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ON THE UNIVERSAL DEFORMATIONS FOR SL_2-REPRESENTATIONS OF KNOT GROUPS

Dedicated to Professor Kunio Murasugi

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Abstract. Based on the analogies between knot theory and number theory, we study a deformation theory for SL_2-representations of knot groups, following after Mazur's deformation theory of Galois representations. Firstly, by employing the pseudo-SL_2-representations, we prove the existence of the universal deformation of a given SL_2-representation of a finitely generated group Π over a perfect field k whose characteristic is not 2. We then show its connection with the character scheme for SL_2-representations of Π when k is an algebraically closed field. We investigate examples concerning Riley representations of 2-bridge knot groups and give explicit forms of the universal deformations. Finally we discuss the universal deformation of the holonomy representation of a hyperbolic knot group in connection with Thurston's theory on deformations of hyperbolic structures.

Introduction. The motivation of this paper is coming from the analogies between knot theory and number theory. The study of those analogies is now called arithmetic topology ([11]). In particular, it has been known that there are close analogies between Alexander-Fox theory and Iwasawa theory, where the Alexander polynomial and the Iwasawa polynomial (p-adic zeta function) are analogous objects, for instance ([8], [11, Chapters 8–12]). As Mazur pointed out ([10], [11, Chapters 13, 14]), from the viewpoint of group representations, Alexander-Fox theory and Iwasawa theory are concerned about 1-dimensional representations of knot and Galois groups, respectively, and it would be interesting to pursue the analogies further for higher dimensional representations.

As a first step to explore this perspective, in this paper, we study a deformation theory for representations of knot groups, following after the deformation theory for Galois representations ([9]). In fact, we develop a general theory on deformations for SL_2-representations of a finitely generated group. We deal with only SL_2-representations, since our main interest is applications to knot theory and 3-dimensional topology (hyperbolic geometry) where the character varieties of SL_2-representations of fundamental groups have often been studied.

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See [3] for example. We note that while Galois deformation theory is concerned with p-adic deformation of a continuous representation of a profinite group over a finite residue field, our work deals with infinitesimal deformation of a representation of any finitely generated group over any perfect residue field whose characteristic is not 2 (for example, the field of complex numbers), and that our universal deformation space may be regarded as an infinitesimal (1-parameter) deformation over a complete local algebra of the character variety over a filed (see Theorem 3.2.1). Thus our work is applicable to geometry and topology.

The contents of this paper are as follows. In Section 1, following Wiles ([23] for GL_2 case) and Taylor ([20] for GL_n case), we introduce the notion of a pseudo-SL_2-representation of Π over a commutative ring and prove the existence of the universal deformation of a given pseudo-SL_2-representation over a perfect field. In Section 2, for a given representation over a perfect field k whose characteristic is not 2

$$\overline{\rho} : \Pi \longrightarrow \mathrm{SL}_2(k)\,,$$

we prove, using the result in Section 1, that there exists the universal deformation of $\overline{\rho}$

$$\boldsymbol{\rho} : \Pi \longrightarrow \mathrm{SL}_2(\boldsymbol{R}_{\overline{\rho}})\,,$$

which parametrizes all lifts of $\overline{\rho}$ to SL_2-representations over complete local $\mathcal{O}$-algebras where $\mathcal{O}$ a complete discrete valuation ring whose residue field is k. A merit to make use of pseudo-representations is to enable us to relate the universal deformation ring with the character scheme/variety of SL_2-representations where the latter has been extensively studied in the context of topology (e.g., [3], [6], [16] etc). In fact, in Section 3, when k is an algebraically closed field, we show the relation between the universal deformation ring $\boldsymbol{R}_{\overline{\rho}}$ and the SL_2-character scheme of Π. In Section 4, we investigate examples concerning Riley representations of 2-bridge knot groups ([17]) and give explicit forms of universal deformations. In Section 5, we apply our deformation theory to the case where Π is the fundamental group of the complement of a hyperbolic knot in the 3-sphere and $\overline{\rho}$ is the associated holonomy representation, and describe the universal deformation ring by Thurston's deformation theory of hyperbolic structures ([21]). We observe that our result is similar to the case of p-adic ordinary Galois representations where the universal deformation is described by Hida's deformation of p-adic ordinary modular forms ([4], [5]).

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NOTATION. For a local ring R, we denote by $\mathfrak{m}_R$ the maximal ideal of R. For an integral domain k, we denote by $\mathrm{char}(k)$ the characteristic of k.

1. Pseudo-representations and their deformations. In Sections 1 and 2, we develop a deformation theory of representations for any finitely generated group. We consider only SL_2-representations, since our main concern is applications to knot theory and 3-dimensional topology where SL_2-representations of fundamental groups have been often studied. See [3] for example. Moreover, while Galois deformation theory is concerned with p-adic deformation of a continuous representation of a profinite group over a finite residue field, we study infinitesimal deformation of a representation of any finitely generated group over any perfect residue field whose characteristic is not 2 (for example, the field of complex numbers). Thus our work is applicable to geometry and topology.

In Subsection 1.1, we introduce the notion of a pseudo-SL_2-representation of a finitely generated group. This notion was originally introduced by Wiles ([23] for GL_2 case) and by Taylor ([20] for GL_n case) with the intention of applications to p-adic Galois representations. In Subsection 1.2, we show the existence of the universal deformation of a given pseudo-SL_2-representation over any perfect field.

1.1. *Pseudo-*SL_2*-representations.* Let Π be a finitely generated group. Let A be a commutative ring with identity. A map $T : \Pi \to A$ is called a *pseudo-*SL_2*-representation* over A if the following four conditions are satisfied:

(P1) $T(1) = 2$,
(P2) $T(g_1 g_2) = T(g_2 g_1)$ for any $g_1, g_2 \in \Pi$,
(P3) $T(g_1)T(g_2)T(g_3) + T(g_1 g_2 g_3) + T(g_1 g_3 g_2) - T(g_1 g_2)T(g_3) - T(g_2 g_3)T(g_1)$
$- T(g_1 g_3)T(g_2) = 0$ for any $g_1, g_2, g_3 \in \Pi$,
(P4) $T(g)^2 - T(g^2) = 2$ for any $g \in \Pi$.

Note that the conditions (P1) through (P3) are nothing but Taylor's conditions for a pseudo-representation of degree 2 ([20]) and that (P4) is the condition for determinant 1. By the invariant theory of matrices ([15, Theorem 4.3]), the trace $\mathrm{tr}(\rho)$ of a representation $\rho : \Pi \to SL_2(A)$ satisfies the conditions (P1) through (P4). Conversely, a pseudo-SL_2-representation is shown to be obtained as the trace of a representation under certain conditions (See Theorem 2.2.1 below).

1.2. *Deformations of pseudo-*SL_2*-representations.* We fix a perfect field k and a complete discrete valuation ring $\mathcal{O}$ with the residue field $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$. We may take $\mathcal{O}$ to be the Witt ring of k if $\mathrm{char}(\mathcal{O}) \neq \mathrm{char}(k)$, and $\mathcal{O} = k[[\hbar]]$, the formal power series ring of a variable $\hbar$ over k, if $\mathrm{char}(\mathcal{O}) = \mathrm{char}(k)$. There is a unique subgroup V of $\mathcal{O}^\times$ such that $k^\times \simeq V$ and $\mathcal{O}^\times = V \times (1 + \mathfrak{m}_{\mathcal{O}})$. The composition map $\varphi : k^\times \simeq V \hookrightarrow \mathcal{O}^\times$ is called the *Teichmüller lift* which satisfies $\varphi(\alpha) \bmod \mathfrak{m}_{\mathcal{O}} = \alpha$ for $\alpha \in k$. It is extended to $\varphi : k \hookrightarrow \mathcal{O}$ by $\varphi(0) := 0$. Let $\mathcal{C}$ be the category of complete local $\mathcal{O}$-algebras with residue field k. A morphism in $\mathcal{C}$ is an $\mathcal{O}$-algebra homomorphism inducing the identity on residue fields.

Let $\overline{T} : \Pi \to k$ be a pseudo-SL_2-representation over k. A couple (R, T) is called an SL_2*-deformation* of $\overline{T}$ if $R \in \mathcal{C}$ and $T : \Pi \to R$ is a pseudo-SL_2-representation over R such that $T \bmod \mathfrak{m}_R = \overline{T}$. In the following, we say simply a *deformation* of $\overline{T}$ for an SL_2-deformation. A deformation $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$ of $\overline{T}$ is called a *universal deformation* of $\overline{T}$ if the following universal property is satisfied: "For any deformation (R, T) of $\overline{T}$ there exists a unique morphism $\psi : \boldsymbol{R}_{\overline{T}} \to R$ in $\mathcal{C}$ such that $\psi \circ \boldsymbol{T} = T$." So the correspondence $\psi \mapsto \psi \circ \boldsymbol{T}$ gives the bijection

$$\mathrm{Hom}_{\mathcal{C}}(\boldsymbol{R}_{\overline{T}}, R) \simeq \{(R, T) \mid \text{deformation of } \overline{T}\}.$$

Note that a universal deformation of $\overline{T}$ is unique (if it exists) up to $\mathcal{O}$-isomorphism in the obvious sense. The $\mathcal{O}$-algebra $\boldsymbol{R}_{\overline{T}}$ is called the *universal deformation ring* of $\overline{T}$.

THEOREM 1.2.1. *For a pseudo-*SL_2*-representation* $\overline{T} : \Pi \to k$*, there exists a universal deformation* $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$ *of* $\overline{T}$*.*

PROOF. Let $\mathcal{R} := \mathcal{O}[[X_g; g \in \Pi]]$ be the ring of formal power series over $\mathcal{O}$ with variables X_g indexed by elements of Π. By definition, the ring $\mathcal{R}$ consists of formal power series of variables X_{g_i}'s where indices g_i's belong to a finite subset of G. Let $\varphi : k \hookrightarrow \mathcal{O}$ be the Teichmüller lift. We set $T_g := X_g + \varphi(\overline{T}(g))$ for $g \in G$. Consider the ideal $\mathcal{I}$ of $\mathcal{R}$ generated by the elements of following type:

(1) $T_1 - 2 = X_1 + \varphi(\overline{T}(1)) - 2$,

(2) $T_{g_1 g_2} - T_{g_2 g_1} = X_{g_1 g_2} - X_{g_2 g_1}$,

(3) $T_{g_1} T_{g_2} T_{g_3} + T_{g_1 g_2 g_3} + T_{g_1 g_3 g_2} - T_{g_1 g_2} T_{g_3} - T_{g_2 g_3} T_{g_1} - T_{g_1 g_3} T_{g_2}$,

(4) $T_g^2 - T_{g^2} - 2$,

where $g, g_1, g_2, g_3 \in \Pi$. We then set $\boldsymbol{R}_{\overline{T}} := \mathcal{R}/\mathcal{I}$ and define a map $\boldsymbol{T} : \Pi \to \boldsymbol{R}_{\overline{T}}$ by $\boldsymbol{T}(g) := T_g \bmod \mathcal{I}$. Then we note that $\boldsymbol{R}_{\overline{T}} \in \mathcal{C}$, and by the conditions (P1) through (P4), $\boldsymbol{T} : \Pi \to \boldsymbol{R}_{\overline{T}}$ is a pseudo-SL_2-representation and $\boldsymbol{T} \bmod \mathfrak{m}_{\boldsymbol{R}_{\overline{T}}} = \overline{T}$. Hence $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$ is a deformation of $\overline{T}$.

Next let (R, T) be any deformation of $\overline{T}$. Define a morphism $\psi : \mathcal{R} \to R$ in $\mathcal{C}$ by $\psi(f(X_g)) := f(T_g - \varphi(\overline{T}(g)))$ for $f(X_g) \in \mathcal{R}$. Note that $T_g - \varphi(\overline{T}(g)) \in \mathfrak{m}_R$ and hence $f(T_g - \varphi(\overline{T}(g)))$ is well-defined since R is complete with respect to the $\mathfrak{m}_R$-adic topology. By (P1) through (P4), $\psi(\mathcal{I}) = 0$ and hence we have the induced $\mathcal{O}$-algebra homomorphism in $\mathcal{C}$, denoted by the same ψ, $\psi : \boldsymbol{R}_{\overline{T}} \to R$. Then we easily see that $\psi \circ \boldsymbol{T} = T$. The uniqueness of ψ follows from the fact that $\boldsymbol{R}_{\overline{T}}$ is generated by X_g $(g \in \Pi)$ as an $\mathcal{O}$-algebra. □

2. The universal deformation for representations. In this section, we are concerned with deformations of SL_2-representations of a finitely generated group Π.

In Subsection 2.1, we recall two theorems due to Carayol [2] and Nyssen [13]. In Subsection 2.2, by using them, we prove that there is a bijective correspondence given by the trace between SL_2-representations (up to strict equivalence) and pseudo-SL_2-representations, and then derive the existence of the universal deformation of an SL_2-representation over a field.

2.1. *Carayol's and Nyssen's theorems.* Two representations $\rho, \rho' : \Pi \to \mathrm{GL}_n(A)$ over a commutative ring A with identity are said to be *equivalent*, written as $\rho \sim \rho'$, if there is $\gamma \in \mathrm{GL}_n(A)$ such that $\rho'(g) = \gamma^{-1}\rho(g)\gamma$ for any $g \in \Pi$. When A is a local ring, ρ, ρ' are said to be *strictly equivalent*, written as $\rho \approx \rho'$, if there is $\gamma \in I_n + \mathrm{M}_n(\mathfrak{m}_A)$ such that $\rho'(g) = \gamma^{-1}\rho(g)\gamma$ for any $g \in \Pi$. We say that a representation $\rho : \Pi \to \mathrm{GL}_n(k)$ over a field k is *absolutely irreducible* if for an algebraic closure $\overline{k}$ of k the composite of ρ with the inclusion $\mathrm{GL}_n(k) \hookrightarrow \mathrm{GL}_n(\overline{k})$ is an irreducible representation. This condition is independent of the choice of an algebraic closure $\overline{k}$. We recall the following theorem due to Carayol and Serre.

THEOREM 2.1.1 ([2, Theorem 1]). *Let $\rho, \rho' : \Pi \to GL_n(A)$ be representations over a local ring A with the residue field $k = A/\mathfrak{m}_A$. If the residual representation ρ mod $\mathfrak{m}_A : \Pi \to GL_n(k)$ is absolutely irreducible and* $tr(\rho) = tr(\rho')$*, then we have $\rho \sim \rho'$.*

Next we recall the degree 2 case of a theorem by Nyssen.

THEOREM 2.1.2 ([13, Theorem 1]). *Let A be a Henselian separated local ring with the residue field $k := A/\mathfrak{m}_A$ and let $T : \Pi \to A$ be a Taylor's pseudo-representation of degree* 2 *over A. Assume that there is an absolutely irreducible representation $\overline{\rho} : \Pi \to GL_2(k)$ such that* $tr(\overline{\rho}) = T$ *mod $\mathfrak{m}_A$. Then there exists a unique representation $\rho : \Pi \to GL_2(A)$ such that* $tr(\rho) = T$.

2.2. *Deformations of an* SL_2*-representations.* As in Subsection 1.2, let us fix a perfect field k and a complete discrete valuation ring $\mathcal{O}$ with the residue field $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$. We assume $\mathrm{char}(k) \neq 2$. Let $\mathcal{C}$ be the category of complete local $\mathcal{O}$-algebras with residue field k where a morphism is an $\mathcal{O}$-algebra homomorphism inducing the identity on residue fields. We note that 2 is invertible in $\mathcal{O}$ and hence in any $R \in \mathcal{C}$. Let $\overline{\rho} : \Pi \to SL_2(k)$ be a given representation. We call a couple (R, ρ) an SL_2-*deformation* of $\overline{\rho}$ if $R \in \mathcal{C}$ and $\rho : \Pi \to SL_2(R)$ is a representation such that $\rho \bmod \mathfrak{m}_R = \overline{\rho}$. In the following, as in the case of pseudo-SL_2-representations, we say simply a *deformation* of $\overline{\rho}$ for an SL_2-deformation. A deformation $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ of $\overline{\rho}$ is called a *universal deformation* of $\overline{\rho}$ if the following universal property is satisfied: "For any deformation (R, ρ) of $\overline{\rho}$ there exists a unique morphism $\psi : \boldsymbol{R}_{\overline{\rho}} \to R$ in $\mathcal{C}$ such that $\psi \circ \boldsymbol{\rho} \approx \rho$." So the correspondence $\psi \mapsto \psi \circ \boldsymbol{\rho}$ gives the bijection

$$\mathrm{Hom}_{\mathcal{C}}(\boldsymbol{R}_{\overline{\rho}}, R) \simeq \{(R, \rho) \mid \text{deformation of } \overline{\rho}\}/\approx .$$

Note that a universal deformation of $\overline{\rho}$ is unique (if it exists) up to $\mathcal{O}$-isomorphism in the obvious sense. The $\mathcal{O}$-algebra $\boldsymbol{R}_{\overline{\rho}}$ is called the *universal deformation ring* of $\overline{\rho}$.

A deformation (R, ρ) of $\overline{\rho}$ gives rise to a deformation $(R, \mathrm{tr}(\rho))$ of the pseudo-SL_2-representation $\mathrm{tr}(\overline{\rho}) : \Pi \to k$. The following theorem asserts that this correspondence is actually bijective under the assumption that $\overline{\rho}$ is absolutely irreducible.

THEOREM 2.2.1. *Let $\overline{\rho} : \Pi \to SL_2(k)$ be an absolutely irreducible representation and let $R \in \mathcal{C}$. Then the correspondence $\rho \mapsto$* $tr(\rho)$ *gives the following bijection*:

$$\{\rho : \Pi \to SL_2(R) \mid \text{deformation of } \overline{\rho} \text{ over } R\}/\approx \\ \longrightarrow \{T : \Pi \to R \mid \text{deformation of } \mathrm{tr}(\overline{\rho}) \text{ over } R\} .$$

PROOF. Firstly let us show the surjectivity. Let $T : \Pi \to R$ be a pseudo-SL_2-representation such that $T \bmod \mathfrak{m}_R = \mathrm{tr}(\overline{\rho})$. By Theorem 2.1.2, there exists a unique representation $\rho_1 : \Pi \to GL_2(R)$ such that $\mathrm{tr}(\rho_1) = T$. Note that ρ_1 is actually an $SL_2(R)$-representation, because we have $2\det(\rho_1(g)) = \mathrm{tr}(\rho_1(g))^2 - \mathrm{tr}(\rho_1(g^2)) = T(g)^2 - T(g^2) = 2$ for any $g \in \Pi$ and $2 \in R^\times$. Since $\mathrm{tr}(\rho_1 \bmod \mathfrak{m}_R) = T \bmod \mathfrak{m}_R = \mathrm{tr}(\overline{\rho})$ and $\overline{\rho}$ is absolutely irreducible, Theorem 2.1.1 implies that $\overline{\rho} \sim \rho_1 \bmod \mathfrak{m}_R$. So there is $\overline{\gamma} \in GL_2(k)$ such that $\overline{\rho}(g) = \overline{\gamma}^{-1}(\rho_1 \bmod \mathfrak{m}_R)(g)\overline{\gamma}$. Choose a lift $\gamma \in GL_2(R)$ of $\overline{\gamma}$ and define a representation

$\rho : \Pi \to \mathrm{SL}_2(R)$ by $\rho(g) := \gamma^{-1}\rho_1(g)\gamma$ for $g \in \Pi$. Then (R, ρ) is a deformation of $\overline{\rho}$ and $\mathrm{tr}(\rho) = \mathrm{tr}(\rho_1) = T$.

Next let us show the injectivity. Let $\rho, \rho' : \Pi \to \mathrm{SL}_2(R)$ be deformations of $\overline{\rho}$ such that $\mathrm{tr}(\rho) = \mathrm{tr}(\rho')$. Since $\overline{\rho}$ is absolutely irreducible, Theorem 2.1.1 implies $\rho \sim \rho'$. So there is $\gamma \in \mathrm{GL}_2(R)$ such that $\rho'(g) = \gamma^{-1}\rho(g)\gamma$ for $g \in \Pi$. Taking mod $\mathfrak{m}_R$, we have $\overline{\rho}(g) = \overline{\gamma}^{-1}\overline{\rho}(g)\overline{\gamma}$ for $g \in \Pi$ where we put $\overline{\gamma} := \gamma$ mod $\mathfrak{m}_R$. Since $\overline{\rho}$ is irreducible, Schur's lemma implies that $\overline{\gamma}$ is a scaler matrix over k, say $\overline{\gamma} = \overline{a}I_2$, $\overline{a} \in k^\times$. Take a lift $a \in R^\times$ of $\overline{a}$ and set $\gamma' := aI_2$. Then $\gamma\gamma'^{-1} \equiv I_2$ mod $\mathfrak{m}_R$ and $\rho'(g) = (\gamma\gamma'^{-1})^{-1}\rho(g)(\gamma\gamma'^{-1})$ for $g \in \Pi$. Hence $\rho' \approx \rho$. □

THEOREM 2.2.2. *Let $\overline{\rho} : \Pi \to \mathrm{SL}_2(k)$ be an absolutely irreducible representation. Then there exists the universal deformation $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ of $\overline{\rho}$, where $\boldsymbol{R}_{\overline{\rho}}$ is given as $\boldsymbol{R}_{\overline{T}}$ for $\overline{T} :=$* $\mathrm{tr}(\overline{\rho})$ *in Theorem* 1.2.1.

PROOF. By Theorem 1.2.1, there exists the universal deformation $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$ of a pseudo-SL_2-representation $\overline{T} = \mathrm{tr}(\overline{\rho})$. By Theorem 2.2.1, we have a deformation $\boldsymbol{\rho} : \Pi \to \mathrm{SL}_2(\boldsymbol{R}_{\overline{T}})$ of $\overline{\rho}$ such that $\mathrm{tr}(\boldsymbol{\rho}) = \boldsymbol{T}$. We claim that $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{\rho})$ is the universal deformation of $\overline{\rho}$ and hence $\boldsymbol{R}_{\overline{\rho}} = \boldsymbol{R}_{\overline{T}}$. Let (R, ρ) be any deformation of $\overline{\rho}$. By the universality of $(\boldsymbol{R}_{\overline{T}}, \boldsymbol{T})$, there exists a unique morphism $\psi : \boldsymbol{R}_{\overline{T}} \to R$ in $\mathcal{C}$ such that $\psi \circ \boldsymbol{T} = \mathrm{tr}(\rho)$. Since $\mathrm{tr}(\psi \circ \boldsymbol{\rho}) = \psi \circ \mathrm{tr}(\boldsymbol{\rho}) = \psi \circ \boldsymbol{T} = \mathrm{tr}(\rho)$, Theorem 2.2.1 implies $\psi \circ \boldsymbol{\rho} \approx \rho$. □

Finally we recall a basic fact on a presentation of a complete local $\mathcal{O}$-algebra, which will be used later. For $R \in \mathcal{C}$, we define the *relative cotangent space* $\mathfrak{t}^*_{R/\mathcal{O}}$ of R by the k-vector space $\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}}R)$ and the *relative tangent space* $\mathfrak{t}_{R/\mathcal{O}}$ of R by the dual k-vector space of $\mathfrak{t}^*_{R/\mathcal{O}}$. We note that they are the same as the cotangent and tangent spaces of $R/\mathfrak{m}_{\mathcal{O}}R = R \otimes_{\mathcal{O}} k$, respectively. The following Lemma 2.2.3 may be a well-known fact which is proved using Nakayama's lemma (cf. [22, Lemma 5.1]).

LEMMA 2.2.3. *Let d be the dimension of $\mathfrak{t}_{R/\mathcal{O}}$ over k and assume $d < \infty$. For a given system of parameters $x_1, \ldots, x_d$ of the local k-algebra $R \otimes_{\mathcal{O}} k$, there is a surjective $\mathcal{O}$-algebra homomorphism*

$$\lambda : \mathcal{O}[[X_1, \ldots, X_d]] \longrightarrow R$$

in $\mathcal{C}$ such that the image of $\lambda(X_i)$ in $R \otimes_{\mathcal{O}} k$ is x_i $(1 \le i \le d)$.

3. Character schemes. In this section, we show the relation between the universal deformation ring in Sections 1, 2 and the character scheme of SL_2-representations.

In Subsection 3.1, we recall the constructions and some facts concerning the SL_2-character scheme and the skein algebra over an algebraically closed field and then describe their relation. For the details, we consult [3], [7, Chapter 1], [12] and [19]. In Subsection 3.2, we give the relation between the universal deformation ring and the character scheme via the skein algebra. Our universal deformation ring may be regarded as an infinitesimal (1-parameter) deformation of the universal character algebra.

As in Sections 1 and 2, let Π be a finitely generated group.

3.1. *Character schemes and skein algebras.* Let k be an algebraically closed field and consider the functor $\mathfrak{F}$ from the category of commutative k-algebras to the category of sets defined by

$$A \mapsto \mathfrak{F}(A) := \text{the set of all representations } \Pi \to \mathrm{SL}_2(A)\,.$$

The functor $\mathfrak{F}$ is represented by a commutative k-algebra $\mathfrak{A}(\Pi)$, called the *universal* SL_2-*representation algebra* of Π over k, and we have the *universal* SL_2-*representation* $\rho^{\mathrm{univ}} : \Pi \to \mathrm{SL}_2(\mathfrak{A}(\Pi))$ which satisfies the following property: "For any commutative k-algebra A and a representation $\rho : \Pi \to \mathrm{SL}_2(A)$, there is a unique k-algebra homomorphism $\psi : \mathfrak{A}(\Pi) \to A$ such that $\rho = \psi \circ \rho^{\mathrm{univ}}$." In fact, when Π is given by generators $g_1, \ldots, g_s$ subject to the relations $r_q = 1$ $(q \in Q)$, the universal SL_2-representation algebra $\mathfrak{A}(\Pi)$ is given by the quotient of the polynomial ring $k[X_{ij}^{(m)}\ (1 \le m \le s, 1 \le i, j \le 2)]$ by the ideal J generated by $\det(X^{(m)}) - 1\ (1 \le m \le s)$ and $(r_q)_{ij}$ $(q \in Q, 1 \le i, j \le 2)$, where $X^{(m)} := (X_{ij}^{(m)})$ and $(r_q)_{ij}$ denotes the (i, j)-entry of the matrix $r_q(X^{(1)}, \ldots, X^{(s)})$, and the universal representation ρ^{univ} is given by $\rho^{\mathrm{univ}}(g_m) = X^{(m)} \bmod J$ for $1 \le m \le s$. We denote the affine scheme $\mathrm{Spec}(\mathfrak{A}(\Pi))$ by $\mathfrak{R}(\Pi)$ and call it the SL_2-*representation scheme* of Π over k. We identify a prime ideal $\mathfrak{p} \in \mathfrak{R}(\Pi)$ with the corresponding representation $\rho_{\mathfrak{p}} := \psi_{\mathfrak{p}} \circ \rho^{\mathrm{univ}} : \Pi \to \mathrm{SL}_2(\mathfrak{A}(\Pi)/\mathfrak{p})$, where $\psi_{\mathfrak{p}} : \mathfrak{A}(\Pi) \to \mathfrak{A}(\Pi)/\mathfrak{p}$ is the natural homomorphism. We set $\mathcal{R}(\Pi) := \mathfrak{R}(\Pi)(k) = \mathrm{Spm}(\mathfrak{A}(\Pi))$ and call it the $\mathrm{SL}_2(k)$-*representation variety* of Π. It is an affine algebraic set over k which parametrizes all representations $\Pi \to \mathrm{SL}_2(k)$, obtained as $\rho_{\mathfrak{m}} = \psi_{\mathfrak{m}} \circ \rho^{\mathrm{univ}}$ for $\mathfrak{m} \in \mathcal{R}(\Pi)$, where $\psi_{\mathfrak{m}} : \mathfrak{A}(\Pi) \to \mathfrak{A}(\Pi)/\mathfrak{m} = k$ is the natural homomorphism. We identify a maximal ideal $\mathfrak{m} \in \mathcal{R}(\Pi)$ and the corresponding representation $\rho_{\mathfrak{m}}$. We denote by $k[\mathcal{R}(\Pi)]$ the coordinate ring of $\mathcal{R}(\Pi)$. We note that $k[\mathcal{R}(\Pi)]$ is the quotient of $\mathfrak{A}(\Pi)$ by the nilradical, $k[\mathcal{R}(\Pi)] = \mathfrak{A}(\Pi)/\sqrt{0}$.

The adjoint action of the group scheme GL_2 on $\mathfrak{B}(\Pi)$ is defined by sending the (i, j)-entry of $X^{(m)}$ to the (i, j)-entry of $P^{-1}X^{(m)}P$ for $P \in \mathrm{GL}_2$. Let $\mathfrak{B}(\Pi)$ be the invariant subalgebra of $\mathfrak{A}(\Pi)$ under this action of GL_2, $\mathfrak{B}(\Pi) := \mathfrak{A}(\Pi)^{\mathrm{GL}_2}$, which we call the *universal* SL_2-*character algebra* of Π over k. We denote by $\mathfrak{X}(\Pi)$ the affine scheme $\mathrm{Spec}(\mathfrak{B}(\Pi))$, namely, the algebro-geometric quotient of $\mathfrak{R}(\Pi)$ by the adjoint action of GL_2, and call it the SL_2-*character scheme* of Π over k. We have a morphism $\mathfrak{R}(\Pi) \to \mathfrak{X}(\Pi)$ induced by the inclusion $\mathfrak{B}(\Pi) \hookrightarrow \mathfrak{A}(\Pi)$. We write $[\mathfrak{p}](= [\rho_{\mathfrak{p}}])$ for the image of $\mathfrak{p}(= \rho_{\mathfrak{p}})$. We set $\mathcal{X}(\Pi) := \mathfrak{X}(\Pi)(k) = \mathrm{Spm}(\mathfrak{B}(\Pi))$ and call it the $\mathrm{SL}_2(k)$-*character variety* of Π. It is an algebraic set which parametrizes all characters $\mathrm{tr}(\rho)$ of representations $\rho : \Pi \to \mathrm{SL}_2(k)$. Under the natural morphism $\mathcal{R}(\Pi) \to \mathcal{X}(\Pi)$, we write $[\rho] \in \mathcal{X}(\Pi)$ for the image of $\rho \in \mathcal{R}(\Pi)$. We note that $[\rho] = [\rho']$ if and only if $\mathrm{tr}(\rho) = \mathrm{tr}(\rho')$. We denote by $k[\mathcal{X}(\Pi)]$ the coordinate ring of $\mathcal{X}(\Pi)$. We note that $k[\mathcal{X}(\Pi)]$ is the invariant subring of $k[\mathcal{R}(\Pi)]$ under the conjugate action of $\mathrm{GL}_2(k)$, $k[\mathcal{X}(\Pi)] = k[\mathcal{R}(\Pi)]^{\mathrm{GL}_2(k)}$ and $k[\mathcal{X}(\Pi)] = \mathfrak{B}(\Pi)/\sqrt{0}$. For $g \in \Pi$, define $\tau_g : \mathcal{R}(\Pi) \to k$ by $\tau_g(\rho) := \mathrm{tr}(\rho(g))$. It is known ([LM, Corollary 1.34]) that $k[\mathcal{X}(\Pi)]$ is generated over k by finitely many τ_g's.

According to [19, 3.1] and [16, Definition 2.5], we define the k-algebra $\mathfrak{C}(\Pi)$ by

$$\mathfrak{C}(\Pi) := k[t_g\ (g \in \Pi)]/I\,,$$

where t_g is a variable for each $g \in \Pi$ and I is the ideal of the polynomial ring $k[t_g\ (g \in \Pi)]$ generated by the polynomials of the form

$$t_1 - 2\,, \quad t_{g_1}t_{g_2} - t_{g_1g_2} - t_{g_1^{-1}g_2}\ (g_1, g_2 \in \Pi)\,.$$

We call $\mathfrak{C}(\Pi)$ the *skein algebra* of Π over k. We note that $\mathfrak{C}(\Pi)$ is Noetherian since Π is finitely generated. We denote the affine scheme $\mathrm{Spec}(\mathfrak{C}(\Pi))$ by $\mathfrak{X}^{\mathrm{skein}}(\Pi)$ and call it the *skein scheme* of Π over k. Since $\mathrm{tr}(\rho^{\mathrm{univ}}(g)) \in \mathfrak{B}(\Pi)$ for $g \in \Pi$ and we have the formula, which is derived by the Cayley-Hamilton theorem,

$$\mathrm{tr}(\rho^{\mathrm{univ}}(g_1))\mathrm{tr}(\rho^{\mathrm{univ}}(g_2)) - \mathrm{tr}(\rho^{\mathrm{univ}}(g_1g_2)) - \mathrm{tr}(\rho^{\mathrm{univ}}(g_1^{-1}g_2)) = 0\ \ (g_1, g_2 \in \Pi)\,,$$

we obtain a k-algebra homomorphism

$$\iota_\Pi : \mathfrak{C}(\Pi) \longrightarrow \mathfrak{B}(\Pi)$$

defined by $\iota(t_g) := \mathrm{tr}(\rho^{\mathrm{univ}}(g))$ for $g \in \Pi$, and hence a morphism of schemes

$$\iota_\Pi^a : \mathfrak{X}(\Pi) \longrightarrow \mathfrak{X}^{\mathrm{skein}}(\Pi)\,.$$

We define the *discriminant ideal* $\Delta(\Pi)$ of $\mathfrak{C}(\Pi)$ by the ideal generated by the images of the elements in $k[t_g\ (g \in \pi)]$ of the form

$$\Delta(g_1, g_2) := t_{g_1g_2g_1^{-1}g_2^{-1}} - 2 = t_{g_1}^2 + t_{g_2}^2 + t_{g_1g_2}^2 - t_{g_1}t_{g_2}t_{g_1g_2} - 4\ \ (g_1, g_2 \in \pi)\,,$$

and the *discrimiant subscheme* by $V(\Delta(\Pi)) = \mathrm{Spec}(\mathfrak{C}(\Pi)/\Delta(\Pi))$. Since $\mathfrak{C}(\Pi)$ is Noetherian, Δ is generated by finitely many $\Delta(g_1^{(i)}, g_2^{(i)})$, $i = 1, \ldots, n$. We set $\Delta := \Delta(g_1^{(1)}, g_2^{(2)}) \cdots \Delta(g_1^{(n)}, g_2^{(n)}) \in \mathfrak{C}(\Pi)$ and define the open subschemes $\mathfrak{X}^{\mathrm{skein}}(\Pi)_{\mathrm{irr}}$ and $\mathfrak{X}(\Pi)_{\mathrm{irr}}$ of $\mathfrak{X}^{\mathrm{skein}}(\Pi)$ and $\mathfrak{X}(\Pi)$, respectively, by

$$\mathfrak{X}^{\mathrm{skein}}(\Pi)_{\mathrm{irr}} := \mathfrak{X}^{\mathrm{skein}}(\Pi) \setminus V(\Delta(\Pi)) = \mathfrak{X}^{\mathrm{skein}}(\Pi)_\Delta\,,$$
$$\mathfrak{X}(\Pi)_{\mathrm{irr}} := \mathfrak{X}(\Pi) \setminus (\iota_\Pi^a)^{-1}(V(\Delta(\Pi))) = \mathfrak{X}(\Pi)_{\iota_\Pi(\Delta)}\,.$$

In fact, it is shown ([19, 4.1], [12, §3]) that $\mathfrak{p} \in \mathfrak{X}(\Pi)$ belongs to $\mathfrak{X}(\Pi)_{\mathrm{irr}}$ if and only if $\rho_\mathfrak{p}$ is an absolutely irreducible representation. Here a representation $\rho : \Pi \to \mathrm{SL}_2(A)$ with a commutative ring A is said to be absolutely irreducible if the composite of ρ with the natural map $\mathrm{SL}_2(A) \to \mathrm{SL}_2(\kappa(\mathfrak{p}))$ is absolutely irreducible over the residue field $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$.

THEOREM 3.1.1 ([19, 4.3], [12, Corollary 6.8]). *The restriction of ι_Π^a to $\mathfrak{X}(\Pi)_{\mathrm{irr}}$ gives an isomorphism*

$$\mathfrak{X}(\Pi)_{\mathrm{irr}} \simeq \mathfrak{X}^{\mathrm{skein}}(\Pi)_{\mathrm{irr}}\,.$$

In terms of algebras, ι_Π induces an isomorphism between $\mathfrak{C}(\Pi)$ and $\mathfrak{B}(\Pi)$ if Δ is inverted:

$$\mathfrak{C}(\Pi)_\Delta \simeq \mathfrak{B}(\Pi)_{\iota_\Pi(\Delta)}\,.$$

COROLLARY 3.1.2. *Let $\rho : \Pi \to \mathrm{SL}_2(k)$ be an irreducible representation and let $[\rho] \in \mathcal{X}(\Pi)$ also denote the corresponding maximal ideal of $\mathfrak{B}(\Pi)$. Then we have an isomorphism of local rings*:

$$\mathfrak{C}(\Pi)_{\iota_\Pi^a([\rho])} \simeq \mathfrak{B}(\Pi)_{[\rho]}\,.$$

3.2. *The relation between the universal deformation ring and the character scheme.* Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$ and let $\mathcal{O}$ be a discrete valuation ring with residue field k. Let $\overline{\rho} : \Pi \to \mathrm{SL}_2(k)$ be an irreducible representation and let $\overline{T} : \Pi \to k$ be a pseudo-SL$_2$-representation over k given by the character $\mathrm{tr}(\overline{\rho})$. Let $\boldsymbol{R}_{\overline{T}}(= \boldsymbol{R}_{\overline{\rho}})$ be the universal deformation ring of $\overline{T}$ (or $\overline{\rho}$) as in Sections 1 and 2. Recall that the universal deformation ring $\boldsymbol{R}_{\overline{T}}$ is a complete local $\mathcal{O}$-algebra whose residue field is k. On the other hand, let $\mathfrak{B}(\Pi)$ be the universal SL$_2$-character algebra of Π over k. Then we have the following

THEOREM 3.2.1. *Assume that $\overline{\rho}$ is irreducible and let $[\overline{\rho}]$ denote the corresponding maximal ideal of $\mathfrak{B}(\Pi)$. We have an isomorphism of k-algebras*

$$\boldsymbol{R}_{\overline{T}} \otimes_{\mathcal{O}} k \ \simeq \ \mathfrak{B}(\Pi)^{\wedge}_{[\overline{\rho}]} ,$$

where $\mathfrak{B}(\Pi)^{\wedge}_{[\overline{\rho}]}$ denotes the $[\overline{\rho}]$-adic completion of $\mathfrak{B}(\Pi)$. So, the universal deformation ring can be considered as an infinitesimal deformation of the universal character algebra. For the case that $\mathrm{char}(\mathcal{O}) = \mathrm{char}(k)$, *we have an isomorphism of $\mathcal{O}$-algebras*

$$\boldsymbol{R}_{\overline{T}} \ \simeq \ \mathfrak{B}(\Pi)^{\wedge}_{[\overline{\rho}]} \hat{\otimes}_k \mathcal{O} ,$$

where $\mathcal{O}$ is considered as a k-algebra by the natural inclusion $k \hookrightarrow \mathcal{O}$.

PROOF. By the construction of $\boldsymbol{R}_{\overline{T}}$ in the proof of Theorem 1.2.1, we have

$$\boldsymbol{R}_{\overline{T}} = \mathcal{O}[[X_g \ (g \in \Pi)]]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal of the power series ring $\mathcal{O}[[X_g \ (g \in \Pi)]]$ generated by elements of the form: setting $T_g := X_g + \varphi(\overline{T}(g))$, φ being the Teichmüller lift,

(1) $T_1 - 2$,

(2) $T_{g_1 g_2} - T_{g_2 g_1}$,

(3) $T_{g_1} T_{g_2} T_{g_3} + T_{g_1 g_2 g_3} + T_{g_1 g_3 g_2} - T_{g_1 g_2} T_{g_3} - T_{g_2 g_3} T_{g_1} - T_{g_1 g_3} T_{g_2}$,

(4) $T_g^2 - T_{g^2} - 2$,

where $g, g_1, g_2, g_3 \in \Pi$.

On the other hand, since the maximal ideal $[\overline{\rho}]$ of $\mathfrak{B}(\Pi)$ corresponds to the maximal ideal $(t_g - \overline{T}(g) \ (g \in \Pi))$ of $\mathfrak{C}(\Pi)$, Corollary 3.1.2 yields

$$\mathfrak{B}(\Pi)^{\wedge}_{[\overline{\rho}]} \simeq k[[x_g \ (g \in \Pi)]]/I^{\wedge} ,$$

where $x_g := t_g - \overline{T}(g)$ $(g \in \Pi)$ and $I^{\wedge}$ is the ideal of the power series ring $k[[x_g \ (g \in \Pi)]]$ generated by elements of the form $t_1 - 2$, $t_{g_1} t_{g_2} - t_{g_1 g_2} - t_{g_1^{-1} g_2}$ $(g_1, g_2 \in \Pi)$. So, in order to show that the correspondence $X_g \otimes 1 \mapsto x_g$ (resp. $X_g \mapsto x_g \otimes 1$) gives the desired isomorphism $\boldsymbol{R}_{\overline{T}} \otimes_{\mathcal{O}} k \simeq \mathfrak{B}(\Pi)^{\wedge}_{[\overline{\rho}]}$ (resp. $\boldsymbol{R}_{\overline{T}} \simeq \mathfrak{B}(\Pi)^{\wedge}_{[\overline{\rho}]} \hat{\otimes}_k \mathcal{O}$ for the case that $\mathrm{char}(\mathcal{O}) = \mathrm{char}(k)$), it suffices to show the following

LEMMA 3.2.2. *Let T be a function on Π with values in an integral domain whose characteristic is not* 2. *Let* (P) *be the relations given by*

(P1) $T(1) = 2$,

(P2) $T(g_1 g_2) = T(g_2 g_1)$,

(P3) $T(g_1)T(g_2)T(g_3) + T(g_1g_2g_3) + T(g_1g_3g_2) - T(g_1g_2)T(g_3)$
$- T(g_2g_3)T(g_1) - T(g_1g_3)T(g_2) = 0$,

(P4) $T(g)^2 - T(g^2) = 2$,

and let (C) *be the relations given by*

(C1) $T(1) = 2$,

(C2) $T(g_1)T(g_2) = T(g_1g_2) + T(g_1^{-1}g_2)$,

where g, g_1, g_2, g_3 *are any element in* Π.

Then (P) *and* (C) *are equivalent.*

PROOF OF LEMMA 3.2.2. (P) $\Rightarrow$ (C): Letting $g_2 = g_1$ in (P3), we have

$$T(g_1)^2T(g_3) - T(g_1^2)T(g_3) + T(g_1^2g_3) + T(g_1g_3g_1) - 2T(g_1g_3)T(g_1) = 0\,.$$

Using (P2) and (P4), we have

$$2(T(g_3) + T(g_1^2g_3) - T(g_1g_3)T(g_1)) = 0\,.$$

Letting g_3 be replaced by $g_1^{-1}g_2$ in the above equation and noting T has the value in an integral domain whose characteristic is not 2, we obtain (C2).

(C) $\Rightarrow$ (P). Letting $g_2 = 1$ in (C2) and using (C1), we have

$$T(g) = T(g^{-1}) \text{ for any } g \in \Pi\,.$$

Exchanging g_1 and g_2 in (C2) each other and using the above relation, we have

$$T(g_2)T(g_1) = T(g_2g_1) + T(g_2^{-1}g_1) = T(g_2g_1) + T(g_1^{-1}g_2)$$

and hence we obtain (P2). Next letting g_1 be replaced by g_1g_3 in (C2), we have

$$-T(g_1g_3)T(g_2) + T(g_1g_3g_2) + T(g_3^{-1}g_1^{-1}g_2) = 0\,, \tag{3.2.2.1}$$

and letting g_2 be replaced by g_2g_3 in (C2), we have

$$-T(g_1)T(g_2g_3) + T(g_1g_2g_3) + T(g_1^{-1}g_2g_3) = 0\,. \tag{3.2.2.2}$$

By (C2), we have

$$\begin{aligned} T(g_3^{-1}g_1^{-1}g_2) &= T(g_3)T(g_1^{-1}g_2) - T(g_3g_1^{-1}g_2) \\ &= T(g_3)T(g_1)T(g_2) - T(g_1g_2)T(g_3) - T(g_3g_1^{-1}g_2)\,. \end{aligned}$$

Hence, using (P2) proved already, we have

$$\begin{aligned} T(g_3^{-1}g_1^{-1}g_2) + T(g_1^{-1}g_2g_3) &= T(g_1)T(g_2)T(g_3) - T(g_1g_2)T(g_3) \\ &\quad -T(g_3g_1^{-1}g_2) + T(g_1^{-1}g_2g_3) \\ &= T(g_1)T(g_2)T(g_3) - T(g_1g_2)T(g_3)\,. \end{aligned} \tag{3.2.2.3}$$

Summing up (3.2.2.1) and (3.2.2.2) together with (3.2.2.3), we obtain (P3). Finally putting $g_1 = g_2$ in (C2) and using (C1), we obtain (P4). □

By Lemma 2.2.3 and Theorem 3.2.1, we have the following

COROLLARY 3.2.3. *Assume that $[\overline{\rho}]$ is a regular point of the scheme $\mathfrak{X}(\Pi)$, namely, $\mathfrak{B}(\Pi)_{[\overline{\rho}]}$ is a regular local ring. Then the dimension d of the relative tangent space $\mathfrak{t}_{\boldsymbol{R}_{\overline{T}}/\mathcal{O}}$ of $\boldsymbol{R}_{\overline{T}}$ is equal to the dimension of the irreducible componennt of $\mathfrak{X}(\Pi)$ containing $[\overline{\rho}]$, and $\mathfrak{B}(\Pi)^{\wedge}_{[\overline{\rho}]}$ is a power series ring over k on a regular system of parameters $x_1, \ldots, x_d$. Hence we have a surjective $\mathcal{O}$-algebra homomorphism*

$$\lambda : \mathcal{O}[[X_1, \ldots, X_d]] \longrightarrow \boldsymbol{R}_{\overline{T}}$$

in $\mathcal{C}$ such that the image of $\lambda(X_i)$ in $\boldsymbol{R}_{\overline{T}} \otimes_{\mathcal{O}} k \simeq \mathfrak{B}(\Pi)^{\wedge}_{[\overline{\rho}]}$ is x_i $(1 \leq i \leq d)$.

4. Examples for 2-bridge knot groups.

In this section, we investigate examples concerning Riley representations of 2-bridge knot groups.

In Subsection 4.1, we recall some results on the Riley representations of 2-bridge knot groups. We refer to [1] for basic information on 2-bridge knots and [17], [18] for the details on Riley representations. In Subsection 4.2, we describe the character scheme/variety of SL_2-representations of a 2-bridge knot group. For this, we refer to [6]. In Subsection 4.3, we give an explicit form of the universal deformation of a Riley representation.

4.1. *2-bridge knots and Riley representations.* Let K be a 2-bridge knot in the 3-sphere S^3, given as the Schubert form $\mathfrak{b}(m, n)$ where m and n are odd integers with $m > 0$, $-m < n < m$ and $\mathrm{g.c.d}(m, n) = 1$. Let Π_K be the knot group $\pi_1(S^3 \setminus K)$. The group Π_K is known to have a presentation of the form

$$\Pi_K = \langle a, b \mid wa = bw \rangle ,$$

where w is a word $w(a, b)$ of a and b which has the following symmetric form

$$w = w(a, b) = a^{\varepsilon_1} b^{\varepsilon_2} \cdots a^{\varepsilon_{m-2}} b^{\varepsilon_{m-1}} ,$$
$$\varepsilon_i = (-1)^{[in/m]} = \varepsilon_{m-i} \quad ([\,\cdot\,] = \text{Gauss symbol}) .$$

Let F be the free group generated by a and b, and let $\pi : F \to \Pi_K$ be the natural homomorphism.

Let A be a commutative ring with identity. For $\alpha \in A^{\times}$ and $\beta \in A$, we consider two matrices $C(\alpha)$ and $D(\alpha, \beta)$ in $\mathrm{SL}_2(A)$ defined by

$$C(\alpha) := \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad D(\alpha, \beta) := \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \tag{4.1.1}$$

and we set

$$W(\alpha, \beta) := C(\alpha)^{\varepsilon_1} D(\alpha, \beta)^{\varepsilon_2} \cdots C(\alpha)^{\varepsilon_{m-2}} D(\alpha, \beta)^{\varepsilon_{m-1}} .$$

It is easy to see that there are (Laurent) polynomials $w_{ij}(t, u) \in \mathbb{Z}[t^{\pm}, u]$ $(1 \leq i, j \leq 2)$ such that $W(\alpha, \beta) = (w_{ij}(\alpha, \beta))$. Let $f_{(\alpha,\beta)} : F \to \mathrm{SL}_2(A)$ be the homomorphism defined by

$$f_{(\alpha,\beta)}(a) := C(\alpha), \quad f_{(\alpha,\beta)}(b) := D(\alpha, \beta) .$$

We call a representation

$$\rho : \Pi_K \longrightarrow \mathrm{SL}_2(A)$$

the *Riley representation over A of type* (α,β), denoted by $\mathfrak{r}_{(\alpha,\beta)}$, if $f_{(\alpha,\beta)}$ factors through ρ, namely, $\rho\circ\pi = f_{(\alpha,\beta)}$.

Let k be an algebraically closed field. The following Theorem 4.1.2 was proved by Riley [17], [18] for the case where k is the field of complex numbers. The proof therein works as well for any algebraically closed field.

THEOREM 4.1.2 ([17], [18]). *Let* $\varphi(t,u) := w_{11}(t,u)+(t^{-1}-t)w_{12}(t,u)\in\mathbb{Z}[t^{\pm},u]$.

(1) *There is a unique polynomial* $\Phi(x,u)\in\mathbb{Z}[x,u]$ *such that*

$$\Phi(t+t^{-1},u) = t^l\varphi(t,u)$$

for an integer l.

(2) *The homomorphism* $f_{(\alpha,\beta)}$ $(\alpha\in k^\times,\beta\in k)$ *factors through the Riley representation* $\mathfrak{r}_{(\alpha,\beta)}$ *over* k *if and only if we have*

$$\Phi(\alpha+\alpha^{-1},\beta)=0\,.$$

(3) *Any non-Abelian* $\mathrm{SL}_2(k)$*-representation of* Π_K *is equivalent to a Riley representation* $\mathfrak{r}_{(\alpha,\beta)}$ *for some* $\alpha\in k^\times$ *and* $\beta\in k$.

For the properties of the polynomial $\Phi(x,u)$, Riley showed, among others, the following

PROPOSITION 4.1.3 ([17]). *The polynomial* $\Phi(2,u)=\varphi(1,u)\in\mathbb{Z}[u]$ *is monic up to multiplication by* ±1 *and its discriminant* $\mathrm{disc}(\Phi(2,u))$ *is an odd integer. If* $\mathrm{char}(k)$ *does not divide* $\mathrm{disc}(\Phi(2,u))$*, then any root of* $\Phi(2,u)=0$ *in* k *is non-zero and simple.*

By Hensel's lemma, we have the following

COROLLARY 4.1.4. *Let* $\mathcal{O}$ *be a complete discrete valuation ring with residue field* k. *For any root* β *of* $\Phi(2,u)=0$ *in* k, *there is a unique power series* $u(x)\in\mathcal{O}[[x-2]]^\times$ *such that* $\beta=u(2) \bmod \mathfrak{m}_{\mathcal{O}}$ *and* $\Phi(x,u(x))\equiv 0$ *in* $\mathcal{O}[[x-2]]$.

EXAMPLE 4.1.5. (1) Let K be the trefoil $\mathfrak{b}(3,1)$. Then we have $w=ab$ and $\varphi(t,u)=t^2(t^2+t^{-2}+u-1)$. Hence $\Phi(x,u)=x^2+u-3$ and $\Phi(2,u)=u+1$. Therefore $\beta=-1$ and $u(x)=3-x^2$ for any k.

(2) Let K be the figure eight $\mathfrak{b}(5,3)$. Then we have $w=ab^{-1}a^{-1}b$ and $\varphi(t,u)=u^2+(t^2+t^{-2}-3)u-(t^2+t^{-2}-3)$. Hence $\Phi(x,u)=u^2+(x^2-5)u-(x^2-5)$ and $\Phi(2,u)=u^2-u+1$. Therefore, if $\mathrm{char}(k)\neq 2,3$, then $\beta=\frac{1}{2}(1\pm\sqrt{-3})\in k$ and $u(x)=\frac{1}{2}\{5-x^2\pm\sqrt{(x^2-1)(x^2-5)}\}\in\mathcal{O}[[x-2]]^\times$ where $\sqrt{(x^2-1)(x^2-5)}$ stands for an element of $\mathcal{O}[[x-2]]$ whose square is $(x^2-1)(x^2-5)$.

4.2. *Character varieties.* We keep the notations in Subsection 3.1. Let k denote an algebraically closed field and let $\mathcal{X}(\Pi_K)$ denote the $\mathrm{SL}_2(k)$-character variety of Π_K. The proof of Proposition 1.4.1 of [3] tells us that any τ_g $(g\in\Pi_K)$ is given as a polynomial of $\tau_a(=\tau_b)$ and τ_{ab} with coefficients in $\mathbb{Z}$. In particular, the coordinate ring $k[\mathcal{X}(\Pi_K)]$ is generated by τ_a and τ_{ab}. We let x and y denote the variables corresponding, respectively, to the coordinate functions τ_a and τ_{ab} on $\mathcal{X}(\Pi_K)$ embedded in k^2. This variable x is consistent with the variable x of $\Phi(x, u)$ in Theorem 4.1.2 (and so causes no confusion). In fact, the coordinate variables x and y are related with t and u in Theorem 4.1.2 by

$$x = t + t^{-1}, \quad y = t^2 + t^{-2} + u = x^2 + u - 2\,,$$

since we have

$$\tau_a(\mathfrak{r}_{(\alpha,\beta)}) = \mathrm{tr}(C(\alpha)) = \alpha + \alpha^{-1}\,, \quad \tau_{ab}(\mathfrak{r}_{(\alpha,\beta)}) = \mathrm{tr}(C(\alpha)D(\alpha, \beta)) = \alpha^2 + \alpha^{-2} + \beta\,.$$

By Theorem 4.1.2 (2), (3), characters of irreducible $\mathrm{SL}_2(k)$-representations of Π_K correspond bijectively to points on the algebraic curve in k^2 defined by the equation

$$\Phi(x, y - x^2 + 2) = 0\,,$$

except the finitely many intersection points with the algebraic curve $y - x^2 + 2 = 0$. The points on the latter curve $y-x^2+2 = 0$ correspond (not bijectively) to characters of reducible $\mathrm{SL}_2(k)$-representations of Π_K. It is also shown ([6, Proposition 3.4.1]) that the ideal generated by $\Phi(x, y - x^2 + 2)$ in $k[x, y]$ is a radical ideal. Thus we have the following

THEOREM 4.2.1 ([6, Theorem 3.3.1]). *The character variety $\mathcal{X}(\Pi_K)$ is the affine algebraic curve in k^2 defined by the equation*

$$(y - x^2 + 2)\Phi(x, y - x^2 + 2) = 0\,,$$

and the coordinate ring of $\mathcal{X}(\Pi_K)$ is given by

$$k[\mathcal{X}(\Pi_K)] \simeq k[x, y]/((y - x^2 + 2)\Phi(x, y - x^2 + 2))\,.$$

Here the points on the algebraic curve $\Phi(x, y - x^2 + 2) = 0$ correspond bijectively to irreducible $\mathrm{SL}_2(k)$-characters of Π_K except the finitely many intersection points with the algebraic curve $y-x^2+2 = 0$, and the points on $y-x^2+2 = 0$ correspond (not bijectively) to reducible $\mathrm{SL}_2(k)$-characters of Π_K.

EXAMPLE 4.2.2. (1) When K is the trefoil $\mathfrak{b}(3, 1)$, we see $\Phi(x, y-x^2+2) = y-1$. Hence $\mathcal{X}(\Pi_K)$ is given by $(y - x^2 + 2)(y - 1) = 0$.

(2) When K is the figure eight $\mathfrak{b}(5, 3)$, we have $\Phi(x, y - x^2 + 2) = y^2 - (1 + x^2)y + 2x^2 - 1$. Hence $\mathcal{X}(\Pi_K)$ is given by $(y - x^2 + 2)\{y^2 - (1 + x^2)y + 2x^2 - 1\} = 0$.

Przytycki and Sikora proved the following theorem for the case where k is the field of complex numbers. Their proof works well for any algebraically closed field whose characteristic is not 2.

THEOREM 4.2.3 ([16, Theorem 7.3]). *Assume* $\mathrm{char}(k) \neq 2$. *Then the universal* SL_2-*character algebra $\mathfrak{B}(\Pi_K)$ is reduced and hence $\mathfrak{B}(\Pi_K) = k[\mathcal{X}(\Pi_K)]$.*

4.3. *The universal deformation.* As in Subsection 3.2, let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$ and let $\mathcal{O}$ be a complete discrete valuation ring with residue field k. We assume further that $\mathrm{char}(k)$ does not divide the discriminant of $\Phi(2, u) \in \mathbb{Z}[u]$.

Let $\overline{\rho} : \Pi_K \to \mathrm{SL}_2(k)$ be a Riley representation $\mathfrak{r}_{(1,\beta)}$ so that

$$\overline{\rho}(a) := C(1), \quad \overline{\rho}(b) := D(1, \beta)\,,$$

where β is a root of $\Phi(2,\beta)=0$. By Proposition 4.1.3, $\overline{\rho}$ is irreducible and $(x,y)=(2,\beta+2)$ is a non-singular point on $\mathcal{X}(\Pi_K)$.

Let $u(x)$ be the power series in Corollary 4.1.4 and set $\overline{u}(x) := u(x) \bmod \mathfrak{m}_{\mathcal{O}}$. By Theorem 4.2.1, we have the isomorphism

$$k[\mathcal{X}(\Pi_K)]^{\wedge}_{[\overline{\rho}]} \simeq (k[x,y]/(\Phi(x,y-x^2+2)))^{\wedge}_{(x-2,y-(\beta+2))} \simeq k[[x-2]], \tag{4.3.1}$$

where the second isomorphism is given by $y \mapsto x^2+\overline{u}(x)-2$. So $x-2$ is a local parameter of $\mathcal{X}(\Pi_K)$ at $[\overline{\rho}]$.

Let $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ be the universal deformation of $\overline{\rho}$. By Theorem 3.2.1, Theorem 4.2.3 and (4.3.1), we have

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \simeq k[[x-2]]\,,$$

where $T_a \bmod \mathcal{I} = \mathrm{tr}(\boldsymbol{\rho}(a)) \in \boldsymbol{R}_{\overline{\rho}}$ corresponds to x. By Corollary 3.2.3, we have

LEMMA 4.3.2. *The dimension of the relative tangent space* $\mathfrak{t}_{\boldsymbol{R}_{\overline{\rho}}/\mathcal{O}}$ *is* 1 *and there is a surjective* $\mathcal{O}$*-algebra homomorphism*

$$\lambda : \mathcal{O}[[X]] \longrightarrow \boldsymbol{R}_{\overline{\rho}}$$

in $\mathcal{C}$ *such that* $\lambda(X) = \mathrm{tr}(\boldsymbol{\rho}(a)) - 2$.

In the following Theorem 4.3.3, we show that the map λ in Lemma 4.3.2 is in fact an isomorphism, and we give an explicit form of the universal deformation $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$. We remark on the notation used in the following: For $p(x) \in \mathcal{O}[[x-2]]$ with $p(2) \in \mathcal{O}^{\times}$, $\sqrt{p(x)}$ stands for an element in $\mathcal{O}[[x-2]]$ whose square is $p(x)$. If $p(2)=1$, we adopt the unique one normalized by $\sqrt{p(2)}=1$. For $p(x) \in \mathcal{O}[[x-2]]$ with $p(2) \in \mathfrak{m}_{\mathcal{O}}$, $\sqrt{p(x)}$ is an element of a quadratic extension of $E((x-2))$ whose square is $p(x)$, where E is the field of fractions of $\mathcal{O}$. In particular, $\sqrt{x-2}$ is a prime element of a quadratic extension L of $E((x-2))$ and we denote by $\mathcal{O}[[\sqrt{x-2}]]$ the integral closure of $\mathcal{O}[[x-2]]$ in L. For $p(x) \in \mathcal{O}[[x-2]]$ with $p(2)=0$, we may write $p(x)=(x-2)p_1(x)$ with $p_1(x) \in \mathcal{O}[[x-2]]$ and so we have $\frac{p(x)}{\sqrt{x-2}} = \sqrt{x-2}\,p_1(x) \in \mathcal{O}[[\sqrt{x-2}]]$.

THEOREM 4.3.3. *We let* $v(x) := \sqrt{1+\frac{x^2-4}{u(x)}} \in \mathcal{O}[[x-2]]^{\times}$ *and define* $U(x) \in \mathrm{SL}_2(\mathcal{O}[[\sqrt{x-2}]])$ *by*

$$U(x) := \begin{pmatrix} \dfrac{1}{\sqrt{v(x)}} & \dfrac{1-v(x)}{\sqrt{v(x)}\sqrt{x^2-4}} \\ \dfrac{\sqrt{x^2-4}}{2\sqrt{v(x)}} & \dfrac{1+v(x)}{2\sqrt{v(x)}} \end{pmatrix}.$$

We define $A(x), B(x) \in \mathrm{SL}_2(\mathcal{O}[[x-2]])$ *by*

$$A(x) := U(x)C(t)U(x)^{-1} = \begin{pmatrix} \dfrac{x}{2} & 1 \\ \dfrac{x^2-4}{4} & \dfrac{x}{2} \end{pmatrix},$$

$$B(x) := U(x)D(t,u(x))U(x)^{-1} = \begin{pmatrix} \dfrac{x}{2} & \dfrac{(1-v(x))^2u(x)}{x^2-4} \\ \dfrac{(1+v(x))^2u(x)}{4} & \dfrac{x}{2} \end{pmatrix},$$

where t is an element $\mathcal{O}[[\sqrt{x-2}]]$ such that $t + t^{-1} = x$ and $C(t)$, $D(t, u(x))$ are the matrices over $\mathcal{O}[[\sqrt{x-2}]]$ defined in (4.1.1). *We define the deformation of $\overline{\rho}$*

$$\boldsymbol{\rho}^u : \Pi_K \longrightarrow \mathrm{SL}_2(\mathcal{O}[[x-2]])$$

by

$$\boldsymbol{\rho}^u(a) := A(x), \quad \boldsymbol{\rho}^u(b) := B(x).$$

Then there is an isomorphism $\psi : \boldsymbol{R}_{\overline{\rho}} \xrightarrow{\sim} \mathcal{O}[[x-2]]$ in $\mathcal{C}$ such that $\psi \circ \boldsymbol{\rho} \approx \boldsymbol{\rho}^u$.

PROOF. Firstly, let us check that $U(x) \in \mathrm{SL}_2(\mathcal{O}[[\sqrt{x-2}]])$ and $A(x), B(x) \in \mathrm{SL}_2(\mathcal{O}[[x-2]])$. Since $v(2) = 1$, $\sqrt{v(x)} \in \mathcal{O}[[x-2]]^\times$ and $1 - v(x) = (x-2)p(x)$ with some $p(x) \in \mathcal{O}[[x-2]]$ and hence all entries of $U(x)$ are lying in $\mathcal{O}[[\sqrt{x-2}]]$. Further we easily see $\det U(x) = 1$ and also $U(2) = I$. As for $A(x), B(x)$, we see immediately that $\det A(x) = \det C(t) = 1$ and $\det B(x) = \det D(t, u(x)) = 1$. The straightforward computations of $U(x)C(t)U(x)^{-1}$ and $U(x)D(t, u(x))U(x)^{-1}$ using $t + t^{-1} = x, t - t^{-1} = \sqrt{x^2-4}$, $x^2 - 4 + u(x) = v(x)^2u(x)$ yield the desired matrices in the statement, from which we easily see that all entries of $A(x)$ and $B(x)$ are lying in $\mathcal{O}[[x-2]]$.

Next, let us show that $\boldsymbol{\rho}^u$ is a deformation of $\overline{\rho}$ over $\mathcal{O}[[x-2]]$. Since $\boldsymbol{\rho}^u$ is equivalent to the Riley representation $\mathfrak{r}_{(t,u(x))}$ over $\mathcal{O}[[\sqrt{x-2}]]$, $\boldsymbol{\rho}^u$ is indeed a representation. Since $A(2) \bmod \mathfrak{m}_{\mathcal{O}[[x-2]]} = C(1)$ and $B(2) \bmod \mathfrak{m}_{\mathcal{O}[[x-2]]} = D(1, \beta)$, we find that $\boldsymbol{\rho}^u \bmod \mathfrak{m}_{\mathcal{O}[[x-2]]} = \overline{\rho}$.

Finally, by the universality of $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$, we have a homomorphism $\psi : \boldsymbol{R}_{\overline{\rho}} \to \mathcal{O}[[x-2]]$ in $\mathcal{C}$ such that $\psi \circ \boldsymbol{\rho} \approx \boldsymbol{\rho}^u$. So we have $\psi(\mathrm{tr}(\boldsymbol{\rho}(a))) = \mathrm{tr}(\boldsymbol{\rho}^u(a)) = x$. On the other hand, by Lemma 4.3.2, we have $\lambda(X) = \mathrm{tr}(\boldsymbol{\rho}(a)) - 2$. Therefore $\psi \circ \lambda(X) = x - 2$ and hence $\psi \circ \lambda$ is an isomorphism $\mathcal{O}[[X]] \simeq \mathcal{O}[[x-2]]$. Since λ is surjective, ψ and λ must be isomorphisms in $\mathcal{C}$. □

REMARK 4.3.4. (1) By the construction of $\boldsymbol{\rho}^u$, if $\overline{\rho}$ is defined over a subfield k' of k, the representation $\boldsymbol{\rho}^u$ is also defined over a ring $\mathcal{O}'[[x-2]]$, where $\mathcal{O}'$ is a complete discrete valuation ring with residue field k'. For example, if $\overline{\rho}$ is defined over a prime field $\mathbb{F}_p$ of p elements, $\boldsymbol{\rho}^u$ can be defined over $\mathbb{Z}_p[[x-2]]$, where $\mathbb{Z}_p$ is the ring of p-adic integers.

(2) Suppose $\mathrm{char}(\mathcal{O}) = \mathrm{char}(k)$ so that $\mathcal{O} = k[[\hbar]]$. Then the representation $\boldsymbol{\rho}^u$ is independent of $\hbar$. However, there are deformations of $\overline{\rho}$ which depend on $\hbar$. For example, letting $\rho_n(a) := A((1+\hbar)^n + (1+\hbar)^{-n})$ and $\rho_n(b) := B((1+\hbar)^n + (1+\hbar)^{-n})$ for $n \in \mathbb{Z}$, we have a family of deformations ρ_n of $\bar{\rho}$ over $k[[\hbar]]$.

EXAMPLE 4.3.5. (1) Let K be the trefoil $\mathfrak{b}(3,1)$ and assume $\mathrm{char}(k) \neq 2$. We then have $\beta = -1$, $u(x) = 3 - x^2$ and $v(x) = \frac{1}{\sqrt{x^2-3}}$. For example, we can consider $\overline{\rho} = \mathfrak{r}_{(1,-1)} : \Pi_K \to \mathrm{SL}_2(\mathbb{F}_p)$ for an odd prime number p, where $\mathbb{F}_p \subset k$. Then $\boldsymbol{\rho}^u$ can be a representation into $\mathrm{SL}_2(\mathbb{Z}_p[[x-2]])$, strictly equivalent to $\boldsymbol{\rho}$ over $\mathcal{O}[[x-2]]$.

(2) Let K be the figure eight $\mathfrak{b}(5,3)$ and assume $\mathrm{char}(k) \neq 2, 3$. We then have $\beta = \frac{1}{2}(1 \pm \sqrt{-3})$, $u(x) = \frac{1}{2}\{5 - x^2 \pm \sqrt{(x^2-1)(x^2-5)}\}$ and $v(x)^2 = \frac{1}{2}\{x^2 - 2 \pm \frac{(x^2-4)\sqrt{x^2-1}}{\sqrt{x^2-5}}\}$. For example, we can consider $\overline{\rho} = \mathfrak{r}_{(1,\beta)} : \Pi_K \to \mathrm{SL}_2(\mathbb{F}_p(\beta))$ for $p \neq 2, 3$, where $\mathbb{F}_p(\beta) \subset k$. Then $\boldsymbol{\rho}^u$ can be a representation into $\mathrm{SL}_2(\mathbb{Z}_p[\beta][[x-2]])$, strictly equivalent to $\boldsymbol{\rho}$ over $\mathcal{O}[[x-2]]$.

5. The universal deformation of a holonomy representation. In this section, we apply our deformation theory to the case where Π is the fundamental group of the complement of a hyperbolic knot in the 3-sphere and $\overline{\rho}$ is (a lift of) the holonomy representation.

In Subsection 5.1, we recall Thurston's theorem on deformations of hyperbolic structures ([21]). In Subsection 5.2, we then describe the universal deformation of $\overline{\rho}$ by using Thurston's theorem, and discuss some analogies with p-adic Galois deformations.

In this section, we work over the field $k = \mathbb{C}$ of complex numbers.

5.1. *Holonomy representation and Thurston's theorem.* Let K be a hyperbolic knot in the 3-sphere S^3 and let $\Pi_K := \pi_1(S^3 \setminus K)$ be the knot group. The complement $S^3 \setminus K$ is a complete hyperbolic 3-manifold of finite volume with a cusp, which is given as a quotient of the hyperbolic 3-space $\mathbb{H}^3$ by a discrete, torsion free subgroup of $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\} = \mathrm{Aut}(\mathbb{H}^3)$. To the complete hyperbolic structure on $S^3 \setminus K$ we can associate a faithful representation $\rho_{\mathrm{hol}} : \Pi_K \to \mathrm{PSL}_2(\mathbb{C})$, called the *holonomy representation*. The holonomy representation ρ_{hol} can be lifted to an $\mathrm{SL}_2(\mathbb{C})$-representation, and thus we fix such a lift

$$\overline{\rho}_{\mathrm{hol}} : \Pi_K \longrightarrow \mathrm{SL}_2(\mathbb{C}),$$

which is known to be irreducible.

Let $\mathcal{X}(\Pi_K)$ be the $\mathrm{SL}_2(\mathbb{C})$-character variety of Π_K as defined in Subsection 3.1 and let $\mathcal{X}(\Pi_K)^{\mathrm{hol}}$ be the irreducible component of $\mathcal{X}(\Pi_K)$ containing $[\overline{\rho}_{\mathrm{hol}}]$. We choose any meridian μ of the knot K and consider the map $\tau_\mu : \mathfrak{X}(\Pi_K)^{\mathrm{hol}} \to \mathbb{C}$ defined by $\tau_\mu([\rho]) = \mathrm{tr}(\rho)(\mu)$. Then Thurston has proved the following

THEOREM 5.1.1 ([21]). *The map τ_μ is bianalytic in a neighborhood of* $[\overline{\rho}_{\mathrm{hol}}]$.

Therem 5.1.1 implies that $\mathcal{X}(\Pi_K)^{\mathrm{hol}}$ is a complex algebraic curve and τ_μ gives a local parameter around the smooth point $[\overline{\rho}_{\mathrm{hol}}]$. Hence we have the following

COROLLARY 5.1.2. *We have the following isomorphism of $\mathbb{C}$-algebras*

$$\mathbb{C}[\mathcal{X}(\Pi_K)]^{\wedge}_{[\overline{\rho}_{\mathrm{hol}}]} \simeq \mathbb{C}[[z]],$$

where z is a variable corresponding to $\tau_\mu - \tau_\mu(\overline{\rho}_{\mathrm{hol}}) = \tau_\mu - 2$.

5.2. *The universal deformation of the holonomy representation.* Let

$$\boldsymbol{\rho}_{\mathrm{hol}} : \Pi_K \longrightarrow \mathrm{SL}_2(\boldsymbol{R}_{\overline{\rho}_{\mathrm{hol}}})$$

be the universal deformation of $\overline{\rho}_{\mathrm{hol}}$ in Theorem 2.2.2, where the universal deformation ring $\boldsymbol{R}_{\overline{\rho}_{\mathrm{hol}}}$ is a complete local algebra over $\mathcal{O} = \mathbb{C}[[\hbar]]$. We assume that the universal SL_2-character algebra $\mathfrak{B}(\Pi_K)$ is reduced so that $\mathfrak{B}(\Pi_K) = k[\mathcal{X}(\Pi_K)]$. Then, by Theorem 3.2.1 and Corollary 5.1.2, we have the following

THEOREM 5.2.1. *Under the above assumption, we have the following isomorphism of $\mathcal{O}$-algebras*

$$\boldsymbol{R}_{\overline{\rho}_{\mathrm{hol}}} \simeq \mathcal{O}[[z]].$$

REMARK 5.2.2. The analogy between the structures of $\mathcal{X}(\Pi_K)^{\mathrm{hol}}$ and the deformation space of nearly ordinary p-adic Galois representations was firstly pointed out by Kazuhiro Fujiwara (cf. [11, Chapter 14]). Theorem 5.2.1 may make this analogy more precise as follows: Let $\mathfrak{D}$ be the irreducible component of the deformation space of hyperbolic structures on $S^3 \setminus K$ containing the complete hyperbolic structure, say z°. By Thurston's theory on hyperbolic Dehn filling ([21]), a neighborhood of z° in $\mathfrak{D}$ is homeomorphic to a neighborhood of $[\overline{\rho}_{\mathrm{hol}}]$ in $\mathfrak{X}(\Pi_K)^{\mathrm{hol}}$, associating to an (incomplete) hyperbolic structure the holonomy representation. So Theorem 5.2.1 gives the isomorphism between the universal deformation ring $\boldsymbol{R}_{\overline{\rho}_{\mathrm{hol}}}$ and the complete local ring of $\mathfrak{D}$ at z°, where the parameter z in Theorem 5.2.1 may also be considered as hyperbolic structure (Dehn filling coefficient). Noting that the restriction of ρ to the peripheral group of K (the fundamental group of the boundary of a tubular neighborhood of K) is equivalent to an uppertriangular representation, this isomorphism is quite analogous to the isomorphism between the universal deformation ring for p-adic ordinary Galois representatios and the p-adic ordinary Hecke algebra, which implies that any p-adic ordinary deformation of a given modular Galois representation over $\mathbb{F}_p$ is associated to a p-adic ordinary modular form (cf. [5]). Here we may observe that hyperbolic structures (Dehn filling coefficients) correspond to p-adic ordinary modular forms (p-adic weights). See also Ohtani's article [14] for a related analogy.

REFERENCES

[1] G. BURDE AND H. ZIESCHANG, Knots, Second edition. de Gruyter Studies in Mathematics, 5. Walter de Gruyter & Co., Berlin, 2003.

[2] H. CARAYOL, Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet, p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), 213–237, Contemp. Math., 165, Amer. Math. Soc., Providence, RI, 1994.

[3] M. CULLER AND P. SHALEN, Varieties of group representations and splittings of 3-manifolds, Ann. of Math. 117 (1983), 109–146.

[4] H. HIDA, Galois representations into $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms, Invent. Math. 85 (1986), 545–613.

[5] H. HIDA, Modular forms and Galois cohomology, Cambridge Studies in Advanced Mathematics, 69, Cambridge University Press, Cambridge, 2000.

[6] T. LE, Varieties of representations and their subvarieties of cohomology jumps for certain knot groups, Russian Acad. Sci. Sb. Math. 78 (1994), 187–209.

[7] A. LUBOTZKY AND A. MAGID, Varieties of representations of finitely generated groups, Mem. Amer. Math. Soc. 58 (1985), no. 336, xi+117 pp.

[8] B. MAZUR, Remarks on the Alexander polynomial, available at http://www.math.harvard.edu/~mazur/older.html.

[9] B. MAZUR, Deforming Galois representations, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 385–437, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.

[10] B. MAZUR, The theme of p-adic variation, Mathematics: frontiers and perspectives, 433–459, Amer. Math. Soc., Providence, RI, 2000.

[11] M. MORISHITA, Knots and Primes—An Introduction to Arithmetic Topology, Universitext. Springer, London, 2012.

[12] K. NAKAMOTO, Representation varieties and character varieties, Publ. Res. Inst. Math. Sci. 36 (2000), no. 2, 159–189.

[13] L. NYSSEN, Pseudo-représentations, Math. Ann. 306 (1996), no. 2, 257–283.

[14] S. OHTANI, An analogy between representations of knot groups and Galois groups, Osaka J. Math. 48 (2011), no. 4, 857–872.

[15] C. PROCESI, The invariant theory of $n \times n$ matrices, Advances in Math. 19 (1976) no. 3, 306–381.

[16] J. PRZYTYCKI AND A. SIKORA, On skein algebras and $Sl_2(\mathbb{C})$-character varieties, Topology 39 (2000), 115–148.

[17] R. RILEY, Nonabelian representations of 2-bridge knot groups, Quar. J. Oxford 35 (1984), 191–208.

[18] R. RILEY, Holomorphically parameterized families of subgroups of SL(2, $\mathbb{C}$), Mathematika 32 (1985), no. 2, 248–264.

[19] K. SAITO, Character variety of representations of a finitely generated group in SL_2, Topology and Teichmüller spaces (Katinkulta, 1995), 253–264, World Sci. Publ., River Edge, NJ, 1996.

[20] R. TAYLOR, Galois representations associated to Siegel modular forms of low weight, Duke Math. J. 63 (1991), no. 2, 281–332.

[21] W. THURSTON, The geometry and topology of 3-manifolds, Lect. Note, Princeton, 1979.

[22] J. TILOUINE, Deformations of Galois representations and Hecke algebras, Published for The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad; by Narosa Publishing House, New Delhi, 1996.

[23] A. WILES, On ordinary λ-adic representations associated to modular forms, Invent. Math. 94 (1988), no.3, 529–573.

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