

SURFACES OF GLOBALLY F -REGULAR TYPE ARE OF FANO TYPE

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Abstract. We prove that a projective surface of globally F -regular type defined over a field of characteristic zero is of Fano type.

1. Introduction. Projective varieties of Fano type form a natural and important class of varieties from the point of view of the minimal model theory (see Definition 2.2 for definition). In this paper, we discuss a conjectural characterization of this class proposed by Schwede and Smith:

CONJECTURE 1.1 (= [SS10, Question 7.1]). *Let X be a normal projective variety defined over a field of characteristic zero. Then X is of Fano type if and only if it is of globally F -regular type.*

For the context behind this conjecture, see [SS10]. The ‘only if’ direction is already proven in [SS10, Theorem 1.2] as an application of [Tak04, Corollary 3.4]. That is, a variety of Fano type defined over a field of characteristic zero is of globally F -regular type. Aiming at the other direction, the authors constructed a boundary divisor Δ for any globally F -regular variety X (defined in a fixed positive characteristic) such that (X, Δ) is log Fano [SS10, Theorem 4.3 (i)]. The difficulty lies in that it is not clear if the boundary divisor Δ can be lifted to characteristic zero or not.

On the other hand, it is proven in [GOST15, Theorem 1.2] that a Mori dream space of globally F -regular type is of Fano type. Note that since a variety of Fano type in characteristic zero is a Mori dream space by [BCHM10, Corollary 1.3.2], if Conjecture 1.1 was solved affirmatively, then it would imply that a variety of globally F -regular type is a Mori dream space.

The purpose of this paper is to prove Conjecture 1.1 in dimension 2, *without* assuming X is a Mori dream space:

THEOREM 1.2. *Let X be a projective surface defined over a field of characteristic zero. Suppose that X is of globally F -regular type. Then X is of Fano type.*

Here we describe the outline of our proof. By Lemma 2.3 and Lemma 2.4, it is enough to show that X admits a $(-K_X)$ -MMP whose final model is a weak Fano variety. Note that in

the proof of [GOST15, Theorem 1.2], the existence of such a $(-K_X)$ -MMP was guaranteed by the assumption that X is a Mori dream space.

To show the existence of such a nice MMP, we first check in Proposition 2.8 that a globally F -regular surface (defined in a fixed positive characteristic) is a Mori dream space. This implies that X_μ , the reduction of our X to a positive characteristic, admits a $(-K_{X_\mu})$ -MMP.

Next we lift the $(-K_{X_\mu})$ -MMP to characteristic zero to obtain a $(-K_X)$ -MMP (strictly speaking we have to pass to a base change of X in this step, but it is a minor point). Since X_μ is a surface, each step of the $(-K_{X_\mu})$ -MMP is a divisorial contraction. Hence it is enough to show that the line bundle together with the global sections which define the morphism lift to characteristic zero (see Proposition 2.10). This follows from the standard deformation theory of line bundles (Lemma 2.9) and the vanishing theorem for cohomology of nef line bundles on globally F -regular varieties (Lemma 2.6). The rest of the argument is the same as that of the proof of [GOST15, Theorem 1.2].

Our proof does not use any classification result on surfaces (the author would like to thank Yoshinori Gongyo for pointing it out). In the end of this paper we clarify the obstructions which show up when we try to use our approach in higher dimensions.

After the author finished writing a draft of this paper, which is a part of the the author's PhD thesis, two relevant papers appeared. Gongyo and Takagi gave another proof for the main theorem of this paper in [GT13], and they also proved that surfaces of dense globally F -split type are of log Calabi–Yau type. There is also the paper [HP13], in which Hwang and Park gave yet another proof for the main theorem of this paper based on results of Sakai [Sak84].

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2. Preliminaries. For the notions of singularities in characteristic zero and Mori dream spaces which will be used below without definition, see [KM98] and [HK00] respectively. Readers can also refer to [GOST15, Sections 2.1–2.3].

DEFINITION 2.1. A normal projective variety X is called a *Mori dream space* provided that the following conditions hold.

- (1) X is \mathbb{Q} -factorial and \mathbf{Pic}_X^0 has dimension zero.
- (2) The nef cone of X is the affine hull of finitely many semi-ample line bundles.
- (3) There is a finite collection of small \mathbb{Q} -factorial modifications $f_i: X \dashrightarrow X_i$ such that each X_i satisfies (1)(2) and the movable cone of X is the union of the nef cones of X_i .

The symbol \mathbf{Pic}_X^0 in (1) stands for the identity component of the Picard scheme of X over the base field k (see [FGI⁺05, Chapter 9] for details). To make sense of (3), note that the Weil divisors on X_i are identified with those on X via the small birational map f_i .

Consider a normal projective variety X and an effective \mathbb{Q} -divisor Δ on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Such a pair (X, Δ) is called a *(log) pair*.

DEFINITION 2.2. A pair (X, Δ) is called a *log Fano pair* if its singularity is at worst klt and $-(K_X + \Delta)$ is ample. A normal projective variety X is said to be of *Fano type* if there exists an effective \mathbb{Q} -divisor Δ on X such that (X, Δ) is a log Fano pair. X is called a *weak Fano variety* if it admits at worst log terminal singularities and $-K_X$ is nef and big.

We will repeatedly use

LEMMA 2.3 (= [GOST15, Remark 2.8]). *A weak Fano variety is a variety of Fano type.*

The following claim, whose proof is implicitly given in the proof of [GOST15, Theorem 3.2], will also be used in this paper.

LEMMA 2.4. *Let*

$$(2.1) \quad X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_\ell$$

be a $(-K_X)$ -MMP such that X_ℓ is a variety of Fano type. Then so is X .

Next we explain about singularities in positive characteristics. Let R be a commutative algebra defined over a field of characteristic $p > 0$ such that the (absolute) Frobenius map $F: R \rightarrow R$ is finite. By an abuse of notation, the corresponding morphism between the spectra will be also denoted by

$$(2.2) \quad F: \text{Spec } R \rightarrow \text{Spec } R.$$

The finiteness of F is preserved under morphisms of finite type, and hence it comes for free if we are dealing with algebraic schemes over a perfect field. For an R -module M , the e -times pushforward $F_*^e M$ is isomorphic to M as an abelian group, whereas the action of R is twisted as

$$(2.3) \quad r \cdot F_*^e m = F_*^e(r^{p^e} m) \in F_*^e M.$$

In the rest of this paper, the element $F_*^e m \in F_*^e M$ will be simply denoted by m if it can be read off from the context.

DEFINITION 2.5. (i) An F -finite ring R of characteristic $p > 0$ is *strongly F -regular* if for every $c \neq 0 \in R$, there exists an integer $e \geq 1$ such that the homomorphism of R -modules

$$(2.4) \quad cF^e: R \xrightarrow{F^e} F_*^e R \xrightarrow{F_*^e(c \cdot)} F_*^e R; \quad a \mapsto ca^{p^e}$$

splits (i.e. admits a left inverse) as a homomorphism of R -modules.

(ii) Let X be a variety defined over an F -finite field of characteristic $p > 0$. We say that X is *globally F -regular* if for every effective divisor D on X , there exists an integer $e \geq 1$ such that

$$(2.5) \quad s_D F^e: \mathcal{O}_X \rightarrow F_*^e(\mathcal{O}_X(D)); \quad a \mapsto a^{p^e} s_D$$

splits as a morphism of \mathcal{O}_X -modules, where $s_D \in H^0(X, \mathcal{O}_X(D))$ is the global section of $\mathcal{O}_X(D)$ defined by the effective divisor D .

- (iii) Let X be a variety defined over a field of characteristic zero. X is said to be of *globally F -regular type* if there exists a model $\pi: X_A \rightarrow \text{Spec } A$ of X such that $X_\mu = X_A \otimes_A k(\mu)$ is globally F -regular for any closed point $\mu \in \text{Spec } A$ (see [GOST15, page 7] for the notion of models). A ring R of finite type over a field of characteristic zero is of strongly F -regular type if $X = \text{Spec } R$ is of globally F -regular type.

Globally F -regular varieties share nice properties with varieties of Fano type. In particular, we will use the following

LEMMA 2.6 (= [Smi00, Corollary 4.3]). *Let X be a globally F -regular projective variety. Then for any nef line bundle L on X , the vanishing*

$$(2.6) \quad H^i(X, L) = 0$$

holds for any $i > 0$.

In order to illustrate how the definition of global F -regularity is used in practice, we include a proof.

PROOF. Pick an ample line bundle A on X . Then $B_n = A \otimes L^{\otimes n}$ is ample for any $n > 0$. Take $q = p^e$ such that $F^e: \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ splits. Taking tensor product with B_n , we see that B_n is a direct summand of $F_*^e \mathcal{O}_X \otimes B_n \cong F_*^e (B_n^{\otimes q})$. Thus we obtain the sequence of inclusions $H^i(X, B_n) \subset H^i(X, B_n^{\otimes q}) \subset H^i(X, B_n^{\otimes q^2}) \subset \dots$, and this should vanish eventually by the Serre's vanishing theorem.

Now assume A is effective. So far we checked $H^i(X, A \otimes L^{\otimes n}) = 0$ for all $n > 0$. On the other hand, we know $s_A F^e: \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ splits for some $e > 0$. Taking tensor product with L , we obtain the inclusion $H^i(X, L) \subset H^i(X, A \otimes L^{\otimes q})$ to conclude the proof. \square

REMARK 2.7. For the sake of completeness, here we include a remark on \mathbb{Q} -factoriality of the varieties we are interested in. Log terminal singularities and strongly F -regular singularities are rational singularities. Hence the \mathbb{Q} -factoriality of surfaces with those singularities comes for free by [Lip69, Proposition 17.1].

The following proposition is the first step toward the proof of Theorem 1.2.

PROPOSITION 2.8. *Let X be a projective and globally F -regular surface defined over an F -finite field. Then X is a Mori dream space.*

PROOF. We directly check the conditions of Definition 2.1. First of all, see Remark 2.7 for the \mathbb{Q} -factoriality. Since $H^1(X, \mathcal{O}_X)$, the tangent space of \mathbf{Pic}_X^0 at the origin, is trivial by Lemma 2.6, the condition (1) is satisfied. To see (2), note that by [SS10, Theorem 4.3 (i)] there exists an effective \mathbb{Q} -divisor Δ on X such that (X, Δ) is a log Fano pair. By the cone and contraction theorems for surface pairs in positive characteristics (see [Tan14, Theorem 3.13] and [Tan14, Theorem 3.21] respectively), we see that the effective cone (hence the nef

cone) of X is rational polyhedral and that every extremal ray of the nef cone is generated by a semi-ample line bundle which defines the extremal contraction of the supporting facet of the effective cone. Finally, since X is a surface, any movable divisor is semi-ample by the Zariski–Fujita theorem. Therefore the condition (3) is automatically satisfied. \square

We next prove the liftability of morphisms from globally F -regular varieties.

LEMMA 2.9. *Let A be a complete local ring with the closed point $\mu \in \operatorname{Spec} A$, and $f: X_A \rightarrow \operatorname{Spec} A$ a flat projective morphism. Suppose $H^2(X_\mu, \mathcal{O}_{X_\mu}) = 0$, where X_μ is the closed fiber. Then any line bundle L_μ on X_μ extends to a line bundle L_A on X_A so that $L_A|_{X_\mu} \simeq L_\mu$.*

PROOF. See [FGI⁺05, Corollary 8.5.6(a)]. \square

PROPOSITION 2.10. *Let A and $f: X_A \rightarrow \operatorname{Spec} A$ be as in Lemma 2.9. Assume that X_μ is globally F -regular. Suppose that there exists a morphism with connected fibers $g_\mu: X_\mu \rightarrow Y_\mu$ to a normal projective variety Y_μ . Then g_μ lifts to a morphism with connected fibers $g_A: X_A \rightarrow Y_A$ over A , where Y_A is a normal projective scheme over A .*

PROOF. Pick a very ample line bundle $\mathcal{O}_{Y_\mu}(1)$ on Y_μ , so that the complete linear system corresponding to the line bundle $L_\mu = g_\mu^* \mathcal{O}_{Y_\mu}(1)$ defines the morphism g_μ : to see this use the equality $H^0(X_\mu, L_\mu) = H^0(Y_\mu, \mathcal{O}_{Y_\mu}(1))$, which follows from the assumption that g_μ has connected fibers.

Since X_μ is globally F -regular, we see $H^2(X_\mu, \mathcal{O}_{X_\mu}) = 0$ by Lemma 2.6. Hence by Lemma 2.9 we obtain a line bundle L_A on X_A which is a lift of L_μ .

Also, since L_μ is nef, we see $H^1(X_\mu, L_\mu) = 0$ again by Lemma 2.6. Then [Har77, Chapter III, Corollary 12.9] implies $H^1(X_A, L_A) = 0$. Applying [Har77, Chapter III, Theorem 12.11 (b)] for $i = 1$ and then [Har77, Chapter III, Theorem 12.11 (a)] for $i = 0$, we can check that $H^0(X_A, L_A)$ is a free A module satisfying

$$(2.7) \quad H^0(X_A, L_A) \otimes_A k(\mu) \xrightarrow{\sim} H^0(X_\mu, L_\mu).$$

Since L_μ is globally generated and the base locus of a linear system is closed, this implies that L_A is globally generated and that the corresponding complete linear system defines a morphism $g'_A: X_A \rightarrow Y'_A$, which restricts to the morphism g_μ over $\mu \in \operatorname{Spec} A$. By taking the Stein factorization $g_A: X_A \rightarrow Y_A$ of g'_A , we obtain the desired lift of g_μ . \square

In the final step of the proof of Theorem 1.2, we use the following fact that the property of being of Fano type descends under base field extension.

PROPOSITION 2.11. *Let X be a normal projective variety defined over a field k , and $k \subset K$ an extension of fields. If the base change $X_K = X \otimes_k K$ is a Mori dream space, so is X .*

PROOF. Note first that the condition $\dim \mathbf{Pic}^0 = 0$ is stable under the change of base field, because of the functorial behaviour $\mathbf{Pic}_{X_K/K} \cong \mathbf{Pic}_{X_k/k} \otimes_k K$. Hence by the characteri-

zation of Mori dream spaces [HK00, Proposition 2.9], it is enough to show that a Cox ring of X is of finite type over k .

Let $\Gamma \subset \text{Div } X$ be a finitely generated group of Cartier divisors on X which defines a Cox ring of X . Then we have the canonical isomorphism

$$(2.8) \quad R(X, \Gamma) \otimes_k K \xrightarrow{\sim} R(X_K, \Gamma_K),$$

where Γ_K is the pull-back of Γ to X_K .

As explained in [Oka11, Lemma 2.20], the multi-section ring on the RHS of (2.8) is of finite type over K . Using the descent of finite generation with respect to the field extension $k \subset K$, we obtain the finite generation of $R(X, \Gamma)$. \square

COROLLARY 2.12. *Let X be a normal projective variety defined over a field k of characteristic zero, and $k \subset K$ an extension of fields. If the base change $X_K = X \otimes_k K$ is of Fano type, so is X .*

PROOF. By [BCHM10, Corollary 1.3.2] and Proposition 2.11, X is a Mori dream space. Since $-K_{X_K}$ is big, so is $-K_X$. Hence we find a $(-K_X)$ -MMP $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_\ell$ such that $-K_{X_\ell}$ is semi-ample and big by [HK00, Proposition 1.11(1)]. Taking the base change by $k \subset K$, we get a $(-K_{X_K})$ -MMP. Since X_K is of Fano type, so are $X_i \otimes_k K$ by [GOST15, Lemma 3.1] and [FG12, Theorem 5.1]. Hence X_ℓ also has at worst log terminal singularities, which means that X_ℓ is a weak Fano variety. Finally we can use Lemma 2.4 to conclude the proof. \square

3. Proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Let $\pi : X_A \rightarrow \text{Spec } A$ be a model of X , so that X_μ is globally F -regular for a closed point $\mu \in \text{Spec } A$. By Proposition 2.8, X_μ is a Mori dream space.

Since X_μ is a Mori dream space, there exists a $(-K_{X_\mu})$ -MMP

$$(3.1) \quad X_\mu = X_{\mu,0} \rightarrow X_{\mu,1} \rightarrow \cdots \rightarrow X_{\mu,\ell}$$

such that $-K_{X_{\mu,\ell}}$ is semi-ample and big by [HK00, Proposition 1.11(1)]. Since each morphism is a projective morphism with connected fibers between normal varieties, which in particular is an algebraic fiber space, we see that $X_{\mu,i}$ are also globally F -regular by [GOST15, Lemma 2.14].

Let A_μ be the localisation of A at μ . Consider its completion \widehat{A} at the maximal ideal, and take the base change of $X_A \rightarrow \text{Spec } A$ by $A \subset \widehat{A}$. In the rest, we will write A to mean \widehat{A} .

The generic (resp. closed) point of $\text{Spec } A (= \widehat{A})$ will be denoted by ξ (resp. μ). Note that the generic fiber X_ξ is still of globally F -regular type.

By Proposition 2.10, the morphisms in (3.1) lift over $\text{Spec } A$. They will be denoted by $X_A = X_{A,0} \rightarrow X_{A,1} \rightarrow \cdots \rightarrow X_{A,\ell}$. Since $-mK_{X_{\mu,\ell}}$ is globally generated for sufficiently divisible m , by the same arguments as in the proof of Proposition 2.10, we see that all the global sections of $-mK_{X_{\mu,\ell}}$ lift over $\text{Spec } A$. Thus, restricting to the generic fiber, we obtain a $(-K_{X_\xi})$ -MMP $X_\xi = X_{\xi,0} \rightarrow X_{\xi,1} \rightarrow \cdots \rightarrow X_{\xi,\ell}$ such that $-K_{X_{\xi,\ell}}$ is semi-ample and big.

Using [GOST15, Lemma 2.14], we see that $X_{\xi, \ell}$ is also of globally F -regular type. Hence by [GOST15, Proposition 2.17], it has only log terminal singularities. Thus we see that it is a weak Fano variety. Finally we can use Lemma 2.3 and Lemma 2.4 to conclude the proof. \square

4. Concluding discussion: toward higher dimensions. When we try to generalize our result to higher dimensions, the following issues show up. First of all, the existence of an anti-canonical MMP was essential in our arguments. For that purpose we have to settle the following question affirmatively.

QUESTION 4.1. Is a projective and globally F -regular variety over a finite field always a Mori dream space?

Now let X be a projective variety of globally F -regular type. If Question 4.1 is answered affirmatively, then we obtain an anti-canonical MMP for a reduction X_μ of X to a positive characteristic ending up with a weak log Fano model. Suppose that $f_{\mu, n}: X_{\mu, n} \dashrightarrow X_{\mu, n+1}$ is a step of such an anti-canonical MMP. If it is a divisorial contraction, we can lift it to characteristic zero as before.

Suppose that it is a $(-K_{X_{\mu, n}})$ -flip. In this case we can still lift the flipping contraction, say $X_{\mu, n} \rightarrow Y_\mu$, to characteristic zero. Let $g_A: X_{A, n} \rightarrow Y_A$ be the $(-K_{X_{A, n}})$ -flipping contraction thus obtained. Now the question is if one can lift the flip $X_{\mu, n+1} \rightarrow Y_\mu$ to characteristic zero as well.

The existence of the $(-K_{X_{\mu, n}})$ -flip is equivalent to the finite generation of the \mathcal{O}_{Y_μ} -algebra

$$(4.1) \quad \bigoplus_{m \geq 0} g_{\mu, n*} \mathcal{O}_{X_{\mu, n}}(-mK_{X_{\mu, n}}) = \bigoplus_{m \geq 0} \mathcal{O}_{Y_\mu}(-mK_{Y_\mu}).$$

The corresponding statement for the existence of $(-K_{A, n})$ -flip holds as well.

If one could show that the restrictions of the sheaves $\omega_{Y_A}^{[m]} = (\omega_{Y_A}^m)^{**}$ to Y_μ are reflexive for all m which is divisible by a fixed integer $N > 0$, then that would imply that the natural homomorphism

$$(4.2) \quad \bigoplus_{m \geq 0, N|m} \mathcal{O}_{Y_A}(-mK_{Y_A}) \rightarrow \bigoplus_{m \geq 0, N|m} \mathcal{O}_{Y_\mu}(-mK_{Y_\mu})$$

is surjective. Then, using the Nakayama's lemma, we can derive the desired finite generation from that of (4.1).

Unfortunately the author does not see how to verify the necessary reflexivity.

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