x,r)\cap\Omega} [Mg(y)]^p\, dy \le C|B(x,r)|\kappa(x,r)^{-1}$$

for all $x \in \Omega$ and $r > 0$. Thus, by (10), there exists a constant $C_4 \ge 1$ such that

$$\int_{B(x,r)\cap\Omega} \Phi\big(y, Mf(y)/(3C_1)\big)\, dy \le C_4|B(x,r)|\kappa(x,r)^{-1}$$

for all $x \in \Omega$ and $r > 0$, so that

$$\int_{B(x,r)\cap\Omega} \overline{\Phi}\big(y, Mf(y)/(3A_2C_1C_4)\big)\, dy \le |B(x,r)|\kappa(x,r)^{-1}$$

for all $x \in \Omega$ and $r > 0$. This completes the proof of the theorem. □

4. Sharpness of conditions. We next show that q/N and p in condition (4) are sharp. For $p > 1$, $q > 1$, $\delta > 0$, $\eta > 0$ and $\zeta > 0$, consider the function

$$\begin{aligned}\Phi(x,t) &= \Phi_{[p,q;\delta,\eta;\zeta]}(x,t) \\ &= \begin{cases} t^p\big[(1-h(x_1))t + h(x_1)\max(t, g_\delta(x))\big]^\eta & \text{if } 0 \le t \le 1\,, \\ t^q \max\big(1,\ h(x_1)t^\zeta\big) & \text{if } t \ge 1\,, \end{cases}\end{aligned}$$

where $x = (x_1, \dots, x_N) \in \mathbf{R}^N$,

$$h(x_1) = \max(0, \min(1, x_1)) \quad \text{and} \quad g_\delta(x) = [\max(2, |x|)]^{-N/\delta}\,.$$

This $\Phi(x,t)$ satisfies (Φ1), (Φ2), (Φ3; 0; p) and (Φ3; ∞; q). We shall show:

(I) $\Phi(x,t)$ satisfies (Φ5; ζ);
(II) $\Phi(x,t)$ satisfies (Φ6; ω) for $\omega > \delta$;
(III) If $\zeta > q/N$, then there is $f \in L^\Phi(\mathbf{R}^N)$ such that $Mf \notin L^\Phi(\mathbf{R}^N)$;
(IV) If $\delta > p + \eta$, then there is $f \in L^\Phi(\mathbf{R}^N)$ such that $Mf \notin L^\Phi(\mathbf{R}^N)$.

These show the sharpness of q/N and p in (4).

Proof of (I). Suppose $|x - y| \le \gamma t^{-\zeta}$ and $t \ge 1$. Then

$$\Phi(y,t) \le \Phi(x,t) + t^{q+\zeta}|x_1 - y_1| \le \Phi(x,t) + \gamma t^q \le (1+\gamma)\Phi(x,t),$$

since $\Phi(x,t) \ge t^q$ for $t \ge 1$. This shows that $\Phi(x,t)$ satisfies (Φ5; ζ). □

PROOF OF (II). Suppose $g_\delta(x) \le t \le 1$ and $|x'| \ge |x|$. Then $\max(t, g_\delta(x)) = \max(t, g_\delta(x')) = t$, so that $\Phi(x,t) = \Phi(x',t) = t^{p+\eta}$. Since $g_\delta \in L^\omega(\mathbf{R}^N)$ for $\omega > \delta$, this shows that $\Phi(x,t)$ satisfies $(\Phi 6; \omega)$ if $\omega > \delta$. □

To prove (III) and (IV), we prepare the following lemma.

LEMMA 12. (1) *For $0 < a < N$, let $f_{0,a}(x) = |x|^{-a}\chi_{B(0,1)\cap\{x_1<0\}}(x)$. Then there is a constant $C_0(N,a) > 0$ depending only on N and a such that*

$$Mf_{0,a}(x) \ge C_0(N,a)|x|^{-a} \quad \textit{for} \quad |x| \le 1 . \tag{11}$$

(2) *For $0 < b < N$, let $f_{\infty,b}(x) = |x|^{-b}\chi_{\{x_1<0\}\setminus B(0,1)}(x)$. Then there is a constant $C_\infty(N,b) > 0$ depending only on N and b such that*

$$Mf_{\infty,b}(x) \ge C_\infty(N,b)|x|^{-b} \quad \textit{for} \quad |x| \ge 2 . \tag{12}$$

PROOF. (1) Let $|x| \le 1$. Since $B(x, 2|x|) \supset B(0, |x|)$,

$$\begin{aligned}\frac{1}{|B(x,2|x|)|}&\int_{B(x,2|x|)} f_{0,a}(y)\,dy \\ &\ge \frac{1}{2^N|B(0,1)||x|^N}\int_{B(0,|x|)\cap\{y_1<0\}} |y|^{-a}\,dy = \frac{N}{2^{N+1}(N-a)}|x|^{-a}.\end{aligned}$$

(2) Let $|x| \ge 2$. Then

$$\begin{aligned}\frac{1}{|B(x,2|x|)|}&\int_{B(x,2|x|)} f_{\infty,b}(y)dy \\ &\ge \frac{1}{2^N|B(0,1)||x|^N}\int_{(B(0,|x|)\setminus B(0,1))\cap\{y_1<0\}} |y|^{-b}\,dy \\ &= \frac{N}{2^{N+1}(N-b)}|x|^{-N}(|x|^{N-b}-1) \ge \frac{(1-2^{b-N})N}{2^{N+1}(N-b)}|x|^{-b} .\end{aligned}$$

□

PROOF OF (III). Assume $\zeta > q/N$. Set $a = (N+1)/(q+\zeta)$. Then $0 < a < aq < N$. Since $f_{0,a} \ge 1$ on $B(0,1) \cap \{x_1 < 0\}$ and $\Phi(x,t) = t^q$ if $t \ge 1$ and $x_1 < 0$,

$$\Phi(x, f_{0,a}(x)) = |x|^{-aq}\chi_{B(0,1)\cap\{x_1<0\}}(x) ,$$

so that $f_{0,a} \in L^\Phi(\mathbf{R}^N)$.

On the other hand, by (11), $Mf_{0,a}(x) \ge 1$ if $|x| \le c_0 := \min(1, C_0(N,a)^{1/a})$, and hence

$$\begin{aligned}\Phi(x, Mf_{0,a}(x)) &\ge [Mf_{0,a}(x)]^{q+\zeta}x_1 \\ &\ge 2^{-1}C_0(N,a)^{q+\zeta}|x|^{-a(q+\zeta)+1} = 2^{-1}C_0(N,a)^{q+\zeta}|x|^{-N}\end{aligned}$$

if $|x| \le c_0$ and $x_1 \ge |x|/2$. It follows that $\int_{\mathbf{R}^N} \Phi(x, Mf_{0,a}(x))\,dx = \infty$, which means that $Mf_{0,a} \notin L^\Phi(\mathbf{R}^N)$. □

PROOF OF (IV). Assume $\delta > p+\eta$. Set $b = N(\delta-\eta)/(p\delta)$. Then $N/\delta < N/(p+\eta) < b < N/p < N$. Since $0 \le f_{\infty,b} \le 1$ on $\{x_1 < 0\} \setminus B(0,1)$ and $\Phi(x,t) = t^{p+\eta}$ if $0 \le t \le 1$ and $x_1 < 0$,

$$\Phi(x, f_{\infty,b}(x)) = |x|^{-b(p+\eta)} \chi_{\{x_1<0\}\setminus B(0,1)}(x)\,,$$

so that $f_{\infty,b} \in L^{\Phi}(\mathbf{R}^N)$.

On the other hand, since $Mf_{\infty,b} \le 1$, by (12) we have

$$\begin{aligned}\Phi(x, Mf_{\infty,b}(x)) &= [Mf_{\infty,b}(x)]^p \big[\max(Mf_{\infty,b}(x), |x|^{-N/\delta})\big]^\eta \\ &\ge C_\infty(N,b)^p |x|^{-pb} \big[\max(C_\infty(N,b)|x|^{-b}, |x|^{-N/\delta})\big]^\eta\end{aligned}$$

if $|x| \ge 2$ and $x_1 \ge 1$. Since $b > N/\delta$,

$$C_\infty(N,b)|x|^{-b} \le |x|^{-N/\delta} \quad \text{for} \quad |x| \ge R_b := \max(2,\ C_\infty(N,b)^{1/(b-N/\delta)})\,.$$

Since $pb + N\eta/\delta = N$, it follows that

$$\Phi(x, Mf_{\infty,b}(x)) \ge C_\infty(N,b)^p |x|^{-N}$$

whenever $|x| \ge R_b$ and $x_1 \ge 1$. Hence $\int_{\mathbf{R}^N} \Phi(x, Mf_{\infty,b}(x))\,dx = \infty$, namely $Mf_{\infty,b} \notin L^{\Phi}(\mathbf{R}^N)$. $\square$

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