

## THE MAXIMAL IDEAL CYCLES OVER NORMAL SURFACE SINGULARITIES WITH $\mathbb{C}^*$ -ACTION

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**Abstract.** The maximal ideal cycles and the fundamental cycles are defined on the exceptional sets of resolution spaces of normal complex surface singularities. The former (resp. later) is determined by the analytic (resp. topological) structure of the singularities. We study such cycles for normal surface singularities with  $\mathbb{C}^*$ -action. Assuming the existence of a reduced homogeneous function of the minimal degree, we prove that these two cycles coincide if the coefficients on the central curve of the exceptional set of the minimal good resolution coincide.

**1. Introduction.** Let  $(X, o)$  be a normal complex surface singularity and  $\pi : (\tilde{X}, E) \rightarrow (X, o)$  a good resolution, where  $E = \bigcup_{i=1}^r E_i$  is the irreducible decomposition of the exceptional set  $E$ . Then  $\sum_{i=1}^r E_i$  is a simple normal crossing divisor. A divisor on  $\tilde{X}$  supported in  $E$  is called a cycle. Let us consider a cycle  $M_E := \min \{(f \circ \pi)_E \mid f \in \mathfrak{m}, f \neq 0\}$  on  $E$  is called the *maximal ideal cycle* on  $E$  ([24, p.279]), where  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}_{X,o}$  and  $(f \circ \pi)_E = \sum_{i=1}^r v_{E_i}(f \circ \pi)E_i$  ( $v_{E_i}(f \circ \pi)$  is the vanishing order of  $f \circ \pi$  on  $E_i$ ). In [23, Theorem 2.7], Ph. Wagreich showed that if  $\mathfrak{m}_{\tilde{X}}$  is locally principal, then  $\mathfrak{m}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-M_E)$  and the multiplicity of  $(X, o)$  is equal to  $-M_E^2$ . The *fundamental cycle*  $Z_E$  is defined in [1] by  $Z_E = \min\{D = \sum_{i=1}^r a_i E_i \mid a_i > 0 \text{ and } DE_i \leq 0 \text{ for any } i\}$ . From the definition, we can easily see that  $M_E \cdot E_i \leq 0$  for any irreducible component  $E_i$  of  $E$  and the relation  $M_E \geq Z_E$  follows from the minimality of  $Z_E$ .

The equality  $M_E = Z_E$  are playing the important roles in various stages of normal surface singularity theory. For example, when M. Artin [1] proved that the multiplicity of any Kleinian singularity is equal to two, the key point of the proof is the fact of  $M_E = Z_E$ . Also,  $M_E = Z_E$  is a necessary condition for normal surface singularity to be a Kodaira singularity ([6]). For rational singularities, these two cycles coincide for any resolution ([1]). However, for non-rational singularities, it is a natural and delicate problem to ask whether  $M_E = Z_E$ . In [9, p.322], H. Laufer remarked that  $M_E > Z_E$  on the minimal resolution of a hypersurface singularity defined by  $z^2 = y(x^4 + y^6)$ .

For minimally elliptic singularities,  $M_E$  and  $Z_E$  coincide for the minimal good resolution ([8]). This result was generalized to maximally elliptic singularities by the first named author ([15]). However, for those singularities, it is not always true that  $M_E$  and  $Z_E$  coincide

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for any resolution. We explain it through an example. Let  $(X, o)$  be a quasi-homogeneous hypersurface singularity defined by  $z^2 = x^3 + y^7$ . It is a minimally elliptic singularity which has a natural  $\mathbb{C}^*$ -action. Let  $\pi : (\tilde{X}, E) \rightarrow (X, o)$  be the minimal good resolution. The weighted dual graph of  $E$  is given by  $[1; 0, (2, 1), (3, 1), (7, 1)]$  (see (2.2)). Let  $E_0, E_1, E_2$  and  $E_3$  be irreducible components of  $E$  with  $E_0^2 = -1, E_1^2 = -2, E_2^2 = -3$  and  $E_3^2 = -7$ . Let  $L$  be the strict transform of the divisor defined by  $y = 0$  in  $X$ . Then the divisor  $(y \circ \pi)_{\tilde{X}}$  is equal to  $(y \circ \pi)_E + L$ . We can easily check that  $M_E = (y \circ \pi)_E = 6E_0 + 3E_1 + 2E_2 + E_3 = Z_E$  and  $EL = E_3L = 1$ . Let  $\sigma : (X', E') \rightarrow (\tilde{X}, E)$  be the blowing-up at a point  $P := L \cap E_3$ . Since  $\text{Coeff}_{E_3}(z \circ \pi)_E = 3$  and  $\text{Coeff}_{E_3}(x \circ \pi)_E = 2, M_{E'}$  is given by  $6E'_0 + 3E'_1 + 2E'_2 + E'_3 + 2E'_4$ , where  $E'_i$  is the strict transform of  $E_i$  for  $i = 0, 1, 2, 3$  and  $E'_4 := \sigma^{-1}(P)$ . Then  $M_{E'} = Z_{E'} + E'_4$  on  $E'$ .

Any double point  $(X, o)$  (i.e., a normal surface singularity of multiplicity two) is a hypersurface singularity defined by  $z^2 = f(x, y)$ . In [3], D. J. Dixon proved that if the order of  $f$  is even, then  $M_E = Z_E$  for any resolution  $(\tilde{X}, E)$  of  $(X, o)$ . In [6], using pencils of curves, U. Karras introduced the notion of Kodaira singularities. If  $(X, o)$  is a Kodaira singularity, then  $M_E = Z_E$  for the minimal resolution. Let  $(X, o)$  be a normal hypersurface singularity defined by  $z^n = f(x, y)$ . In [19], the second named author gave two sufficient conditions for  $(X, o)$  to be a Kodaira singularity. This result was generalized to the case of cyclic coverings of normal surface singularity ([21, Theorem 4.14]).

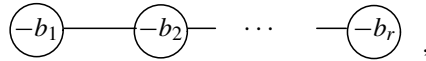
In general, it is not so easy to identify the maximal ideal cycle for a given normal surface singularity. Because it depends on the analytic structure of the singularity. When we compute the maximal ideal cycle, we need to know the detail of resolution process of the singularity. In [7], for Brieskorn hypersurface singularities, K. Konno and D. Nagashima gave the necessary and sufficient numerical condition that  $M_E$  and  $Z_E$  coincide on the minimal good resolution space. Further, F. N. Meng-T. Okuma [11] generalized Konno and Nagashima's result to Brieskorn complete intersection surface singularities (Theorem 3.1 of this paper).

Brieskorn complete intersection surface singularities form a special subclass of normal surface singularities with  $\mathbb{C}^*$ -action. The weighted dual graph of exceptional set of the minimal  $\mathbb{C}^*$ -good resolution (see § 2 for the definition) of a normal surface singularity with  $\mathbb{C}^*$ -action is star-shaped (the exceptional set has the one central curve of genus  $g$  and several  $\mathbb{P}^1$ -chains intersecting the central curve). In this paper, we consider the problem above for the minimal  $\mathbb{C}^*$ -good resolutions of normal surface singularities with  $\mathbb{C}^*$ -action. In Section 2, we shall prepare some facts on normal surface singularities with  $\mathbb{C}^*$ -action. In Section 3, assuming the existence of a reduced function of the minimal degree, we prove that  $M_E = Z_E$  if and only if  $\text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E$  for the central curve  $E_0$  for the minimal  $\mathbb{C}^*$ -good resolution space  $(\tilde{X}, E)$  of  $(X, o)$  (Theorem 3.5 (iii)). From Theorems 2.3 and 3.5 (iii), it is natural to ask whether  $M_E$  coincides to  $Z_E$  if the central curve of  $E$  is  $\mathbb{P}^1$ . However, it is not always true. We give a counterexample for it (Example 3.8). In Section 4, we consider the maximal ideal cycles of Kummer coverings over normal surface singularities with  $\mathbb{C}^*$ -action. We generalize some results of Meng-Okuma [11] (Theorems 4.1 and 4.2). Using our results,

for Brieskorn complete intersection surface singularities, we reprove the main results due to Konno-Nagashima and Meng-Okuma (Corollary 4.4).

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**2. Preliminaries.** Let us explain cyclic quotient singularities. For two integers  $n, q$  with  $0 < q < n$  and  $\gcd(n, q) = 1$ , a cyclic quotient singularity  $C_{n,q}$  is defined as the quotient space  $\mathbb{C}^2/\langle g \rangle$  by the natural action of  $g = \begin{pmatrix} e_n & 0 \\ 0 & e_n^q \end{pmatrix}$  onto  $\mathbb{C}^2$ , where  $e_n := \exp(2\pi\sqrt{-1}/n)$ . Here, if  $n = 1$  and  $q = 0$ , then  $C_{1,0}$  means a non-singular point. In this paper, we consider that a non-singular point is a sort of cyclic quotient singularities. Let  $(X, o)$  be a cyclic quotient singularity  $C_{n,q}$  ( $q \geq 0$ ). Then there exists a resolution  $\pi : (\tilde{X}, E) \rightarrow (X, o)$  such that the weighted dual graph of  $E$  is given as follows:

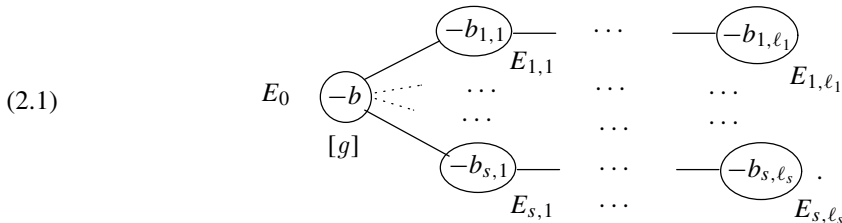


where  $b_i \geq 1$  for any  $i$  and

$$\frac{n}{q} = [[b_1, \dots, b_r]] := b_1 - \frac{1}{\dots - \frac{1}{b_r}} .$$

If  $b_i \geq 2$  for any  $i$ , then the resolution coincides with the minimal resolution of  $C_{n,q}$  ( $n \geq 2$ ).

Now we prepare some facts on normal singularities  $\mathbb{C}^*$ -action ([12]). Assume that there is an embedding  $(X, o) \subset (\mathbb{C}^{N+1}, o)$  such that the  $\mathbb{C}^*$ -action on  $(X, o)$  is induced from a diagonal action  $t \cdot (z_0, \dots, z_N) = (t^{q_0}z_0, \dots, t^{q_N}z_N)$  on  $\mathbb{C}^{N+1}$ , where  $q_i$  is a positive integer for any  $i$ . If  $\gcd(q_0, \dots, q_N) = 1$ , then the action is called a *good  $\mathbb{C}^*$ -action*. In this paper, we abbreviate “good  $\mathbb{C}^*$ -action” to “ $\mathbb{C}^*$ -action”. Then the affine ring  $R_X$  of  $X$  is a graded ring  $\mathbb{C}[z_0, \dots, z_N]/I_X$ , where  $I_X$  is the defining ideal which is generated by quasi-homogeneous polynomials of type  $(q_0, \dots, q_N)$ . In the following, we always assume that  $(X, o)$  is 2-dimensional. Let  $\pi : (\tilde{X}, E) \rightarrow (X, o)$  be the minimal good resolution of a normal surface singularity with  $\mathbb{C}^*$ -action. Then the weighted dual graph of  $E$  is star-shaped which is given as follows:



If we put  $\frac{\alpha_i}{\beta_i} = [[b_{i,1}, \dots, b_{i,\ell_i}]]$  for positive integers  $\alpha_i, \beta_i$  with  $\gcd(\alpha_i, \beta_i) = 1$ , then the intersection matrix of  $E$  is negative definite if and only if  $b - \sum_{i=1}^s \frac{\beta_i}{\alpha_i} > 0$ . The component  $E_0$  is a compact smooth algebraic curve of genus  $g$  and  $E_0^2 = -b$ . It is called the *central curve*. Each  $\mathbb{P}^1$ -chain  $\bigcup_{j=1}^{\ell_i} E_{i,j}$  is contracted to a cyclic quotient singularity  $C_{\alpha_i, \beta_i}$ .

If the minimal good resolution of a normal surface singularity has the exceptional set whose weighted dual graph is given by (2.1), then the singularity is called a *star-shaped surface singularity*. Therefore, any normal surface singularity with  $\mathbb{C}^*$ -action is a star-shaped surface singularity. However, the converse is not always true. Please refer [16] for star-shaped surface singularities. In this paper, the weighted dual graph (2.1) associated with a star-shaped surface singularity  $(X, o)$  is indicated by the following:

$$(2.2) \quad [b; g; (\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s)].$$

DEFINITION 2.1. Under the situation above, we define a positive integer as follows:

$$\alpha_0(X, o) := \begin{cases} \text{lcm}(\alpha_1, \dots, \alpha_s) & \text{if } s > 0 \\ 1 & \text{if } s = 0. \end{cases}$$

THEOREM 2.2 ([12], [13] and [20]). *There exists a  $\mathbb{C}^*$ -equivariant resolution  $\pi : (\tilde{X}, E) \rightarrow (X, o)$  uniquely which satisfies the following:*

- (i) *The weighted dual graph of  $E$  is a star-shaped graph of (2.1).*
- (ii) *The  $\mathbb{C}^*$ -action on  $\tilde{X}$  acts trivially on the central curve  $E_0$ ; each irreducible component of  $E$  except for  $E_0$  contains a one-dimensional  $\mathbb{C}^*$ -orbit.*
- (iii) *Each  $\mathbb{P}^1$ -chain  $\bigcup_{j=1}^{\ell_i} E_{i,j}$  does not contain a  $(-1)$ -curve.*

In this paper, a resolution satisfying the three conditions above is called the *minimal  $\mathbb{C}^*$ -good resolution*. It depends on the  $\mathbb{C}^*$ -action of  $(X, o)$ . For cases aside from cyclic quotient singularities, the minimal  $\mathbb{C}^*$ -good resolution is the minimal good resolution. However, for cyclic quotient singularities, this is not always true.

Here, as a preparation of Sections 3 and 4, we explain the minimal  $\mathbb{C}^*$ -good resolution for cyclic quotient singularities according to [20]. Though the argument in [20] was done under the condition of  $n > 1$ , it is applicable to the case of  $(n, q) = (1, 0)$  (i.e., non-singular point  $(\mathbb{C}^2, o)$ ). Let  $(X, o)$  be a cyclic quotient singularity  $C_{n,q}$  for integers  $n, q$  with  $0 \leq q < n$  and  $\gcd(n, q) = 1$ . Let consider the  $\mathbb{C}^*$ -action on  $\mathbb{C}^2$  defined by  $t \cdot (x, y) = (t^r x, t^s y)$  for any  $t \in \mathbb{C}^*$ , where  $\gcd(r, s) = 1$ . It induces on  $(X, o)$  a natural  $\mathbb{C}^*$ -action. Such  $(X, o)$  is called a *cyclic quotient singularity  $C_{n,q}$  with  $\mathbb{C}^*$ -action of type  $(r, s)$* . We give the minimal  $\mathbb{C}^*$ -good resolution of  $(X, o)$  with respect to the  $\mathbb{C}^*$ -action. For  $q > 0$  (resp.  $q = 0$ ), let  $q'$  be the integer defined by  $qq' \equiv 1 \pmod{n}$  and  $0 < q' < n$  (resp.  $q' = 0$ ). Let  $\mu$  and  $\lambda$  be the integers defined by  $\mu = \gcd(n, qr - s)$  and  $\lambda = \frac{n}{\mu}$ ; and also  $\alpha_1, \alpha_2$  integers defined by the following:

$$\alpha_1 \left( \frac{qr - s}{\mu} \right) \equiv 1 \pmod{\lambda r} \quad \text{and} \quad 0 < \alpha_1 < \lambda r,$$

$$\alpha_2 \left( \frac{q's - r}{\mu} \right) \equiv 1 \pmod{\lambda s} \quad \text{and} \quad 0 < \alpha_2 < \lambda s.$$

Then  $b := \frac{\mu}{\lambda r s} + \frac{\alpha_1}{\lambda r} + \frac{\alpha_2}{\lambda s}$  is an integer. From Theorem 2.3 in [20], the weighted dual graph associated to the minimal  $\mathbb{C}^*$ -good resolution is given by  $[b; 0; (\lambda r, \alpha_1), (\lambda s, \alpha_2)]$ . For example, if  $(X, o)$  is  $C_{3,2}$  (resp.  $C_{1,0}$ ) with  $\mathbb{C}^*$ -action of type  $(5, 2)$ , then the weighted dual graph associated with the minimal  $\mathbb{C}^*$ -good resolution is given  $[1; 0; (15, 2), (6, 5)]$  (resp.  $[1; 0; (5, 2), (2, 1)]$ ).

Let  $\pi : (\tilde{X}, E) \rightarrow (X, o)$  be the minimal  $\mathbb{C}^*$ -good resolution of a normal  $\mathbb{C}^*$ -surface singularity such that the weighted dual graph of  $E$  is given by (2.1). The analytic structure of  $(X, o)$  is determined by the analytic structures of the central curve  $E_0$  and the normal bundle of  $E_0$  in the minimal  $\mathbb{C}^*$ -good resolution and intersection points of  $E_0$  and  $\mathbb{P}^1$ -chains ([4] and [13]). Let  $H$  be a divisor on  $E_0$  satisfying  $[H] \sim N_{E_0/\tilde{X}}^*$  (linearly equivalent to the conormal bundle  $N_{E_0/\tilde{X}}^*$ ) and  $P_i := E_0 \cap E_{i,1}$  for any  $i$ . For affine graded ring  $R_X$  of  $(X, o)$ , Pinkham [13] (see [20] for cyclic quotient singularities) proved the following isomorphism of graded rings

$$(2.3) \quad R_X \cong \bigoplus_{k=0}^{\infty} H^0(E_0, \mathcal{O}_{E_0}(D^{(k)}))t^k,$$

where  $D^{(k)} = kH - \sum_{j=1}^s \lceil \frac{\beta_j k}{\alpha_j} \rceil P_j$  and  $\lceil a \rceil$  is the round up of  $a \in \mathbb{R}$ . This representation is called the *Pinkham–Demazure construction* of  $R_X$ . For  $h \in R_X$ ,  $h$  is a homogeneous element of degree  $k$  if and only if  $h \in H^0(E_0, \mathcal{O}_{E_0}(D^{(k)}))t^k$ . Also, for a homogeneous element  $h$ , we have

$$(2.4) \quad \text{deg}(h) = \text{Coeff}_{E_0}(h \circ \pi)_E.$$

We prepare the following result which was proven by the first named author.

**THEOREM 2.3** ([18, p.282]). *If  $(X, o)$  is a normal surface singularity which has a star-shaped weighted dual graph of (2.1), then  $\text{Coeff}_{E_0} Z_E = \min\{k \in \mathbb{N} \mid \text{deg}(D^{(k)}) \geq 0\}$ . Further, if  $(X, o)$  has a good  $\mathbb{C}^*$ -action and  $E_0 = \mathbb{P}^1$ , then  $\text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E$ .*

If  $(\tilde{X}, E)$  is the minimal  $\mathbb{C}^*$ -good resolution of a normal surface singularity with  $\mathbb{C}^*$ -action, then we can easily see that  $\text{Coeff}_{E_0} M_E = \min\{k \in \mathbb{N} \mid H^0(E_0, \mathcal{O}_{E_0}(D^{(k)})) \neq 0\}$ .

**3. The maximal ideal cycles over normal surface singularities with  $\mathbb{C}^*$ -action.** Let  $(\tilde{X}, E) \rightarrow (X, o)$  be the minimal good resolution of a normal surface singularity with  $\mathbb{C}^*$ -action. In the following, let  $\text{min.deg}(R_X)$  be the minimal degree of nonzero homogeneous elements of  $(R_X)_+$ , where  $(R_X)_+ := \bigoplus_{k \geq 1} (R_X)_k$  (i.e., the homogeneous maximal ideal of  $R_X$ ). Obviously, we have

$$(3.1) \quad \text{min.deg}(R_X) = \text{Coeff}_{E_0} M_E.$$

Now we describe the result of Konno-Nagashima and Meng-Okuma.

**THEOREM 3.1** ([11, Theorem 6.1], [7, Theorem 3.2]). *Let  $(X, o)$  be a Brieskorn complete intersection singularity defined by  $z_3^{a_3} = p_3 z_1^{a_1} + q_3 z_2^{a_2}, \dots, z_m^{a_m} = p_m z_1^{a_1} + q_m z_2^{a_2}$  in  $(\mathbb{C}^m, o)$ , where  $p_i, q_j \in \mathbb{C}^*$  with  $p_i q_j \neq p_j q_i$  for  $i \neq j$ . Assume that  $a_1 \leq a_2 \leq \dots \leq a_m$ .*

Let  $\pi : (\tilde{X}, E) \rightarrow (X, o)$  be the minimal good resolution. Then  $M_E = (z_m \circ \pi)_E$ , and also  $M_E = Z_E$  if and only if  $\text{min.deg}(R_X) \leq \alpha_0(X, o)$ .

In Proposition 3.2, we prove that the ‘‘only if’’ part of Theorem 3.1 is always true for any normal surface singularity with  $\mathbb{C}^*$ -action. It is obtained as a corollary of Theorem 2.3. In fact, we have  $\text{min.deg}(R_X) = \text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E \leq \alpha_0(X, o)$ , since  $D^{(\alpha_0(X, o))}$  is an integral divisor with  $\text{deg}(D^{(\alpha_0(X, o))}) > 0$ . However, we can prove it directly from the definition of  $Z_E$  as follows.

PROPOSITION 3.2. *Let  $(\tilde{X}, E) \rightarrow (X, o)$  be the minimal good resolution of a normal surface singularity with  $\mathbb{C}^*$ -action. If  $\text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E$ , then  $\text{min.deg}(R_X) = \text{Coeff}_{E_0} M_E \leq \alpha_0(X, o)$ .*

PROOF. Assume that the weighted dual graph of  $E$  is given by (2.1). Put  $\alpha_0 := \alpha_0(X, o)$ . Let us consider a  $\mathbb{P}^1$ -chain  $\bigcup_{j=1}^{\ell_i} E_{i,j}$  ( $1 \leq i \leq s$ ), where it is contracted to  $C_{\alpha_i, \beta_i}$ . Let  $\beta_{i,0} := \alpha_0$  and  $\beta_{i,1} := \frac{\beta_i \alpha_0}{\alpha_i}$ . Let  $\beta_{i,k}$  be integers defined inductively by  $\beta_{i,k+1} := b_{i,k} \beta_{i,k} - \beta_{i,k-1}$  for  $k = 1, \dots, \ell_i - 1$ . Let  $D := \alpha_0 E_0 + \sum_{i=1}^s \sum_{k=1}^{\ell_i} \beta_{i,k} E_{i,k}$ . Then we can easily see that  $DE_{i,k} = 0$  for any  $i$  and  $k = 1, \dots, \ell_i$  and also  $DE_0 = -\alpha_0(b - \sum_{i=1}^s \frac{\beta_i}{\alpha_i}) < 0$ . Hence  $Z_E \leq D$ ; thus  $\text{min.deg}(R_X) = \text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E \leq \text{Coeff}_{E_0} D = \alpha_0$ .  $\square$

Let us consider the following three conditions:

- (i)  $M_E = Z_E$ , (ii)  $\text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E$ , (iii)  $\text{min.deg}(R_X) \leq \alpha_0(X, o)$ .

Then, obviously we have (i)  $\Rightarrow$  (ii); also we have (ii)  $\Rightarrow$  (iii) from Proposition 3.2. However, the converse implications (ii)  $\Rightarrow$  (i) and (iii)  $\Rightarrow$  (ii) are not always true. Example 3.8 gives a counterexample of the former implication. For the later implication, let  $(X, o)$  be a quasi-homogeneous hypersurface singularity defined by  $z^2 = y(x^4 + y^6)$  (see [9]). Then  $\text{deg}(x) = 3$ ,  $\text{deg}(y) = 2$  and  $\text{deg}(z) = 7$ . The weighted dual graph of the minimal good resolution  $(\tilde{X}, E)$  is given by  $[1; 1; (3, 2)]$ . Then, since  $\text{Coeff}_{E_0} Z_E = 1 < \text{min.deg}(R_X) = \text{Coeff}_{E_0} M_E = 2 < \alpha_0(X, o) = 3$ ,  $(X, o)$  is a counterexample of the later implication. Theorem 3.1 asserts that three conditions above are equivalent for Brieskorn complete intersection singularities.

From now on we prepare two lemmas to prove Theorems 3.5 and 4.1. Let  $(\tilde{X}, E)$  be the minimal resolution of a cyclic quotient singularity  $C_{n,q}$ . Let  $n/q = [[b_1, \dots, b_r]]$  ( $b_i \geq 2$ ) and consider the following configuration:

$$(3.2) \quad * \text{ --- } \textcircled{-b_1} \text{ --- } \textcircled{-b_2} \text{ --- } \dots \text{ --- } \textcircled{-b_r} \text{ --- } * \\
 E_0 \quad E_1 \quad E_2 \quad \dots \quad E_r \quad E_{r+1}$$

where  $E_0$  (resp.  $E_{r+1}$ ) is a smooth complex curve which intersects with  $E = \bigcup_{i=1}^r E_i$  at only one point in  $E_1$  (resp.  $E_r$ ) transversally.

LEMMA 3.3. *Under the condition above, consider two effective divisors  $D_1 = \sum_{i=0}^{r+1} a_i E_i$  and  $D_2 = \sum_{i=0}^{r+1} c_i E_i$  such that  $D_k E_i = 0$  for  $i = 1, 2, \dots, r$  and  $k = 1, 2$ . If we assume  $a_0 \geq c_0$ , then we have the following.*

- (i) *If  $a_{r+1} \geq c_{r+1}$ , then  $D_1 \geq D_2$ .*
- (ii) *If  $a_{r+1} = 0$  and  $c_{r+1} = 1$ , then  $D_1|_{\bigcup_{i=0}^r E_i} \geq D_2|_{\bigcup_{i=0}^r E_i}$ , where  $D_k|_{\bigcup_{i=0}^r E_i}$  means the restriction of  $D_k$  onto  $\bigcup_{i=0}^r E_i$  ( $k = 1, 2$ ).*

PROOF. Let us put  $\delta_{-1} := 0, \delta_0 := 1$  and  $\delta_i := b_i \delta_{i-1} - \delta_{i-2}$  for  $i = 1, \dots, r$ . Then we can easily check that  $\delta_1 = b_1 < \delta_2 = b_1 b_2 - 1 < \dots < \delta_r = n$ . We have the following relations:  $a_i - a_{i+1} b_{i+1} + a_{i+2} = 0$  and  $c_i - b_{i+1} c_{i+1} + c_{i+2} = 0$  for  $i = 0, 1, \dots, r - 1$ . Then, we have  $a_0 = a_1 \delta_1 - a_2 \delta_0 = \dots = a_i \delta_i - a_{i+1} \delta_{i-1} = \dots = a_r \delta_r - a_{r+1} \delta_{r-1}$ . For  $c_0, \dots, c_{r+1}$ , we have similar equations by the same way. Therefore,

$$(3.3) \quad a_0 = a_i \delta_i - a_{i+1} \delta_{i-1} \text{ and } c_0 = c_i \delta_i - c_{i+1} \delta_{i-1} \text{ for } i = 1, \dots, r.$$

(i) Since  $a_r \delta_r - a_{r+1} \delta_{r-1} = a_0 \geq c_0 = c_r \delta_r - c_{r+1} \delta_{r-1}$ , we have  $(a_r - c_r) \delta_r \geq (a_{r+1} - c_{r+1}) \delta_{r-1}$ . Hence,  $a_r \geq c_r$  from  $a_{r+1} \geq c_{r+1}$ . By the induction on  $i$ , we obtain  $a_i \geq c_i$  for any  $i$ ; consequently  $D_1 \geq D_2$ .

(ii) From (3.3), we have  $a_r \delta_r = a_r \delta_r - a_{r+1} \delta_{r-1} = a_0 \geq c_0 = c_r \delta_r - c_{r+1} \delta_{r-1} = c_r \delta_r - \delta_{r-1}$ . Hence,  $a_r \geq c_r - \frac{\delta_{r-1}}{\delta_r}$  and so  $a_r \geq c_r$  from  $0 < \frac{\delta_{r-1}}{\delta_r} < 1$ . Moreover,  $a_0 = a_i \delta_i - a_{i+1} \delta_{i-1} \geq c_0 = c_i \delta_i - c_{i+1} \delta_{i-1}$ ; hence  $\delta_i (a_i - c_i) \geq \delta_{i-1} (a_{i+1} - c_{i+1})$  for  $i = 1, \dots, r$ . Since  $\delta_{r-1} (a_{r-1} - c_{r-1}) \geq \delta_r (a_r - c_r) \geq 0$ , we have  $a_{r-1} \geq c_{r-1}$ . By the induction on  $i$ , we have  $a_i \geq c_i$  for  $i = 1, \dots, r$ . □

For the figure (3.2) and a positive integer  $\lambda_0$ , let us consider the following set and the divisor.

$$\mathfrak{D}(\lambda_0) := \left\{ D := \lambda_0 E_0 + \sum_{i=1}^r m_i E_i \mid m_i \in \mathbb{N} \text{ and } DE_i \leq 0 \text{ for } i = 1, \dots, r \right\},$$

$$D(\lambda_0) := \min\{D \in \mathfrak{D}(\lambda_0)\}.$$

LEMMA 3.4 (Lemma 2.2 in [10]). *If  $D \in \mathfrak{D}(\lambda_0)$  satisfies  $DE_i = 0$  for  $i = 1, \dots, r - 1$  and  $DE_r \geq -1$ , then  $D = D(\lambda_0)$ .*

Let  $(\tilde{X}, E) \rightarrow (X, o)$  be the minimal  $\mathbb{C}^*$ -good resolution of a normal surface singularity with  $\mathbb{C}^*$ -action; and  $h$  a homogeneous element of  $R_X$ . From (2.4), the degree of  $h$  is equal to  $\text{Coeff}_{E_0}(h \circ \pi)_E$  for the central curve  $E_0$ . In the following,  $(h \circ \pi)_{\tilde{X}}$  means the divisor defined by  $h \circ \pi$  on  $\tilde{X}$ . Then  $(h \circ \pi)_E$  is the restriction of  $(h \circ \pi)_{\tilde{X}}$  (i.e.,  $\text{supp}((h \circ \pi)_E) = E$ ).

THEOREM 3.5. (i) *Let  $h_1, h_2$  be homogeneous elements satisfying  $\deg(h_1) \geq \deg(h_2)$ . If  $h_2$  is reduced, then  $(h_1 \circ \pi)_E \geq (h_2 \circ \pi)_E$ .*

(ii) *If  $h_1, h_2$  are reduced homogeneous elements of the same degree, then  $(h_1 \circ \pi)_E = (h_2 \circ \pi)_E$ .*

(iii) *Suppose that there exists a reduced homogeneous element  $f$  satisfying  $(f \circ \pi)_E = M_E$ . Then,  $M_E = Z_E$  if and only if  $\text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E$  for the central curve  $E_0$  of  $E$ .*

PROOF. (i) Assume that the weighted dual graph of  $E$  is given by (2.1). Let  $E_{i, \ell_i+1}$  be the non-exceptional curve such that  $E_{i, \ell_i} \cap E_{i, \ell_i+1} \neq \emptyset$  and it contains a 1-dimensional  $\mathbb{C}^*$ -orbit. For a  $\mathbb{P}^1$ -chain  $\bigcup_{j=1}^{\ell_i} E_{i, j}$ , let us put

$$a_0 := \text{Coeff}_{E_0}(h_1 \circ \pi)_E, c_0 := \text{Coeff}_{E_0}(h_2 \circ \pi)_E,$$

$$a_j := \text{Coeff}_{E_{i, j}}(h_1 \circ \pi)_E \text{ and } c_j := \text{Coeff}_{E_{i, j}}(h_2 \circ \pi)_E \text{ for } j = 1, \dots, \ell_i + 1.$$

Since  $h_2$  is reduced,  $(a_{\ell_i+1}, c_{\ell_i+1}) = (0, 1)$  or  $a_{\ell_i+1} \geq c_{\ell_i+1}$ . If  $(a_{\ell_i+1}, c_{\ell_i+1}) = (0, 1)$ , we have  $(h_1 \circ \pi)_{\tilde{X}}|_{\bigcup_{j=1}^{\ell_i} E_{i, j}} \geq (h_2 \circ \pi)_{\tilde{X}}|_{\bigcup_{j=1}^{\ell_i} E_{i, j}}$  from Lemma 3.4. If  $a_{\ell_i+1} \geq c_{\ell_i+1}$ , we have  $(h_1 \circ \pi)_{\tilde{X}}|_{\bigcup_{j=1}^{\ell_i} E_{i, j}} \geq (h_2 \circ \pi)_{\tilde{X}}|_{\bigcup_{j=1}^{\ell_i} E_{i, j}}$  from Lemma 3.3 (i). Consequently, we have  $(h_1 \circ \pi)_E \geq (h_2 \circ \pi)_E$ .

(ii) Since  $\text{Coeff}_{E_0}(h \circ \pi)_E = \text{deg}(h)$  by (2.4), (ii) is obvious from (i).

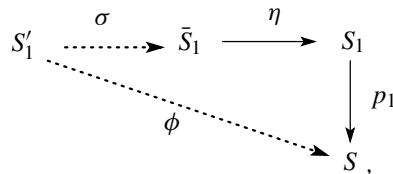
(iii) We need to prove “if part”. If  $E = E_0$  (i.e., there are no  $\mathbb{P}^1$ -chain), then it is obvious. Thus, suppose that there is at least one  $\mathbb{P}^1$ -chain in  $E$ . Let  $\bigcup_{j=1}^{\ell_i} E_{i, j}$  be any  $\mathbb{P}^1$ -chain of  $E$ . Since  $f$  is homogeneous, the cycle  $(f \circ \pi)_E$  satisfies that  $(f \circ \pi)_E E_{i, j} = 0$  for  $i = 1, \dots, s$  and  $j = 1, \dots, \ell_i - 1$ . If we put  $D_i := (f \circ \pi)_{\tilde{X}}|_{\bigcup_{j=0}^{\ell_i} E_{i, j}}$ , then  $D_i E_{i, j} = 0$  for  $j = 1, \dots, \ell_i - 1$ . Since  $f$  is reduced, we have  $0 \leq v_{E_{i, \ell_i+1}}(f \circ \pi) \leq 1$ . From  $(f \circ \pi)_{\tilde{X}} E_{i, \ell_i} = 0$ , we have  $D_i E_{i, \ell_i} \geq -1$  for any  $\mathbb{P}^1$ -chain. Let  $\theta := \text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E$ . From Lemma 3.4, we have  $M_E|_{\bigcup_{j=0}^{\ell_i} E_{i, j}} = D_i = D(\theta) = Z_E|_{\bigcup_{j=0}^{\ell_i} E_{i, j}}$ . Therefore,  $M_E = Z_E$ .  $\square$

COROLLARY 3.6. *Let  $\pi : (\tilde{X}, E) \rightarrow (X, o)$  be the minimal  $\mathbb{C}^*$ -good resolution. Assume that the central curve  $E_0$  is  $\mathbb{P}^1$ . If  $M_E > Z_E$  and  $\text{Coeff}_{E_0}(h_0 \circ \pi)_E = \text{Coeff}_{E_0} M_E$  for a homogeneous element  $h_0 \in R_X$ , then  $h_0$  is non-reduced.*

PROOF. Assume that  $h_0$  is reduced. From Theorem 3.5 (i), we have  $(h_0 \circ \pi)_E \geq (h_0 \circ \pi)_E$  for any homogeneous element  $h$ . Then we have  $(f \circ \pi)_E \geq (h_0 \circ \pi)_E$  for any non-zero element  $f \in \mathfrak{m} \subset \mathcal{O}_{X, o}$ . Hence  $M_E = (h_0 \circ \pi)_E$  and  $\text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E$  from  $E_0 = \mathbb{P}^1$ . Consequently,  $M_E = Z_E$  from Theorem 3.5 (iii). This is a contradiction.  $\square$

Here we prepare the following lemma for the computations in Example 3.8 and the proof of Theorem 4.1.

LEMMA 3.7 ([19, Lemma 3.1]). *Let  $S$  be a complex normal surface. Let  $f, h$  be non-constant holomorphic functions on  $S$ . Let  $S_1 := \{(p, z) \in S \times \mathbb{C} \mid z^n = h(p)\}$  and  $p_1$  the projection map; also consider a resolution map  $\eta : \bar{S}_1 \rightarrow S_1$ . Let  $\sigma : S'_1 \rightarrow \bar{S}_1$  be a birational map from a complex normal surface  $S'_1$ . Then we have the following diagram:*





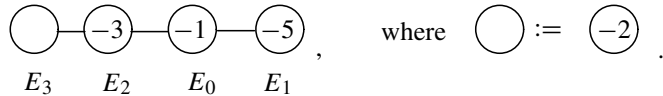
where  $\phi$  is a generically finite map. Let  $C \subset S$  and  $C_1 \subset S'_1$  be irreducible complex curves satisfying  $\phi_*(C_1) = C$  and  $h(C) = 0$ . Then we have

$$(i) \ v_{C_1}(z \circ \eta \circ \sigma) = \frac{v_C(h)}{\gcd(n, v_C(h))}, \quad (ii) \ v_{C_1}(f \circ \phi) = \frac{n \cdot v_C(f)}{\gcd(n, v_C(h))}.$$

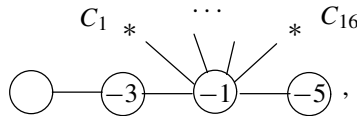
PROOF. Let  $P$  be a general point of  $C$  such that  $(C, P)$  and  $(S, P)$  are non-singular. It is sufficient to discuss our argument locally near  $P$ . Let  $\{U, (u, v)\}$  be a local coordinate open neighborhood of  $P$  in  $S$  such that  $h$  and  $f$  are written as  $h|_U = v^d$  (i.e.,  $C = \{v = 0\}$ ) and  $f|_U = v^m f_1$ , where  $m = v_C(f)$ . Put  $\ell := \gcd(n, d)$ . Then  $p_1^{-1}(U) = \bigcup_{j=1}^{\ell} V_j$ , where  $V_j = \{(u, v, z) \in \Delta^3 \mid z^{n/\ell} = \omega^j v^{d/\ell}\}$  for a small open polydisc  $\Delta^3 := \Delta \times \Delta \times \Delta \subset \mathbb{C}^3$  around the origin and  $\omega := \exp(2\pi\sqrt{-1}/\ell)$ . Then,  $\eta : V'_j = \Delta \times \Delta \rightarrow V_j$  ( $\eta(u, t) := (u, t^{n/\ell}, t^{d/\ell}) = (u, v, z)$ ) is the normalization map of  $V_j$ ; also it is a resolution of  $V_j$ . Since  $v = t^{n/\ell}$  and  $z = t^{d/\ell}$ , we have  $v_{C_1}(z \circ \eta \circ \sigma) = \frac{d}{\ell}$  and  $v_{C_1}(f \circ \phi) = \frac{nm}{\ell}$ .  $\square$

For any resolution of every rational singularity, the maximal ideal cycle coincides the fundamental cycle. Also, if  $(\tilde{X}, E) \rightarrow (X, o)$  is the minimal  $\mathbb{C}^*$ -good resolution of a normal surface singularity with  $\mathbb{C}^*$ -action whose central curve  $E_0$  is  $\mathbb{P}^1$ , we can easily see that  $\text{Coeff}_{E_0} M_E = \text{Coeff}_{E_0} Z_E$  from Theorem 2.3. However, for such singularities, we can not expect to hold that  $M_E = Z_E$  in general. We show it by the following.

EXAMPLE 3.8. Let  $(X, o)$  be a cyclic quotient singularity  $C_{10,3}$  with  $\mathbb{C}^*$ -action of type  $(1, 1)$  (see § 2). Let  $\pi_X : (\tilde{X}, E) \rightarrow (X, o)$  be the minimal  $\mathbb{C}^*$ -good resolution. Then the weighted dual graph of  $E$  is given as follows:



If  $P_i := E_0 \cap E_i$  ( $i = 1, 2$ ) and  $\mathcal{O}_{\tilde{X}}(-E_0)|_{E_0} = \mathcal{O}_{E_0}(R)$  for  $R \in E_0$ , then  $R_X \cong \bigoplus_{k=0}^{\infty} H^0(E_0, \mathcal{O}_{E_0}(D^{(k)}))t^k$ , where  $D^{(k)} = kR - \lceil \frac{k}{5} \rceil P_1 - \lceil \frac{2k}{5} \rceil P_2$ . We can easily see that there is a homogeneous element  $h \in R_X$  whose divisor is given by  $40E_0 + 8E_1 + 16E_2 + 8E_3 + \sum_{i=1}^{16} C_i$  for the following figure:



where each  $C_i$  is a non-exceptional curve which contains a 1-dimensional  $\mathbb{C}^*$ -orbit. Let  $(Y, o)$  be the cyclic cover of  $(X, o)$  defined by  $z^3 = h$  and  $\pi_Y : (\tilde{Y}, F) \rightarrow (Y, o)$  the minimal  $\mathbb{C}^*$ -good resolution; also let  $p : (Y, o) \rightarrow (X, o)$  be the natural covering map. Through some computations (see [22, Theorem 3.4]), we can easily see that the weighted dual graph of  $F$  is given as follows:



$h_{j,k} := h_{j,k-1} \circ \phi_k$  for any  $j$ . By the same way as above, we can see that  $(X_k, o)$  is a normal surface singularity with  $\mathbb{C}^*$ -action. Then we have a sequence of covering maps of normal surface singularities with  $\mathbb{C}^*$ -action as follows:

$$(Y, o) = (X_m, o) \xrightarrow{\phi_m} \cdots \xrightarrow{\phi_2} (X_1, o) \xrightarrow{\phi_1} (X_0, o) = (X, o).$$

Put  $(\tilde{X}_0, \tilde{E}_0) := (\tilde{X}, E)$ . Let  $\eta_0 : (\tilde{X}_0, \tilde{E}_0) \rightarrow (\bar{X}_0, \bar{E}_0)$  be the morphism which contracts the divisor  $\tilde{E}_0 - E_{0,0} (\subset \tilde{X}_0)$ , where  $E_{0,0}$  is the central curve of  $\tilde{E}_0$ . Since any connected component of  $\tilde{E}_0 - E_{0,0}$  is a  $\mathbb{P}^1$ -chain, every singularity of  $\tilde{X}_0$  is a cyclic quotient singularity. Let  $\bar{\pi}_0 : (\bar{X}_0, \bar{E}_0) \rightarrow (X_0, o)$  be the contraction map of  $\bar{E}_0$ ; also let  $\bar{h}_{j,0} := h_j \circ \bar{\pi}_0$  for  $j = 1, \dots, m$ . Suppose that  $\bar{X}_k$  and  $\bar{h}_{j,k}$  are obtained for  $0 \leq k < m$  and  $j = 1, \dots, m$ . Let  $\bar{X}_{k+1}$  be the normalization of a surface  $\{(p, z_{k+1}) \in \bar{X}_k \times \mathbb{C} \mid z_{k+1}^{a_{k+1}} = \bar{h}_{k+1,k}(p)\}$ . Let  $\bar{\phi}_{k+1} : \bar{X}_{k+1} \rightarrow \bar{X}_k$  be the natural morphism and  $\bar{h}_{j,k+1} := \bar{h}_{j,k} \circ \bar{\phi}_{k+1}$  for any  $j$ . Let  $\bar{\pi}_k : (\bar{X}_k, \bar{E}_k) \rightarrow (X_k, o)$  be the contraction map of  $\bar{E}_k$ . All singularities of  $\bar{X}_k$  are cyclic quotient singularities contained in  $\bar{E}_k$ . Let  $\eta_m : (\tilde{X}_m, \tilde{E}_m) \rightarrow (\bar{X}_m, \bar{E}_m)$  be the minimal resolution of all cyclic quotient singularities on  $\tilde{X}_m$ . Then,  $\pi_Y := \bar{\pi}_m \circ \eta_m : (\tilde{Y}, F) \rightarrow (Y, o)$  gives the minimal  $\mathbb{C}^*$ -good resolution. We have the following commutative diagram:

$$(4.1) \quad \begin{array}{ccccccc} (\tilde{X}_m, \tilde{E}_m) = (\tilde{Y}, F) & & & & & & (\tilde{X}_0, \tilde{E}_0) = (\tilde{X}, E) \\ \downarrow \eta_m & \searrow \bar{\phi}_m & \cdots & \xrightarrow{\bar{\phi}_2} & \xrightarrow{\bar{\phi}_1} & & \downarrow \eta_0 \\ (\bar{X}_m, \bar{E}_m) & \xrightarrow{\bar{\phi}_m} & \cdots & \xrightarrow{\bar{\phi}_2} & \xrightarrow{\bar{\phi}_1} & (\bar{X}_0, \bar{E}_0) & \\ \downarrow \pi_Y & \downarrow \bar{\pi}_m & & \downarrow \bar{\pi}_1 & & \downarrow \bar{\pi}_0 & \downarrow \pi_X \\ (Y, o) = (X_m, o) & \xrightarrow{\phi_m} & \cdots & \xrightarrow{\phi_2} & \xrightarrow{\phi_1} & (X_0, o) = (X, o) & \end{array}$$

Since  $\bar{\phi}_k$  is a cyclic covering, the  $\mathbb{C}^*$ -action on  $\bar{X}_k$  can be lifted onto  $\bar{X}_{k+1}$  from [22, Lemma 4.4], and (4.1) is a  $\mathbb{C}^*$ -equivariant diagram.

**THEOREM 4.1.** *Under the situation above, assume that  $\frac{d_1}{a_1} \geq \dots \geq \frac{d_m}{a_m}$ .*

- (i)  $(z_1 \circ \pi_Y)_F \geq \dots \geq (z_m \circ \pi_Y)_F$ .
- (ii)  $(z_i \circ \pi_Y)_F = (z_j \circ \pi_Y)_F$  if and only if  $\frac{d_i}{a_i} = \frac{d_j}{a_j}$ .

**PROOF.** (i) Let  $\phi := \phi_1 \circ \dots \circ \phi_m$  and  $\bar{\phi} := \bar{\phi}_1 \circ \dots \circ \bar{\phi}_m$  for (4.1). Let  $e_1, e_2, \dots, e_m$  be positive integers defined inductively as follows:

$$e_1 := d_1, e_2 := \frac{a_1 d_2}{\gcd(a_1, e_1)}, e_3 := \frac{a_1 a_2 d_3}{\gcd(a_1, e_1) \cdot \gcd(a_2, e_2)}, \dots, e_m := \frac{a_1 a_2 \cdots a_{m-1} d_m}{\prod_{i=1}^{m-1} \gcd(a_i, e_i)}.$$

For a fixed  $j$  with  $1 \leq j \leq m$ , apply Lemma 3.7 (ii) to  $\bar{\phi}_k$  for  $k = 1, \dots, j - 1$ . Then we have

$$v_{\bar{E}_k}(\bar{h}_{j,k}) = v_{\bar{E}_k}(\bar{h}_j \circ \bar{\phi}_1 \circ \dots \circ \bar{\phi}_k) = d_j \prod_{\ell=1}^k \frac{a_\ell}{\gcd(a_\ell, e_\ell)} \text{ for } 1 \leq k \leq j - 1.$$

Thus,  $v_{\bar{E}_{j-1}}(\bar{h}_{j,j-1}) = e_j$ . Therefore, from Lemma 3.7 (i), we have

$$v_{\bar{E}_j}(z_j \circ \bar{\pi}_j) = \frac{d_j}{\gcd(a_j, e_j)} \prod_{k=1}^{j-1} \frac{a_k}{\gcd(a_k, e_k)}.$$

If we put  $\psi_j := \bar{\phi}_{j+1} \circ \dots \circ \bar{\phi}_m$  and apply Lemma 3.7 (ii) to  $\bar{\phi}_k$  for  $k = j + 1, \dots, m$ , then

$$\begin{aligned} v_{\bar{E}_m}(z_j \circ \bar{\pi}_j \circ \psi_j) &= \frac{d_j}{\gcd(a_j, e_j)} \prod_{k=1}^{j-1} \frac{a_k}{\gcd(a_k, e_k)} \cdot \frac{a_{j+1}}{\gcd(a_{j+1}, e_{j+1})} \dots \frac{a_m}{\gcd(a_m, e_m)} \\ &= \frac{d_j}{a_j} \prod_{k=1}^m \frac{a_k}{\gcd(a_k, e_k)} \quad \text{for } j = 1, \dots, m. \end{aligned}$$

Hence, if we put  $L := \prod_{k=1}^m \frac{a_k}{\gcd(a_k, e_k)}$ , then

$$(4.2) \quad \deg(z_j) = \text{Coeff}_{F_0}(z_j \circ \pi_Y)_F = v_{\bar{E}_m}(z_j \circ \bar{\pi}_j \circ \psi_j) = \frac{d_j}{a_j} L.$$

Therefore,  $\text{Coeff}_{F_0}(z_1 \circ \pi_Y)_F \geq \dots \geq \text{Coeff}_{F_0}(z_m \circ \pi_Y)_F$  from the assumption  $\frac{d_1}{a_1} \geq \dots \geq \frac{d_m}{a_m}$ . Each  $h_j$  is a reduced element; hence  $z_j$  is a reduced element in  $R_Y$  from Lemma 3.7 (i); also the degree of  $z_j$  in  $R_Y$  is equal to  $\text{Coeff}_{F_0}(z_j \circ \pi_Y)_F$ . Hence (i) is proved by Theorem 3.5 (i). (ii) is obvious from (i).  $\square$

For Brieskorn complete intersection singularities, Theorem 4.1 (i) was already proved by Meng-Okuma ([11, Theorem 6.1]).

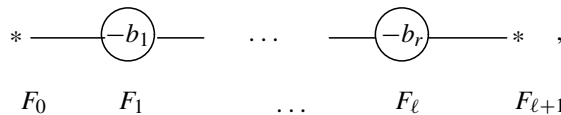
**THEOREM 4.2.** *Under the situation of Theorem 4.1, assume that  $a_m \text{Coeff}_{E_0} M_E \geq d_m$  and  $F_0$  is the central curve of  $F$ . Then we have the following.*

- (i)  $M_F = (z_m \circ \pi_Y)_F$ .
- (ii)  $M_F = Z_F$  if and only if  $\text{Coeff}_{F_0} M_F = \text{Coeff}_{F_0} Z_F$ .
- (iii) If  $F_0 = \mathbb{P}^1$ , then  $M_F = Z_F$ .

**PROOF.** (i) Let  $f$  be an element of  $\mathcal{O}_{X,o}$  with  $(f \circ \pi_X)_E = M_E$ . Let  $f = \sum_j f_j$  be the decomposition to the sum of homogeneous elements of  $\deg(f_j) = \theta_j$  and  $\theta_1 < \theta_2 < \dots$ . Then  $\theta_1 = \text{Coeff}_{E_0} M_E$ . By the same way as above, we can easily see that  $\text{Coeff}_{F_0}(f_j \circ \phi \circ \pi_Y)_F = L\theta_j$ . From the assumption and (4.2), we have

$$\min \{ \text{Coeff}_{F_0}(f_j \circ \phi \circ \pi_Y)_F, \text{Coeff}_{F_0}(z_m \circ \pi_Y)_F \} = \min \left\{ L\theta_j, \frac{Ld_m}{a_m} \right\} = \frac{Ld_m}{a_m}$$

for any  $j$ . Hence,  $\text{Coeff}_{F_0} M_F = \frac{Ld_m}{a_m} = \text{Coeff}_{F_0}(z_m \circ \pi_Y)_F$ . Therefore, it suffices to compare the coefficients of  $M_F$  and  $(z_m \circ \pi_Y)_F$  on irreducible components of  $\mathbb{P}^1$ -chains of  $F$ . Let  $\sum_{i=0}^{\ell+1} F_i$  be a divisor on  $F$  whose weighted dual graph is given as follows:



where  $F_{\ell+1}$  is a curve which contains a 1-dimensional  $\mathbb{C}^*$ -orbit. Put  $a_{j,i} := v_{F_i}(f_j \circ \phi \circ \pi_Y)$ ,  $c_i := v_{F_i}(z_m \circ \pi_Y)$  for  $i = 0, 1, \dots, \ell + 1$ . Then,  $a_{j,0} = L\theta_j \geq L\theta_1 \geq \frac{Ld_m}{a_m} = c_0$ . Since  $z_m$  is reduced,  $(a_{j,\ell+1}, c_{\ell+1}) = (0, 1)$  or  $a_{j,\ell+1} \geq c_{\ell+1}$ . If  $(a_{j,\ell+1}, c_{\ell+1}) = (0, 1)$  (resp.  $a_{j,\ell+1} \geq c_{\ell+1}$ ), then  $a_{j,i} \geq c_i$  for  $i = 1, \dots, \ell$  from Lemma 3.3 (ii) (resp. (i)). Thus  $M_F|_{\bigcup_{i=1}^{\ell} F_i} = (z_m \circ \pi_Y)|_{\bigcup_{i=1}^{\ell} F_i}$ ; consequently  $M_F = (z_m \circ \pi_Y)_F$ .

(ii) Assume that  $\text{Coeff}_{F_0} M_F = \text{Coeff}_{F_0} Z_F$ . Since  $M_F = (z_m \circ \pi_Y)_F$  for a reduced homogeneous element  $z_m$ , we have  $M_F = Z_F$  from Theorem 3.5 (iii).

(iii) From  $E_0 = \mathbb{P}^1$  and Theorem 2.3,  $\text{Coeff}_{F_0} M_F = \text{Coeff}_{F_0} Z_F$ . Then  $M_E = Z_E$  by (ii). □

In the following, let  $(Y, o)$  be a Brieskorn complete intersection singularity defined by  $z_3^{a_3} = h_3, \dots, z_m^{a_m} = h_m$  in  $(\mathbb{C}^m, o)$ , where  $a_1 \leq \dots \leq a_m$  and  $h_j = p_j z_1^{a_1} + q_j z_2^{a_2}$  ( $j = 3, \dots, m$ ) for  $p_i, q_j \in \mathbb{C}^*$  satisfying  $p_i q_j \neq p_j q_i$  for  $i \neq j$ . Let  $(C_j, o)$  be a plane curve singularity defined by  $p_j z_1^{a_1} + q_j z_2^{a_2} = 0$  for any  $j$ . If  $i \neq j$ , then  $C_i \cap C_j = \emptyset$  from  $p_i q_j \neq p_j q_i$ . Put  $r := \frac{a_2}{\gcd(a_1, a_2)}$  and  $s := \frac{a_1}{\gcd(a_1, a_2)}$ . Then,  $(\mathbb{C}^2, o)$  can be considered as a cyclic quotient singularity with  $\mathbb{C}^*$ -action of type  $(r, s)$ . Let us represent it as  $(X, o)$ . Then  $(Y, o)$  is a Kummer covering defined by  $z_3^{a_3} = h_3, \dots, z_m^{a_m} = h_m$  over  $(X, o)$ . Let  $\pi_X : (\tilde{X}, E) \rightarrow (X, o) = (\mathbb{C}^2, o)$  be the minimal  $\mathbb{C}^*$ -good resolution of  $(X, o)$ . Then,  $\pi_X$  coincides with the minimal good embedded resolution of  $(C, o) := \bigcup_{j=1}^m (C_j, o)$  (see [20], [7] or [11]). We can easily see that  $d_1 = \dots = d_m = \text{lcm}(a_1, a_2)$  and  $\text{Coeff}_{E_0} M_E = \text{mult}(C_j, o) = a_1$  from M. Noether's Theorem ([2], p. 518) and  $a_m \geq a_2$ . Hence,

$$(4.3) \quad a_m \text{Coeff}_{E_0} M_E \geq d_m .$$

Therefore, from Theorem 4.2 (iii) and (4.3), we have the following.

**COROLLARY 4.3.** *Let  $(Y, o)$  be a Brieskorn complete intersection singularity as above and  $\pi : (\tilde{Y}, F) \rightarrow (Y, o)$  the minimal  $\mathbb{C}^*$ -good resolution. If the central curve  $F_0$  is  $\mathbb{P}^1$ , then  $M_F = Z_F$ .*

From now on, we prove Theorem 3.1 according to our argument (by Theorem 4.2). Let  $(\tilde{Y}, F) \rightarrow (Y, o)$  be the minimal  $\mathbb{C}^*$ -good resolution. The weighted dual graph of  $F$  was given by M. Jankins and W. Neumann [5] (please refer [11] as a good reference). To review it, let us define some integers as follows:

$$d_0 := \text{lcm}(a_1, \dots, a_m), e_i := \frac{d_0}{a_i}, A_i := \text{lcm}(a_1, \dots, \hat{a}_i, \dots, a_m),$$

$$\hat{g} = \frac{a_1 \cdots a_m}{d_0}, \hat{g}_i := \frac{a_1 \cdots \hat{a}_i \cdots a_m}{A_i} \text{ and } \alpha_i := \frac{a_i}{\gcd(a_i, A_i)} \text{ for } i = 1, \dots, m .$$

(The symbol  $\hat{\phantom{x}}$  in the definition of  $A_i$  and  $\hat{g}_i$  indicates an omitted term.) Then  $e_i$  is equal to the degree of  $z_i$  in  $R_Y$ ; hence  $e_m$  is equal to  $\text{min.deg}(R_Y)$ . Also, let  $\beta_i$  be an integer defined by  $e_i \beta_i + 1 \equiv 0 \pmod{\alpha_i}$  and  $0 \leq \beta_i < \alpha_i$  for each  $i$ , where  $\beta_i = 0$  if and only if  $\alpha_i = 1$ . Let  $g$  be a non-negative integer defined by  $2g - 2 = (m - 2)\hat{g} - \sum_{i=1}^m \hat{g}_i$ ; and  $c_0 := \sum_{i=1}^m \frac{\beta_i \hat{g}_i}{\alpha_i} + \frac{\hat{g}}{d_0}$ .

Then, the type of the weighted dual graph associated to the minimal  $\mathbb{C}^*$ -good resolution is given as follows ([5]):

$$[\overset{\hat{g}_1}{\underbrace{c_0; (\alpha_1, \beta_1), \dots, (\alpha_1, \beta_1)}, \dots, (\alpha_m, \beta_m), \dots, (\alpha_m, \beta_m)}_{\hat{g}_m}].$$

For  $D^{(k)}$  of the Pinkham-Demazure construction, we have

$$\deg(D^{(k)}) = \frac{\hat{g}k}{d_0} - \sum_{i=1}^m \frac{a_1 \cdots a_m}{a_i A_i} \left( \left\lceil \frac{\beta_i k}{\alpha_i} \right\rceil - \frac{\beta_i k}{\alpha_i} \right) = \frac{\hat{g}}{d_0} \left\{ k - \sum_{i=1}^m \frac{d_0}{\gcd(a_i, A_i)} \left( \left\lceil \frac{\beta_i k}{\alpha_i} \right\rceil - \frac{\beta_i k}{\alpha_i} \right) \right\}$$

from  $d_0 = \text{lcm}(a_i, A_i)$ . Since  $\gcd(a_i, A_i) = \frac{a_i}{\alpha_i}$  and  $d_0 = e_i a_i$  for any  $i$ ,

$$(4.4) \quad \deg(D^{(k)}) = \frac{\hat{g}}{d_0} \left\{ k - \sum_{i=1}^m \alpha_i e_i \left( \left\lceil \frac{\beta_i k}{\alpha_i} \right\rceil - \frac{\beta_i k}{\alpha_i} \right) \right\}.$$

If we put  $A := \text{lcm}(a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_m)$  for  $i \neq j$ , then  $\alpha_i = \frac{a_i}{\gcd(a_i, \text{lcm}(a_j, A))}$  and so  $\gcd(\alpha_i, \alpha_j) = 1$  for any  $i \neq j$ . Hence  $\alpha_0(Y, o) = \alpha_1 \cdots \alpha_m$ .

Next we repeat Theorem 3.1 in a slightly different style and prove it according to our argument.

**COROLLARY 4.4** ([11, Theorem 6.1], [7, Theorem 3.2]). *Let  $(Y, o)$  be a Brieskorn complete intersection singularity defined by  $z_3^{a_3} = p_3 z_1^{a_1} + q_3 z_2^{a_2}, \dots, z_m^{a_m} = p_m z_1^{a_1} + q_m z_2^{a_2}$  in  $(\mathbb{C}^m, o)$ , where  $p_i, q_j \in \mathbb{C}^*$  with  $p_i q_j \neq p_j q_i$  for  $i \neq j$ . Assume that  $a_1 \leq a_2 \leq \dots \leq a_m$ . Let  $\pi : (\tilde{Y}, F) \rightarrow (Y, o)$  be the minimal good resolution. Then,  $M_F = (z_m \circ \pi)_F$ . Also, the following three conditions are equivalent.*

- (i)  $M_F = Z_F$ , (ii)  $\text{Coeff}_{F_0} M_F = \text{Coeff}_{F_0} Z_F$ , (iii)  $\min.\deg(R_Y) \leq \alpha_0(Y, o)$ .

**PROOF.** Put  $\alpha := \alpha_0(Y, o)$  and  $\theta := \text{Coeff}_{F_0} Z_F$ . From the assumption  $a_1 \leq a_2 \leq \dots \leq a_m$ , we have  $e_m = \min.\deg(R_Y)$ . If we prove the following:

$$(4.5) \quad \theta = \min\{\alpha, e_m\},$$

then (ii)  $\Leftrightarrow$  (iii) from it and (i)  $\Leftrightarrow$  (ii) from Theorem 4.2 (ii). Therefore, (4.5) completes the proof.

Though (4.5) is already proven in [11, Theorem 5.1], we reprove it by using (4.4) and Theorem 2.3 (i.e.,  $\theta = \min\{k \in \mathbb{N} \mid \deg(D^{(k)}) \geq 0\}$ ). From (4.4), we have

$$(4.6) \quad \frac{d_0}{\hat{g}} \deg(D^{(k)}) = k - \sum_{i=1}^m \alpha_i e_i \left( \left\lceil \frac{\beta_i k}{\alpha_i} \right\rceil - \frac{\beta_i k}{\alpha_i} \right).$$

Then  $\deg(D^{(\alpha)}) = \frac{\hat{g}\alpha}{d_0} > 0$  from (4.4), since  $\alpha_i \mid \alpha$  for any  $i$ . Assume  $\alpha \leq e_m$ . Let  $k$  be any integer with  $0 < k < \alpha$ . Since  $\left\lceil \frac{\beta_{i_0} k}{\alpha_{i_0}} \right\rceil - \frac{\beta_{i_0} k}{\alpha_{i_0}} \geq \frac{1}{\alpha_{i_0}}$  for an  $i_0$  and  $\alpha \leq e_m \leq e_i$  for any  $i$ , we have  $\frac{d_0}{\hat{g}} \deg(D^{(k)}) \leq k - e_{i_0} \leq k - \alpha < 0$  from (4.6). Hence  $\theta = \alpha$  if  $\alpha \leq e_m$ . We have  $\deg(D^{(e_m)}) \geq 0$  because of  $H^0(F_0, \mathcal{O}_{F_0}(D^{(e_m)})) \neq 0$ . Assume  $\alpha \geq e_m$ . Let  $k$  be any integer

with  $0 < k < e_m$ . If  $\alpha_j \mid k$  for any  $j$ , then  $\alpha \mid k$  and  $\alpha < e_m$ ; this is a contradiction. Hence there exists  $j_0$  with  $\alpha_{j_0} \nmid k$ . Since  $\left\lceil \frac{\beta_{j_0} k}{\alpha_{j_0}} \right\rceil - \frac{\beta_{j_0} k}{\alpha_{j_0}} \geq \frac{1}{\alpha_{j_0}}$ ,  $\frac{d_0}{g} \deg(D^{(k)}) \leq k - e_{j_0} \leq k - e_m < 0$ . Hence  $\theta = e_m$  and so we get (4.5) and completes the proof.  $\square$

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