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ON THE GENERALIZED WINTGEN INEQUALITY FOR LEGENDRIAN SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Abstract. The generalized Wintgen inequality was conjectured by De Smet, Dillen, Verstraelen and Vrancken in 1999 for submanifolds in real space forms. It is also known as the DDVV conjecture. It was proven recently by Lu (2011) and by Ge and Tang (2008), independently. The present author established a generalized Wintgen inequality for Lagrangian submanifolds in complex space forms in 2014. In the present paper we obtain the DDVV inequality, also known as generalized Wintgen inequality, for Legendrian submanifolds in Sasakian space forms. Some geometric applications are derived. Also we state such an inequality for contact slant submanifolds in Sasakian space forms.

1. Preliminaries. In 1979, P. Wintgen [24] proved that the Gauss curvature K, the squared mean curvature $\|H\|^2$ and the normal curvature $K^{\perp}$ of any surface M^2 in $\mathbf{E}^4$ always satisfy the inequality

$$K \leq \|H\|^2 - |K^{\perp}|\,;$$

the equality holds if and only if the ellipse of curvature of M^2 in $\mathbf{E}^4$ is a circle. The Whitney 2-sphere satisfies the equality case of the Wintgen inequality identically. A survey containing recent results on surfaces satisfying identically the equality case of Wintgen inequality can be read in [5].

Later, the Wintgen inequality was extended by B. Rouxel [23] and by I.V. Guadalupe and L. Rodriguez [14] independently, to surfaces M^2 of arbitrary codimension m in real space forms $\widetilde{M}^{2+m}(c)$ as follows:

$$K \leq \|H\|^2 - |K^{\perp}| + c\,.$$

The equality case was also investigated.

A corresponding inequality for totally real surfaces in n-dimensional complex space forms was obtained in [17]. The equality case was studied and a non-trivial example of a totally real surface satisfying the equality was given.

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Let M^n be an n-dimensional submanifold of a real space form $\widetilde{M}^{n+m}(c)$. We denote by K and $R^\perp$ the sectional curvature function and the normal curvature tensor on M^n, respectively. Then the normalized scalar curvature is given by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)\,,$$

where τ is the scalar curvature, and the normalized normal scalar curvature by

$$\rho^\perp = \frac{2\tau^\perp}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1 \leq i < j \leq n} \sum_{1 \leq \alpha < \beta \leq m} (R^\perp(e_i, e_j, \xi_\alpha, \xi_\beta))^2}\,.$$

In 1999, P.J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken [9] formulated the conjecture on the Wintgen inequality for submanifolds of real space forms, which is also known as the *DDVV conjecture*.

CONJECTURE. *Let $f : M^n \to \widetilde{M}^{n+m}(c)$ be an isometric immersion, where $\widetilde{M}^{n+m}(c)$ is a real space form of constant sectional curvature c. Then*

$$\rho \leq \|H\|^2 - \rho^\perp + c\,,$$

where ρ is the normalized scalar curvature (intrinsic invariant) and $\rho^\perp$ is the normalized normal scalar curvature (extrinsic invariant).

This conjecture was proven by the authors for submanifolds M^n of arbitrary dimension $n \geq 2$ and codimension 2 in real space forms $\widetilde{M}^{n+2}(c)$ of constant sectional curvature c. Also, a detailed characterisation of the equality case in terms of the shape operators of M^n in $\widetilde{M}^{n+2}(c)$ was given.

T. Choi and Z. Lu [7] proved that this conjecture is true for all 3-dimensional submanifolds M^3 of arbitrary codimension $m \geq 2$ in $\widetilde{M}^{3+m}(c)$. The characterisation of the equality case gives the specific forms of shape operators of M^3 in $\widetilde{M}^{3+m}(c)$.

For normally flat submanifolds, i.e., $R^\perp = 0$, the normal scalar curvature vanishes; B.Y. Chen [4] established the inequality

$$\rho \leq \|H\|^2 + c\,.$$

Hence, the conjecture is true for hypersurfaces of real space forms.

Other extensions of Wintgen inequality have been studied by P.J. De Smet, F. Dillen, J. Fastenakels, A. Mihai, J. Van der Veken, L. Verstraelen, L. Vrancken and the present author for certain submanifolds in Kähler, nearly Kähler and Sasakian spaces (see [10], [11], [18], [20], etc.). Recently, the DDVV-conjecture was finally settled for the general case by Z. Lu [16] and independently by J. Ge and Z. Tang [12].

THEOREM 1.2. *The Wintgen inequality*

$$\rho \leq \|H\|^2 - \rho^\perp + c\,,$$

holds for every submanifold M^n in any real space form $\widetilde{M}^{n+m}(c)$ ($n \geq 2$, $m \geq 2$). The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_i\}$

and $\{\xi_\alpha\}$, the shape operators of M^n in $\widetilde{M}^{n+m}(c)$ take the forms

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \lambda_2+\mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2-\mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix}, \quad A_{\xi_3} = \begin{pmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ and μ are real functions on M^n,

$$A_{\xi_4} = \cdots = A_{\xi_m} = 0\,.$$

Submanifolds satisfying the equality in Wintgen inequality are called *Wintgen ideal submanifolds* (see [15]). We recall that an n-dimensional submanifold M^n of an m-dimensional Riemannian manifold $\widetilde{M}^m$ is said to be a *Chen submanifold* (or an $\mathcal{A}$-submanifold) if its allied mean curvature vector

$$a(H) = \frac{1}{n}\sum_{r=2}^{m-n} \mathrm{Trace}(A_H A_r)e_r$$

vanishes identically, where $\{\xi_1, \xi_2, \ldots, \xi_{m-n}\}$ is an orthonormal frame in the normal bundle with ξ_1 parallel to H (see [22]).

We mention that M^n is a Chen submanifold if and only if

$$\sum_{i,j=1}^{n} g(h(e_i, e_j), H)h(e_i, e_j)$$

is parallel to the mean curvature vector H, where h is the second fundamental form and $\{e_1, \ldots, e_n\}$ an orthonormal frame (see [13]). Any Wintgen ideal submanifold is a Chen submanifold (see [8]).

2. Submanifolds in Sasakian space forms. A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}^{2m+1}, g)$ is said to be a *Sasakian manifold* if it admits an endomorphism ϕ of its tangent bundle $T\widetilde{M}^{2m+1}$, a vector field ξ and a 1-form η, satisfying:

$$\phi^2 = -Id + \eta\otimes\xi,\ \ \eta(\xi) = 1,\ \ \phi\xi = 0,\ \ \eta\circ\phi = 0\,,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \ \eta(X) = g(X, \xi)\,,$$

$$(\tilde{\nabla}_X\phi)Y = -g(X, Y)\xi + \eta(Y)X,\ \ \tilde{\nabla}_X\xi = \phi X\,,$$

for any vector fields X, Y on $\widetilde{M}^{2m+1}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

A plane section π in $T_p\widetilde{M}^{2m+1}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\widetilde{M}^{2m+1}(c)$.

The curvature tensor $\widetilde{R}$ of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ is given by (see [1])

$$\begin{aligned}\widetilde{R}(X,Y)Z =& \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}\\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\}\,,\end{aligned}$$

for any tangent vector fields X, Y, Z on $\widetilde{M}^{2m+1}(c)$. As examples of Sasakian space forms we mention $\mathbf{R}^{2m+1}$ and S^{2m+1}, with standard Sasakian structures (see [1], [26]).

Let M^n be an n-dimensional submanifold in a Sasakian space form $\widetilde{M}^{2m+1}(c)$. We denote by ∇ and h the Riemannian connection of M^n and the second fundamental form respectively. Let R be the Riemann curvature tensor of M^n, i.e., $R(X,Y,Z,W) = g(R(X,Y)W,Z)$, $\forall X,Y,Z,W \in \Gamma(TM^n)$. Then the Gauss equation is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W))\,,$$

for any vectors X, Y, Z, W tangent to M^n.

Let $p \in M^n$ and $\{e_1,\dots,e_n,e_{n+1},\dots,e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}^{2m+1}(c)$, such that $e_1,\dots,e_n$ are tangent to M at p. We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)\,.$$

A submanifold M^n normal to ξ in a Sasakian manifold is said to be a *C-totally real* submanifold. In this case, it follows that ϕ maps any tangent space of M^n into the normal space, that is, $\phi(T_pM^n) \subset T_p^\perp M^n$, for every $p \in M^n$. In particular, if $n = m$, then M^n is called a *Legendrian* submanifold.

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following.

(i) A submanifold M^{2n+1} tangent to ξ is called an *invariant* submanifold if ϕ preserves any tangent space of M^{2n+1}, that is, $\phi(T_pM^{2n+1}) \subset T_pM^{2n+1}$, for every $p \in M^{2n+1}$.

(ii) A submanifold M^n tangent to ξ is called an *anti-invariant* submanifold if ϕ maps any tangent space of M^n into the normal space, that is, $\phi(T_pM^n) \subset T_p^\perp M^n$, for every $p \in M^n$.

(iii) A submanifold M^n tangent to ξ is called a *contact slant* submanifold if for any vector $X \in T_pM^n$ linearly independent on ξ_p the angle between ϕX and T_pM^n is a constant, which is independent of the choice of $p \in M^n$ and $X \in T_pM^n$.

Other studied classes of submanifolds in Sasakian manifolds are the contact CR-submanifolds, contact CR-warped product submanifolds, etc.

For invariant submanifolds in Sasakian space forms, F. Dillen et al. [10] established the following inequalities.

THEOREM 2.1. *Let M^{2n+1} be an invariant submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$. Then:*

i) $4n(\tau^{\perp})^2 \geq [n(n+2)c + 3n^2 - 2\tau]^2 + n^2(m-n-1)(c-1)^2$,
with equality holding identically if and only if M^{2n+1} is an η-Einstein manifold.

ii) $4(\tau^{\perp})^2 \leq [(n^2+n+1)c + 3n^2 + n - 1 - 2\tau]^2 + (mn - n^2 - 1)(c-1)^2$,
with equality holding identically if and only if the rank of $\mathcal{A} = \sum_{\alpha=1}^{2(m-n)} A_{u_\alpha}^2$ is at most 2, where A is the shape operator and $\{u_1, \dots, u_{2(m-n)}\}$ is an orthonormal frame in the normal bundle.

It is well-known that any invariant submanifold of a Sasakian manifold is a minimal submanifold.

COROLLARY 2.2. *For an invariant submanifold M^{2n+1} of S^{2m+1}, we have*

$$\rho \leq 1 - \rho^{\perp}.$$

3. Generalized Wintgen inequality for Legendrian submanifolds. The main result of this paper is the generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms.

Let M^n be an n-dimensional C-totally real submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}^{2m+1}(c)$ and $\{e_1, \dots, e_n\}$ an orthonormal frame on M^n and $\{e_{n+1}, \dots, e_{2m}, e_{2m+1} = \xi\}$ an orthonormal frame in the normal bundle $T^{\perp}M^n$, respectively.

We denote by h and A the second fundamental form and the shape operator of M^n in $\widetilde{M}^{2m+1}(c)$. The Gauss and Ricci equations are

$$\begin{aligned} R(X,Y,Z,W) =& \frac{c+3}{4}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ &+ g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)), \end{aligned}$$

for any vector fields $X, Y, Z, W \in \Gamma(TM^n)$,

$$R^{\perp}(X,Y,\mu,\nu) = \frac{c-1}{4}[g(\phi X,\mu)g(\phi Y,\nu) - g(\phi X,\nu)g(\phi Y,\mu)] - g([A_\mu, A_\nu]X,Y),$$

for any vector fields $X, Y \in \Gamma(TM^n)$ and $\mu, \nu \in \Gamma(T^{\perp}M^n)$.

Following [25] we put

$$K_N = -\frac{1}{4}\sum_{r,s=1}^{2m-n+1} \mathrm{Trace}[A_r, A_s]^2,$$

where $A_r = A_{e_{n+r}}$, $r \in \{1, \dots, 2m-n+1\}$, and call it the *scalar normal curvature* of M^n. The normalized scalar normal curvature is given by $\rho_N = \frac{2}{n(n-1)}\sqrt{K_N}$.

Since $A_\xi = 0$, it follows that

$$K_N = -\frac{1}{2} \sum_{1 \le r < s \le 2m-n} \operatorname{Trace}[A_r, A_s]^2 = \sum_{1 \le r < s \le 2m-n} \sum_{1 \le i < j \le n} (g([A_r, A_s]e_i, e_j))^2 .$$

As usual we denote by $h^r_{ij} = g(h(e_i, e_j), e_{n+r}), i, j \in \{1, \dots, n\}, r \in \{1, \dots, 2m-n+1\}$. In terms of the components of the second fundamental form, we can express ρ_N by the formula

$$\rho_N = \frac{2}{n(n-1)} \left[\sum_{1 \le r < s \le 2m-n} \sum_{1 \le i < j \le n} \left(\sum_{k=1}^{n} (h^r_{jk} h^s_{ik} - h^r_{ik} h^s_{jk}) \right)^2 \right]^{\frac{1}{2}} . \tag{3.1}$$

First we establish an inequality involving the normalized scalar curvature ρ, the normalized scalar normal curvature ρ_N and the squared mean curvature $||H||^2$ for C-totally real submanifolds in Sasakian space forms.

PROPOSITION 3.1. *Let M^n be an n-dimensional C-totally real submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}^{2m+1}(c)$. Then we have*

$$||H||^2 + \frac{c+3}{4} \ge \rho + \rho_N .$$

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_1, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{2m}, e_{2m+1} = \xi\}$, the shape operators of M^n in $\widetilde{M}^{2m+1}(c)$ take the forms

$$A_{e_{n+1}} = \begin{pmatrix} \lambda_1 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix},$$

$$A_{e_{n+2}} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix}, \quad A_{e_{n+3}} = \begin{pmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ and μ are real functions on M^n,

$$A_{e_{n+4}} = \cdots = A_{e_{2m}} = A_{e_{2m+1}} = 0 .$$

PROOF. We see that

$$n^2||H||^2 = \sum_{r=1}^{2m-n} \left(\sum_{i=1}^{n} h^r_{ii} \right)^2 \tag{3.2}$$

$$= \frac{1}{n-1}\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}(h_{ii}^r - h_{jj}^r)^2 + \frac{2n}{n-1}\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r \,.$$

We shall use the following inequality (see [16]).

$$\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}(h_{ii}^r - h_{jj}^r)^2 + 2n\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}(h_{ij}^r)^2 \tag{3.3}$$

$$\ge 2n\left[\sum_{1\le r<s\le 2m-n}\sum_{1\le i<j\le n}\left(\sum_{k=1}^{n}(h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s)\right)^2\right]^{\frac{1}{2}} .$$

Combining (3.2), (3.3) and (3.1), we get

$$n^2||H||^2 - n^2\rho_N \ge \frac{2n}{n-1}\sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]\,. \tag{3.4}$$

On the other hand, the Gauss equation implies

$$\tau = \sum_{1\le i<j\le n} R(e_i, e_j, e_i, e_j) = \frac{n(n-1)(c+3)}{8} + \sum_{r=1}^{2m-n}\sum_{1\le i<j\le n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]\,. \tag{3.5}$$

Substituting (3.5) in (3.4) we obtain

$$||H||^2 - \rho_N \ge \rho - \frac{c+3}{4}\,,$$

which achieves the proof of Proposition 3.1.

The equality case holds identically if and only if the shape operators take the above forms with respect to suitable frames (we use similar arguments as in [16]). □

Next we are able to state a generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms.

THEOREM 3.2. *Let M^n be a Legendrian submanifold of a Sasakian space form $\widetilde{M}^{2n+1}(c)$. Then*

$$(\rho^\perp)^2 \le \left(\|H\|^2 - \rho + \frac{c+3}{4}\right)^2 + \frac{4}{n(n-1)}\left(\rho - \frac{c+3}{4}\right)\cdot\frac{c-1}{4} + \frac{(c-1)^2}{8n(n-1)}\,.$$

PROOF. Let M^n be a Legendrian submanifold of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ and $\{e_1, \ldots, e_n\}$ an orthonormal frame on M^n; then $\{e_{n+1} = \phi e_1, \ldots, e_{2n} = \phi e_n, e_{2n+1} = \xi\}$ is an orthonormal frame in the normal bundle $T^\perp M^n$.

The Ricci equation implies

$$g(R^\perp(e_i, e_j)e_{n+r}, e_{n+s})$$
$$= -\frac{c-1}{4}[g(\phi e_i, e_{n+r})g(\phi e_j, e_{n+s}) - g(\phi e_i, e_{n+s})g(\phi e_j, e_{n+r})] + g([A_r, A_s]e_i, e_j)$$

$$= -\frac{c-1}{4}(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) + g([A_r, A_s]e_i, e_j)\,,$$

for all $i, j \in \{1, \ldots, n\}$, $r, s \in \{1, \ldots, n\}$.

Then we have

$$(\tau^{\perp})^2 = \sum_{1\le r<s\le n}\sum_{1\le i<j\le n} g^2(R^{\perp}(e_i, e_j)e_{n+r}, e_{n+s}) \tag{3.6}$$

$$= \sum_{1\le r<s\le n}\sum_{1\le i<j\le n}\left[-\frac{c-1}{4}(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) + g([A_r, A_s]e_i, e_j)\right]^2$$

$$= K_N + \frac{(c-1)^2}{16}\sum_{1\le r<s\le n}\sum_{1\le i<j\le n}(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})^2$$

$$-\frac{c-1}{2}\sum_{1\le r<s\le n}\sum_{1\le i<j\le n}(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})g([A_r, A_s]e_i, e_j)$$

$$= \frac{n^2(n-1)^2}{4}\rho_N^2 + \frac{n(n-1)(c-1)^2}{32} - \frac{c-1}{4}\|h\|^2 + \frac{c-1}{4}n^2\|H\|^2\,.$$

Obviously by the Gauss equation we get

$$2\tau = n^2||H||^2 - ||h||^2 + n(n-1)\frac{c+3}{4}\,,$$

or, equivalently,

$$n^2||H||^2 - ||h||^2 = n(n-1)\left(\rho - \frac{c+3}{4}\right). \tag{3.7}$$

Substituting (3.7) in (3.6) we obtain

$$(\rho^{\perp})^2 \le \rho_N^2 + \frac{4}{n(n-1)}\left(\rho - \frac{c+3}{4}\right)\cdot\frac{c-1}{4} + \frac{(c-1)^2}{8n(n-1)}\,.$$

Taking account of Proposition 3.1, it follows that

$$(\rho^{\perp})^2 \le \left(\|H\|^2 - \rho + \frac{c+3}{4}\right)^2 + \frac{4}{n(n-1)}\left(\rho - \frac{c+3}{4}\right)\cdot\frac{c-1}{4} + \frac{(c-1)^2}{8n(n-1)}\,.$$

□

COROLLARY 3.3. *Let M^n be a minimal Legendrian submanifold of S^{2n+1}. Then*

$$\rho \le 1 - \rho^{\perp}\,.$$

We give a nontrivial example of a Wintgen ideal Legendrian submanifold in S^{2n+1}, where S^{2n+1} is the unit hypersphere of $\mathbb{C}^{n+1}$ endowed with the standard Sasakian structure.

First we consider $f : N^2 \to S^5$ a minimal, isometric, Legendrian immersion of a surface N^2 in the 5-dimensional sphere S^5.

In particular, if we take $N^2 = S^1 \times S^1$ and

$$f(t^1, t^2) = \frac{1}{\sqrt{3}}(\cos t^1, \sin t^1, \cos t^2, \sin t^2, \cos(t^1 + t^2), -\sin(t^1 + t^2))\,,$$

the shape operators are given by (see [26], page 350)

$$A_{e_3} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \quad A_{e_5} = 0 .$$

Then, by Proposition 3.1 it follows that f satisfies the equality case of Theorem 3.2.

For any minimal, isometric, Legendrian immersion $f : N^2 \to S^5$, we can find a suitable orthonormal frame such that the shape operator takes the above forms.

Next we denote by J the standard complex structure on $\mathbb{C}^{n+1}$ and consider the orthogonal decomposition $\mathbb{C}^{n+1} = \mathbb{C}^3 \oplus J(\mathbb{R}^{n-2}) \oplus \mathbb{R}^{n-2}$ and take the warped product $M^n = (0, \frac{\pi}{2}) \times_{\cos t} N^2 \times_{\sin t} S^{n-3}$. We define

$$x : M^n \to S^{2n+1}, \ (t, p, q) \mapsto (\cos t) f(p) + (\sin t) q ,$$

where S^{n-3} is the hypersphere of $\mathbb{R}^{n-2}$.

It is easily seen that x is a minimal, isometric, Legendrian immersion (see [6]). Moreover it satisfies the equality case of Theorem 3.2 because its shape operators have the desired forms (see also [6]), with $\lambda_1 = \lambda_2 = \lambda_3 = 0$ (x is a minimal immersion).

We remark that the equality case in Theorem 3.2 holds identically if and only if with respect to suitable frames the shape operator takes the forms from Proposition 3.1. This implies the following.

COROLLARY 3.4. *Any Wintgen ideal Legendrian submanifold of a Sasakian space form is a Chen submanifold.*

4. An inequality for contact slant submanifolds in Sasakian space forms. Contact slant submanifolds in Sasakian manifolds were defined by analogy with slant submanifolds in complex space forms (see [3]).

We recall that a submanifold M^n tangent to ξ of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ is said to be a *contact slant* submanifold [2] if for any $p \in M^n$ and $X \in T_pM^n$ linearly independent on ξ_p, the angle between ϕX and T_pM^n is a constant θ, called the slant angle of M^n.

Interesting geometric inequalities involving the scalar curvature and the Ricci curvature respectively on such submanifolds were established in [19].

A generalized Wintgen inequality can be stated for contact slant submanifolds in Sasakian space forms. For 3-dimensional such submanifolds a similar inequality was proven by the present author and Y. Tazawa [21].

THEOREM 4.1. *Let M^n be an n-dimensional contact θ-slant submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}^{2m+1}(c)$. Then*

$$\rho + \rho_N \le ||H||^2 + \frac{c+3}{4} + \frac{(3\cos^2\theta - 2)(c-1)}{4n} .$$

PROOF. Let M^n be a contact θ-slant submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$, $\{e_1, \ldots, e_{n-1}, e_n = \xi\}$ an orthonormal frame on M^n and $\{e_{n+1}, \ldots, e_{2m+1}\}$ an orthonormal frame in the normal bundle $T^{\perp}M^{n+1}$.

The Gauss equation is given by

$$R(X,Y,Z,W) = \frac{c+3}{4}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \tag{4.1}$$
$$+ \frac{c-1}{4}[\eta(X)\eta(W)g(Y,Z) - \eta(Y)\eta(W)g(X,Z)$$
$$+ \eta(Y)\eta(Z)g(X,W) - \eta(X)\eta(Z)g(Y,W)$$
$$+ g(\phi X,Z)g(\phi Y,W) - g(\phi X,W)g(\phi Y,Z) + 2g(\phi X,Y)g(\phi Z,W)]$$
$$+ g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)),$$

for any vector fields $X, Y, Z, W \in \Gamma(TM^n)$.

By using similar arguments as in the proof of Proposition 3.1, we get

$$n^2||H||^2 - n^2\rho_N \geq \frac{2n}{n-1}\sum_{r=1}^{2m-n+1}\sum_{1\leq i<j\leq n}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \tag{4.2}$$

The Gauss equation implies

$$\tau = \sum_{1\leq i<j\leq n} R(e_i,e_j,e_i,e_j) \tag{4.3}$$
$$= n(n-1)\frac{c+3}{8} + (n-1)\Big(\frac{3}{2}\cos^2\theta - 1\Big)\frac{c-1}{4}$$
$$+ \sum_{r=1}^{2m-n+1}\sum_{1\leq i<j\leq n}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2].$$

Substituting (4.3) in (4.2) we obtain

$$\rho + \rho_N \leq ||H||^2 + \frac{c+3}{4} + \frac{(3\cos^2\theta - 2)(c-1)}{4n}.$$

□

COROLLARY 4.2. *Let M^n be an n-dimensional contact slant submanifold of S^{2m+1}. Then*

$$\rho + \rho^{\perp} \leq \|H\|^2 + 1.$$

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