Tohoku Math. J.
69 (2017), 1–23

REAL ANALYTIC COMPLETE NON-COMPACT SURFACES IN EUCLIDEAN SPACE WITH FINITE TOTAL CURVATURE ARISING AS SOLUTIONS TO ODES

PETER V. GILKEY, CHAN YONG KIM AND JEONGHYEONG PARK

(Received November 7, 2014, revised April 20, 2015)

Abstract. We use the solution space of a pair of ODEs of at least second order to construct a smooth surface in Euclidean space. We describe when this surface is a proper embedding which is geodesically complete with finite total Gauss curvature. If the associated roots of the ODEs are real and distinct, we give a universal upper bound for the total Gauss curvature of the surface which depends only on the orders of the ODEs and we show that the total Gauss curvature of the surface vanishes if the ODEs are second order. We examine when the surfaces are asymptotically minimal.

1. Introduction.

1.1. Historical context. Let Σ be a finitely connected non-compact geodesically complete Riemann surface. If the Gauss curvature K is integrable with respect to the Riemannian element of volume, dvol, then the *total Gauss curvature* is given by $K[\Sigma] := \int_\Sigma K\,\text{dvol}$. The total Gauss curvature plays an important role in many settings – and the role is subtly different in each application. Cohn-Vossen [7, 8] showed that

$$K[\Sigma] \le 2\pi\chi(\Sigma)\,. \tag{1.a}$$

Subsequently, Huber [12] reproved this result and showed additionally that if the total volume of Σ was finite, then equality holds. We also refer to a more recent derivation of Equation (1.a) by Bleecker [1] of using work of Chern. Higher dimensional analogues have been studied – see, for example, Dillen and Kühnel [9].

Mafra [20] examined the question of whether a holomorphic curve in $\mathbb{C}^2$ with finite total Gauss curvature is contained in an algebraic curve. Shioya [25] showed that if $K[\Sigma] < 2\pi$, then any maximal geodesic outside a sufficiently large compact set in Σ forms almost the same shape as that of a maximal geodesic in a flat cone. Shioya [26] subsequently considered the case where $K[\Sigma] = 2\pi$ (see also related work in Shiohama et al. [22, 23, 24]). Carron et al. [3] showed the existence of geometrically bound states if $K[\Sigma] < \infty$ and Σ is not homeomorphic to the plane. Li et al. [17] examined conformal maps of the 2-disk into $\mathbb{R}^n$ under the condition that the total Gauss curvature was at most 2π.

2010 *Mathematics Subject Classification.* Primary 53A05; Secondary 53C21.

Key words and phrases. Geodesically complete surface, finite total Gauss curvature, Gauss–Bonnet theorem, asymptotically minimal, constant coefficient ordinary differential equation.

The total Gauss curvature is central to the study of minimal surfaces. If the surface is minimal, Chern and Osserman [5] improved Equation (1.a) to become

$$K[\Sigma] \leq 2\pi(\chi(\Sigma) - e)$$

where e denotes the number of ends. We refer to subsequent work of Jorge and Meeks [14], and Kobuku et al. [15] (among others). We also refer to the discussion in Chen and Cheng [4] or Seo [21] where the ambient space is $\mathbb{H}^n$, to Esteve and Palmer [10] where the ambient manifold is a Cartan–Hadamard manifold, and to Ma [18] and Ma, Wang, and Wang [19] where the ambient space is Lorentzian.

Integrals of the Gauss curvature are not only important in the 2-dimensional setting. For example, Willerton [28] used the total Gauss curvature to examine the leading terms in the magnitude of an arbitrary homogeneous Riemannian manifold. Hwang et al. [13] used the total Gauss curvature to study the eigenvalues of the Laplacian. The total Gauss curvature plays an important role in Ricci flow. Li [16] showed the lowest eigenvalue in a family of geometric operators was monotonic under the normalized Ricci flow if the initial manifold had nonpositive total Gauss curvature. Chow et al. [6] gave a necessary and sufficient condition for the asymptotic volume ratio to be positive that involved the average Gauss curvature. That total Gauss curvature has also been studied for connections other than the Levi–Civita connection, see, for example, the discussion in Stephanov et al. [27].

1.2. Outline of the paper. In this paper, we shall discuss a family of non-compact real analytic isometric embeddings Σ of the plane in Euclidean space $\mathbb{R}^n$ which arise as the solution space to a pair of ODE's. The condition that Σ is real analytic is, of course, important as otherwise one could simple take a flat plane and put a small bump in it; this would, of course produce $K[\Sigma] = 0$ and for many of our examples, $K[\Sigma]$ is strictly negative.

We shall assume that all the roots of the associated characteristic polynomials are simple to avoid notational complexities with the multiplicities; the second author is investigating what happens when the roots have higher multiplicities in his thesis. We shall also assume that the real roots of the associated characteristic polynomials are dominant, i.e. control the asymptotic behavior of the embedding at infinity. Under these conditions, we will show in Theorem 1.7 that the surface Σ is properly embedded, is geodesically complete, and has infinite volume. We will also show in Theorem 1.8 that the Gauss curvature $K \in L^1(\Sigma, \text{dvol})$ and hence the total Gauss curvature $K[\Sigma]$ is well defined. In Example 8.2, we show that $|K|[\Sigma]$ can be infinite if the real roots are not dominant.

In Theorem 1.11, we use the Gauss–Bonnet theorem to express $K[\Sigma]$ in terms of integrals along the coordinate curves. The case where the two ODE's are second order is particularly tractable; we will use Theorem 1.11 to prove Theorem 1.12 which shows that $K[\Sigma] = 0$ if $n_1 = n_2 = 2$. In Example 8.5, we show $K[\Sigma]$ can be negative if $n_1 = n_2 = 3$ so this result is non-trivial. If all the roots of the associated ODE's are real, we will show in Theorem 1.13 that there is a uniform upper bound for $|K|[\Sigma]$ which depends only on the dimension; again, this uses Theorem 1.11. In Example 8.6, we will provide a family of examples Σ_k where this condition fails and where $\lim_{k\to\infty} K[\Sigma_k] = -\infty$. If all the roots are real and if there are at least two positive and at least two negative roots for each ODE, we show in Theorem 1.15 that the mean curvature vector H goes to zero at infinity and that $H \in L^3(\Sigma, \text{dvol})$ so Σ is asymptotically minimal; in Example 8.7 we show the condition that there are at least two roots of each sign is essential in this regard and in Example 8.8 we show that $p = 3$ is optimal if a uniform estimate is required.

The present paper grew out of the study of curves of finite total first curvature given by an ODE with two other authors [11]. We begin by reviewing these results for the convenience of the reader as many of our subsequent theorems depend on these results. The rest of the introduction is then a careful statement of the main results of the paper. Section 2 is an introduction to the geometry of surfaces embedded in $\mathbb{R}^n$ and expresses the relevant geometric quantities we shall need in terms of the exterior algebra as this is a convenient formalism for our purposes. In Section 3, we demonstrate Theorem 1.7, in Section 4 we establish Theorem 1.8, in Section 5 we derive Theorem 1.11, and in Section 6 we use the Gauss–Bonnet theorem to prove Theorem 1.12; this express $K[\Sigma]$ in terms of the curves defined by the two ODE's and plays a central role in the proof of Theorem 1.13 which gives a uniform estimate for $K[\Sigma]$. In Section 7, we examine the norm of the mean curvature vector and prove Theorem 1.15. We conclude the paper in Section 8 by presenting some Mathematica calculations using a Mathematica program constructed by M. Brozos-Vazquez [2] to discuss various illustrative examples.

1.3. Curvature. If $\vec{u}, \vec{v} \in \mathbb{R}^n$, let $(\vec{u}, \vec{v}) := u^1 v^1 + \dots + u^n v^n$ and $||\vec{u}||^2 := (\vec{u}, \vec{u})$. We extend $(\cdot, \cdot)$ to an inner product on tensors on all types and, in particular, to the exterior algebra on $\mathbb{R}^n$. If $\sigma : \mathbb{R} \to \mathbb{R}^n$ is an immersed curve, then the *element of arc length* $ds = ds_\sigma$, the *first curvature* $\kappa = \kappa_\sigma$, and the *total first curvature* $\kappa[\sigma]$ are defined, respectively, by:

$$ds := ||\dot{\sigma}(t)||dt, \quad \kappa_\sigma(t) := \frac{||\dot{\sigma}(t) \wedge \ddot{\sigma}(t)||}{||\dot{\sigma}(t)||^3},$$
$$\kappa[\sigma] := \int_\sigma \kappa ds = \int_{-\infty}^{\infty} \frac{||\dot{\sigma}(t) \wedge \ddot{\sigma}(t)||}{||\dot{\sigma}(t)||^2} dt.$$

If Σ is an immersed surface in $\mathbb{R}^n$, let dvol be the Riemannian measure and let K be the Gauss curvature. If $|K|$ is in $L^1(M, \text{dvol})$, let $K[\Sigma] := \int_\Sigma K \,\text{dvol}$.

1.4. Curves defined by ODEs. We review briefly some previous results that we shall need and refer to the discussion in [11] for further details. If $\phi = \phi(t)$ is a smooth real valued function, let $\phi^{(i)}$ be the i^{th} derivative. Let

$$P(\phi) := \phi^{(n)} + c_{n-1}\phi^{(n-1)} + \dots + c_0\phi$$

be a real constant coefficient ordinary differential operator of order $n \geq 2$. Let $\mathcal{S} = \mathcal{S}(P)$ be the solution space of P, let $\mathcal{P} = \mathcal{P}(P)$ be the characteristic polynomial of P, and let $\mathcal{R} = \mathcal{R}(P)$ be the roots of $\mathcal{P}$:

$$\mathcal{S} := \{\phi \in C^\infty(\mathbb{R}) : P(\phi) = 0\}, \quad \mathcal{P}(\lambda) := \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_0,$$
$$\mathcal{R} := \{\lambda \in \mathbb{C} : \mathcal{P}(\lambda) = 0\}.$$

Enumerate the roots in the form $\mathcal{R} = \{s_1, \ldots, s_k, z_1, \ldots, z_u, \bar{z}_1, \ldots, \bar{z}_u\}$ where the $\{s_i\}$ are the distinct real roots of $\mathcal{P}$ for $1 \le i \le k$ and where the $\{z_j = a_j + b_j\sqrt{-1}\}$ are the distinct complex roots of $\mathcal{P}$ for $1 \le j \le u$ where $b_j > 0$. We order the real roots so $s_1 > \cdots > s_k$ and the complex roots so $a_1 \ge a_2 \ge \cdots$. If there are no real roots, we set $k = 0$; if there are no complex roots, we set $u = 0$.

DEFINITION 1.1. Let $\Re(\cdot)$ be the real part of a complex number. We shall say that a root $\lambda \in \mathcal{R}$ is *dominant* if $\Re(\lambda) > \Re(\mu)$ for all $\mu \in \mathcal{R} - \{\lambda\}$ and if $\Re(\lambda) > 0$ or if $\Re(\lambda) < \Re(\mu)$ for all $\mu \in \mathcal{R} - \{\lambda\}$ and if $\Re(\lambda) < 0$; note that a dominant root is necessarily real and is either s_1 or s_k. One has that s_1 is dominant if $s_1 > 0$ and if $s_1 > a_1$ and similarly that s_k is dominant if $0 > s_k$ and if $a_u > s_k$. If s_1 is dominant and if $\lambda \in \mathcal{R} - \{s_1\}$ satisfies $\Re(\lambda) \ge \Re(\mu)$ for all $\mu \in \mathcal{R} - \{s_1\}$, then we say λ is *sub-dominant*. Similarly if s_k is dominant and if $\lambda \in \mathcal{R} - \{s_k\}$ satisfies $\Re(\lambda) \le \Re(\mu)$ for all $\mu \in \mathcal{R} - \{s_k\}$, then we say λ is *sub-dominant*.

If all the roots are simple (i.e. have multiplicity 1 so $n = k + 2u$), then the canonical basis for the solution space $\mathcal{S}$ consists of the functions:

$$\{e^{s_i t},\ e^{a_j t}\cos(b_j t),\ e^{a_j t}\sin(b_j t)\} \text{ for } 1 \le i \le k \text{ and } 1 \le j \le u\,. \tag{1.b}$$

The functions $\{e^{s_i t}\}$ do not appear, of course, if there are no real roots and, similarly, the functions $\{e^{a_j t}\cos(b_j t), e^{a_j t}\sin(b_j t)\}$ do not appear if there are no complex roots. More generally, if s_i is a real root of multiplicity $\nu \ge 2$, then we must replace the single function $e^{s_i t}$ in Equation (1.b) by the ν functions

$$\{e^{s_i t},\ te^{s_i t}, \ldots,\ t^{\nu-1}e^{s_i t}\}$$

while if z_j is a complex root of multiplicity $\nu \ge 2$, then we must replace the pair of functions $\{e^{a_j t}\cos(b_j t), e^{a_j t}\cos(b_j t)\}$ in Equation (1.b) by the 2ν functions:

$$\begin{gathered}\{e^{a_j t}\cos(b_j t),\ te^{a_j t}\cos(b_j t), \ldots, t^{\nu-1}e^{a_j t}\cos(b_j t)\,,\\ e^{a_j t}\sin(b_j t),\ te^{a_j t}\sin(b_j t), \ldots, t^{\nu-1}e^{a_j t}\sin(b_j t)\}\,.\end{gathered}$$

Let $\{\phi_1, \ldots, \phi_n\}$ be an enumeration of the canonical basis for $\mathcal{S}$ described above. We define the *associated curve* $\sigma = \sigma_P$ by setting:

$$\sigma(t) := (\phi_1(t), \ldots, \phi_n(t)) : \mathbb{R} \to \mathbb{R}^n\,.$$

If $\{s_1, s_k\}$ are dominant roots, then these roots control the behavior of $\|\sigma\|$ at infinity, i.e.:

$$\lim_{t\to\infty} e^{-s_1 t}\|\sigma\| = 1\,, \qquad \text{and} \qquad \lim_{t\to-\infty} e^{-s_k t}\|\sigma\| = 1\,.$$

We refer to [11] for the proof of the following result:

THEOREM 1.2. *If all the roots of $\mathcal{P}$ are simple and if $\{s_1, s_k\}$ are dominant roots, then σ is a proper embedding of $\mathbb{R}$ in $\mathbb{R}^n$ of infinite length with $\kappa[\sigma] < \infty$.*

REMARK 1.3. In Example 8.1 we will see that $\kappa[\sigma]$ can be infinite if there exists a complex root $\lambda \in \mathcal{R}$ with $\Re(\lambda)$ maximal or minimal.

We have taken the standard inner product on $\mathbb{R}^n$ to define the element of arc length ds and the geodesic curvature κ. The precise inner product is irrelevant; Theorem 1.2 continues to hold for an arbitrary positive definite inner product on $\mathbb{R}^n$. Equivalently, this shows that it is not necessary to choose the standard basis for $\mathcal{S}$ in defining σ; any basis will do. Consequently, Theorem 1.2 is really a result about the solution space $\mathcal{S}$. If $a_1 \geq s_1$ or $s_k \geq a_u$, then the dominant exponential involves sin and cos. This implies that the total first curvature is infinite. There are analogous results when multiple roots are permitted; as they are a bit more complicated to state, we shall refer to [11] for details.

If all the roots of P are real and simple, then the associated curve is of the form

$$\sigma(t) = \sigma_{s_1,\dots,s_n}(t) := (e^{s_1 t}, \dots, e^{s_n t}) \text{ where } s_1 > \cdots > s_n\,.$$

There is a uniform estimate for the total first curvature [11] of such a curve:

THEOREM 1.4. $\kappa[\sigma_{s_1,\dots,s_n}] \leq 2(n-1)n$.

REMARK 1.5. Let $\sigma_k(t) := (e^t, \cos(kt), \sin(kt), e^{-t})$ so $\mathcal{R}_k = \{\pm 1, \pm\sqrt{-1}\}$. In Example 8.4, we will show that $\lim_{k\to\infty} \kappa[\sigma_k] = \infty$ so the assumption all the roots are real is essential to establish a uniform upper bound.

1.5. Surfaces defined by a pair of ODEs. We establish some basic notational conventions for the remainder of the paper. Let n_1 (resp. n_2) be the order, let $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) be the roots, and let σ_1 (resp. σ_2) be the curve defined by the ODE P_1 (resp. P_2). We assume that all the roots are simple and express:

$$\begin{aligned}
&\mathcal{R}_1 = \{r_1, \dots, r_k, a_1 \pm b_1\sqrt{-1}, \dots, a_p \pm b_p\sqrt{-1}\}\,,\\
&\mathcal{R}_2 = \{s_1, \dots, s_\ell, c_1 \pm d_1\sqrt{-1}, \dots, c_q \pm d_q\sqrt{-1}\}\,,\\
&\sigma_1(t_1) := (e^{r_1 t_1}, \dots, e^{r_k t_1}, e^{a_1 t_1}\cos(b_1 t_1), e^{a_1 t_1}\sin(b_1 t_1), \dots)\,,\\
&\sigma_2(t_2) := (e^{s_1 t_2}, \dots, e^{s_\ell t_2}, e^{c_1 t_2}\cos(d_1 t_2), e^{c_1 t_2}\sin(d_1 t_2), \dots)\,.
\end{aligned}$$

Let $n = n_1 n_2$ and let $\Sigma : \mathbb{R}^2 \to \mathbb{R}^n$ be defined by:

$$\Sigma(t_1, t_2) := \sigma_1(t_1) \otimes \sigma_2(t_2)\,.$$

If $\{\phi_{1,1}, \dots, \phi_{n_1,1}\}$ (resp. $\{\phi_{1,2}, \dots, \phi_{n_2,2}\}$) is the standard basis for the solution space of P_1 (resp. P_2), then the coordinates of Σ are the collection of functions $\{\phi_{i,1}(t_1)\phi_{j,2}(t_2)\}$ for $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$.

DEFINITION 1.6. Let P_1 and P_2 be real ODEs with simple roots. We say that *the real roots are dominant* if $\{r_1, r_k\}$ are dominant roots for P_1 and $\{s_1, s_\ell\}$ are dominant roots for P_2.

We shall establish the following generalization of Theorem 1.2 in Section 2.

THEOREM 1.7. *If all the roots of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are simple and if the real roots are dominant roots, then Σ is a proper embedding of $\mathbb{R}^2$ in $\mathbb{R}^n$ which is geodesically complete and which has infinite volume.*

We shall establish the following result in Section 4:

THEOREM 1.8. *If all the roots of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are simple and if the real roots are dominant, then there exist $\varepsilon = \varepsilon(\Sigma) > 0$ and $C = C(\Sigma) > 0$ so*

(1) $|K(t_1, t_2)| \le Ce^{-\varepsilon\|(t_1,t_2)\|}$.
(2) $g|K(t_1, t_2)| \le Ce^{-\varepsilon\|(t_1,t_2)\|}$.
(3) $|K|[\Sigma] < \infty$.

REMARK 1.9. We will show in Example 8.2 that this can fail if the real roots are not dominant.

DEFINITION 1.10. Let σ be an immersed curve in $\mathbb{R}^n - \{0\}$ so that $\sigma \wedge \dot\sigma \neq 0$; this is the case if σ is defined by a constant coefficient ODE of course. We define:

$$\Theta_\sigma(t) := \frac{(\dot\sigma(t) \wedge \sigma(t), \dot\sigma(t) \wedge \ddot\sigma(t))}{\|\dot\sigma(t) \wedge \sigma(t)\| \cdot \|\dot\sigma(t)\|^3} .$$

If Θds is integrable, we set

$$\Theta[\sigma] := \int_{-\infty}^{\infty} \Theta(\sigma) ds = \int_{-\infty}^{\infty} \frac{(\dot\sigma(t) \wedge \sigma(t), \dot\sigma(t) \wedge \ddot\sigma(t))}{\|\dot\sigma(t) \wedge \sigma(t)\| \cdot \|\dot\sigma(t)\|^2} dt .$$

We use the Cauchy–Schwarz inequality to see

$$|\Theta_\sigma(t)| \le \kappa_\sigma(t) \quad \text{so} \quad |\Theta|[\sigma] \le \kappa[\sigma] . \tag{1.c}$$

Consequently, if all the roots of $\mathcal{P}$ are simple and if the real roots are dominant, then $\Theta[\sigma] := \int_\sigma \Theta ds$ is well defined. For example, if $\sigma(t) = e^{at}e_1 + e^{bt}e_2$ is a curve in $\mathbb{R}^2$ for $a > 0 > b$, then

$$\begin{aligned}
\Theta(\sigma)ds &= \frac{e^{(2a+2b)t}\big((ae_1 + be_2) \wedge (e_1 + e_2), (ae_1 + be_2) \wedge (a^2e_1 + b^2e_2)\big)}{e^{(a+b)t}\|(ae_1 + be_2) \wedge (e_1 + e_2)\| \cdot \{a^2e^{2at} + b^2e^{2bt}\}} dt \\
&= e^{(a+b)t}\frac{\big((a-b)e_1 \wedge e_2, (ab^2 - a^2b)e_1 \wedge e_2\big)}{\|(a-b)e_1 \wedge e_2\| \cdot \{a^2e^{2at} + b^2e^{2bt}\}} dt \\
&= \frac{(a-b)ab(b-a)}{|a-b|} \frac{e^{(a+b)t}}{a^2e^{2at} + b^2e^{2bt}} dt .
\end{aligned} \tag{1.d}$$

Since $a > 0 > b$, the coefficient is $|(a-b)ab| > 0$ and $\Theta[\sigma] = \kappa[\sigma] > 0$.

If P_1 and P_2 are admissible, then $|K|[\Sigma]$ is finite and we set $K[\Sigma] := \int_\Sigma K$ dvol. We will use the Gauss–Bonnet theorem to establish the following result in Section 5:

THEOREM 1.11. *If all the roots of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are simple and if the real roots are dominant, then* $0 = K[\Sigma] - 2\Theta[\sigma_1] - 2\Theta[\sigma_2] + 2\pi$.

The 4-dimensional setting is particularly tractable. We will establish the following result in Section 6:

THEOREM 1.12. *If all the roots of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are simple, if the real roots are dominant, and if $n_1 = n_2 = 2$, then $K[\Sigma] = 0$.*

NOTE. In Example 8.3, we will present an example where $n_1 = n_2 = 2$ and where $|K|[\Sigma] \neq 0$ so this result is non-trivial.

Although Theorem 1.8 shows $K[\Sigma]$ is well defined, it does not provide a useful upper bound for the total Gauss curvature of Σ. Suppose the roots of $\mathcal{P}_1$ and $\mathcal{P}_2$ are real, simple, and dominant. This means that

$$\begin{aligned} &\sigma_1(t_1) = (e^{r_1 t_1}, \dots, e^{r_k t_1}) \text{ for } r_1 > 0 > r_k\,, \\ &\sigma_2(t_2) = (e^{s_1 t_2}, \dots, e^{s_\ell t_2}) \text{ for } s_1 > 0 > s_\ell\,. \end{aligned} \tag{1.e}$$

In this setting, we combine Theorem 1.4, Equation (1.c), and Theorem 1.11 to obtain:

THEOREM 1.13. *If the roots of $\mathcal{P}_1$ and $\mathcal{P}_2$ are real and simple, and if the real roots are dominant, then $|K|[\Sigma] \leq 2\pi + 4n_1(n_1 - 1) + 4n_2(n_2 - 1)$.*

REMARK 1.14. We will show in Example 8.5 that this result is non-trivial; $K[\Sigma]$ does not vanish identically if $n_1 > 2$ and $n_2 > 2$. Furthermore, we will give a family of surfaces Σ_k in $\mathbb{R}^8$ where one of the sub-dominant roots is complex where $\lim_{k\to\infty} K[\Sigma_k] = -\infty$ so there is no universal bound in this setting.

Let L_{ij} be the second fundamental form; this is vector valued and takes values in $T\Sigma^\perp$ (see Section 2 for details). The unnormalized *mean curvature* vector H is given by:

$$H = g^{ij} L_{ij} \in T\Sigma^\perp\,.$$

We omit the normalizing factor of $\frac{1}{n}$ as it plays no role. The surface is minimal if and only if $H = 0$. In Section 7 we will show the surface is asymptotically minimal if there are at least two positive and at least two negative roots for each ODE:

THEOREM 1.15. *Assume that the roots of $\mathcal{P}_1$ and $\mathcal{P}_2$ are real and simple, and that the real roots are dominant. Assume that $r_1 > r_2 > 0 > r_{k-1} > r_k$ and that $s_1 > s_2 > 0 > s_{\ell-1} > s_\ell$.*

(1) *There exists $\varepsilon = \varepsilon(\Sigma) > 0$ and $C = C(\Sigma) > 0$ so $\|H\| \leq Ce^{-\varepsilon\|(t_1,t_2)\|}$.*
(2) $H \in L^3(\Sigma, \text{dvol})$.

REMARK 1.16. In Example 8.7, we will show H need not be bounded if $0 > r_2$ and $0 > s_2$. Fix $p < 3$. In Example 8.8, we will exhibit a surface Σ_p satisfying the hypotheses of Theorem 1.15 where H does not belong to L^p. This shows that $p = 3$ is the best universal estimate.

Throughout this paper, we will let $C = C(\Sigma)$ denote a generic positive constant that can depend on Σ but not on (t_1, t_2).

2. The geometry of surfaces embedded in $\mathbb{R}^n$. Let $\Sigma(t_1, t_2)$ be an immersed surface in $\mathbb{R}^n$. The components g_{ij} of the Riemannian metric and the Riemannian measure dvol on Σ are defined by setting:

$$g_{ij} := (\partial_{t_i}\Sigma, \partial_{t_j}\Sigma) \text{ and } \text{dvol} := g\,dt_1 dt_2 \text{ where } g := \sqrt{g_{11}g_{22} - g_{12}g_{12}}\,.$$

Let ∇ be the Levi–Civita connection of Σ. If π_Σ denotes orthogonal projection on the tangent space of Σ and if X and Y are tangent vector fields along Σ, then:

$$\nabla_X Y = \pi_\Sigma\{XY(\Sigma)\}\,.$$

The curvature tensor is given by $R(X,Y)Z := (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$. The second fundamental form $L(X,Y)$ is defined to be:

$$L(X,Y) = (1-\pi_\Sigma)\{XY(\Sigma)\}\,. \tag{2.a}$$

The second fundamental form is vector valued and takes values in $T\Sigma^\perp$. Let $\{X,Y\}$ be linearly independent tangent vector fields along Σ. The Gauss curvature K is given by:

$$K := (R(X,Y)Y,X)g^{-2}\,.$$

One has the Theorema Egregium of Gauss:

$$K = \{(L(X,X),L(Y,Y)) - (L(X,Y),L(X,Y))\}g^{-2}\,. \tag{2.b}$$

If σ is a curve in Σ and if ν is a unit normal to $\dot\sigma$ in Σ, the geodesic curvature is:

$$\kappa_g(\sigma) := (\nabla_{\dot\sigma}\dot\sigma,\nu)\|\dot\sigma\|^{-2}\,.$$

This vanishes if and only if σ is a geodesic and changes sign if we change the sign of the normal.

We now introduce a convenient formalism to discuss various geometric quantities in terms of wedge products. Although the formulas are well-known, we shall give the proofs to establish notation. Fix a point (a,b) of Σ. Let $\gamma_1(t_1) := \Sigma(t_1,b)$ and $\gamma_2(t_2) := \Sigma(a,t_2)$ be the coordinate curves through (a,b). Let

$$\Sigma_{/i} := \partial_{t_i}\Sigma\,,\quad \Sigma_{/ij} := \partial_{t_i}\partial_{t_j}\Sigma\,,\quad \tilde L_{ij} := \Sigma_{/1}\wedge\Sigma_{/2}\wedge\Sigma_{/ij}\,.$$

LEMMA 2.1. *Let $\{e_1,e_2\}$ be an orthonormal frame for $T\Sigma$ so $\Sigma_{/1}\wedge\Sigma_{/2}$ is a positive multiple of $e_1\wedge e_2$. Choose the normal to γ_1 in Σ which points in the direction of $\Sigma_{/2}$.*

(1) $g = ||\Sigma_{/1}\wedge\Sigma_{/2}||$.
(2) $\kappa_g(\gamma_1) = (\Sigma_{/1}\wedge\Sigma_{/2},\Sigma_{/1}\wedge\Sigma_{/11})\cdot g^{-1}\|\Sigma_{/1}\|^{-3}$.
(3) $\Sigma_{/1}\wedge\Sigma_{/2}\wedge\Sigma_{/ij} = ge_1\wedge e_2\wedge L_{ij}$.
(4) $K = g^{-4}\{(\tilde L_{11},\tilde L_{22}) - (\tilde L_{12},\tilde L_{12})\}$.

PROOF. Fix a point $P\in\Sigma$ and let $\{e_1,e_2\}$ be an orthonormal basis for $T_P\Sigma$. Complete $\{e_1,e_2\}$ to an orthonormal basis $\{e_1,\dots,e_n\}$ for $\mathbb{R}^n$. We may assume the basis chosen so that $\Sigma_{/1} = a_1e_1$ and $\Sigma_{/2} = b_1e_1 + b_2e_2$ where $b_2 > 0$. Then

$$\Sigma_{/1}\wedge\Sigma_{/2} = a_1b_2e_1\wedge e_2 \quad\text{and}\quad ||\Sigma_{/1}\wedge\Sigma_{/2}||^2 = a_1^2b_2^2\,.$$

We show that $g = \|\Sigma_{/1}\wedge\Sigma_{/2}\|$ and establish Assertion 1 by computing:

$$g_{11} = a_1^2\,,\quad g_{22} = b_1^2 + b_2^2\,,\quad g_{12} = a_1b_1\,,$$
$$g^2 = g_{11}g_{22} - g_{12}^2 = a_1^2(b_1^2+b_2^2) - a_1^2b_1^2 = a_1^2b_2^2\,.$$

With our normalizations, e_2 is the normal to $\dot{\gamma}_1$ in Σ which points in the direction of $\Sigma_{/2}$. Further normalize the orthonormal frame so that $\Sigma_{/11} = c_1e_1 + c_2e_2 + c_3e_3$. We prove Assertion 2 by computing:

$$\begin{aligned}
&\Sigma_{/1} \wedge \Sigma_{/2} = a_1b_2e_1 \wedge e_2 = ge_1 \wedge e_2\,,\\
&\Sigma_{/1} \wedge \Sigma_{/11} = a_1c_2e_1 \wedge e_2 + a_1c_3e_1 \wedge e_3\,,\\
&(\Sigma_{/1} \wedge \Sigma_{/2}, \Sigma_{/1} \wedge \Sigma_{/11}) = a_1c_2g\,,\\
&\kappa_g(\gamma_1) = c_2a_1^{-2} = (\Sigma_{/1} \wedge \Sigma_{/2}, \Sigma_{/1} \wedge \Sigma_{/11})a_1^{-3}g^{-1}\,.
\end{aligned}$$

The second fundamental form L_{ij} of Equation (2.a) is the projection of $\Sigma_{/ij}$ on $T\Sigma^\perp$. Expand

$$\Sigma_{/ij} = \Gamma_{ij}{}^1e_1 + \Gamma_{ij}{}^2e_2 + L_{ij}{}^3e_3 + \cdots + L_{ij}{}^ne_n$$

where the $\Gamma_{ij}{}^k$ are the Christoffel symbols of the Levi–Civita connection and where the second fundamental form is given by $L_{ij} = L_{ij}{}^3e_3 + \cdots + L_{ij}{}^ne_n$. By Assertion 1, $\Sigma_{/1} \wedge \Sigma_{/2} = ge_1 \wedge e_2$. We derive Assertion 3 and Assertion 4 from Equation (2.b) and complete the proof by computing:

$$\begin{aligned}
&\tilde{L}_{ij} = ge_1 \wedge e_2 \wedge \sum_{\nu=1}^{n} L_{ij}{}^\nu e_\nu = g\sum_{\nu=3}^{n} L_{ij}{}^\nu e_1 \wedge e_2 \wedge e_\nu\,,\\
&(\tilde{L}_{ij}, \tilde{L}_{kl}) = g^2\sum_{\nu=3}^{n} L_{ij}{}^\nu L_{kl}{}^\nu = g^2(L_{ij}, L_{kl})\,,\\
&K = g^{-2}\{(L_{11}, L_{22}) - (L_{12}, L_{12})\} = g^{-4}\{(\tilde{L}_{11}, \tilde{L}_{22}) - (\tilde{L}_{12}, \tilde{L}_{12})\}\,.
\end{aligned}$$

□

3. The proof of Theorem 1.7. Assume that all the roots of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are simple and that the real roots are dominant. We adopt the notation of Section 1.5 throughout. We shall concentrate on the first quadrant $t_1 \geq 0$ and $t_2 \geq 0$ for the most part as the remaining quadrants can be handled similarly by reparametrizing Σ to set $\tilde{t}_i = \pm t_i$ as necessary. Set

(3.a) $$\varepsilon_1 = \varepsilon_1(\Sigma) = \min\{r_1, s_1, -r_k, -s_\ell\} > 0\,.$$

Choose $\alpha_1 \in \mathcal{R}_1 - \{r_1\}$ so $a_1 := \Re(\alpha_1)$ is maximal. Similarly, choose $\beta_1 \in \mathcal{R}_2 - \{s_1\}$ so $c_1 := \Re(\beta_1)$ is maximal; both α_1 and β_1 are sub-dominant. Let

(3.b) $$\mathcal{G}(t_1, t_2) := e^{2r_1t_1+(s_1+c_1)t_2} + e^{(r_1+a_1)t_1+2s_1t_2}\,.$$

The following estimates are fundamental:

LEMMA 3.1. *Assume that all the roots of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are simple and that the real roots are dominant. There exist $C_i = C_i(\Sigma) > 0$ so that:*

(1) *If $t_1 \geq 0$ and $t_2 \geq 0$, then $C_1\mathcal{G}(t_1, t_2) \leq g(t_1, t_2) \leq C_2\mathcal{G}(t_1, t_2)$.*
(2) *For any $(t_1, t_2) \in \mathbb{R}^2$, $\|\Sigma(t_1, t_2)\| \geq \varepsilon_1\|(t_1, t_2)\|$.*

PROOF. Assertion 1 will show that g and $\mathcal{G}$ grow at approximately the same rate on the first quadrant. We begin the proof of Assertion 1 by estimating g from below. Suppose first that $\beta_1 = c_1$ is real. We consider two of the coordinate functions which define Σ, $\{\psi_1(t_1,t_2) := e^{r_1t_1}e^{s_1t_2},\ \psi_2(t_1,t_2) := e^{r_1t_1}e^{c_1t_2}\}$. We use Lemma 2.1 to estimate:

$$\begin{aligned} g &= \|\Sigma_{/1}\wedge\Sigma_{/2}\| \geq \left|\partial_{t_1}\psi_1\cdot\partial_{t_2}\psi_2 - \partial_{t_1}\psi_2\cdot\partial_{t_2}\psi_1\right| \\ &= r_1(s_1-c_1)e^{2r_1t_1+(s_1+c_1)t_2}\,. \end{aligned}$$

If, on the other hand, $\beta_1 = c_1 + d_1\sqrt{-1}$ for $d_1 \neq 0$, then we consider the three coordinate functions:

$$\begin{aligned} &\psi_1(t_1,t_2) := e^{r_1t_1+s_1t_2}\,, \qquad \psi_2(t_1,t_2) := e^{r_1t_1+c_1t_2}\cos(d_1t_2)\,, \\ &\psi_3(t_1,t_2) := e^{r_1t_1+c_1t_2}\sin(d_1t_2) \end{aligned}$$

and estimate similarly

$$\begin{aligned} g &\geq \sum_{1\leq i<j\leq 3}\left\{(\partial_{t_1}\psi_i\cdot\partial_{t_2}\psi_j - \partial_{t_1}\psi_j\cdot\partial_{t_2}\psi_i)^2\right\}^{1/2} \\ &\geq r_1(s_1-c_1)e^{2r_1t_1+(s_1+c_1)t_2}\,. \end{aligned}$$

We have shown $g \geq Ce^{2r_1t_1+(s_1+c_1)t_2}$ for some C. By reducing C if necessary, we have similarly that $g \geq Ce^{(r_1+a_1)t_1+2s_1t_2}$. We average these two estimates to establish the lower bound of Assertion 1 by showing:

$$g \geq \tfrac{1}{2}C\mathcal{G}(t_1,t_2)\,.$$

To establish the upper estimate of Assertion 1, we shall assume, for the sake of simplicity, that all the roots are real as that is the case in which we shall use it; the general case can be dealt with using the arguments above. The coordinate functions of Σ take the form $\phi_{ij}(t_1,t_2) = e^{r_it_1+s_jt_2}$. Then

$$\begin{aligned} g^2 &= \|\Sigma_{/1}\wedge\Sigma_{/2}\|^2 = \tfrac{1}{2}\textstyle\sum_{(i,j)\neq(a,b)}\{\partial_{t_1}\phi_{ij}\partial_{t_2}\phi_{ab} - \partial_{t_1}\phi_{ab}\partial_{t_2}\phi_{ij}\}^2 \\ &= \tfrac{1}{2}\textstyle\sum_{(i,j)\neq(a,b)} e^{2(r_i+r_a)t_1+2(s_j+s_b)t_2}(r_is_b - r_as_j)^2\,. \end{aligned}$$

If $i = a$, then $j \neq b$. Choose the notation so $1 \leq j < b$. We then have that $2r_i + 2r_a \leq 4r_1$ and $2s_j + 2s_b \leq 2s_1 + 2s_2$. Thus we may bound

$$\{\partial_{t_1}\phi_{ij}\partial_{t_2}\phi_{ab} - \partial_{t_1}\phi_{ab}\partial_{t_2}\phi_{ij}\}^2 \leq Ce^{4r_1t_1+(2s_1+2s_2)t_2}\,. \tag{3.c}$$

On the other hand, if $i \neq a$, choose the notation so that $1 \leq i < a$. We then have $2r_i + 2r_a \leq 2r_1 + 2r_2$ and $2s_j + 2s_b \leq 4s_1$. The upper bound of Assertion 1 then follows Equation (3.c) and from the estimate:

$$\{\partial_{t_1}\phi_{ij}\partial_{t_2}\phi_{ab} - \partial_{t_1}\phi_{ab}\partial_{t_2}\phi_{ij}\}^2 \leq Ce^{(2r_1+2r_2)t_1+4s_1t_2}\,.$$

Suppose $t_1 \geq 0$ and $t_2 \geq 0$. Since $r_1 > 0$ and $s_1 > 0$, we may estimate

$$\begin{aligned} \|\Sigma(t_1,t_2)\|^2 &\geq e^{2r_1t_1+2s_1t_2} \geq \tfrac{1}{2}(2r_1t_1+2s_1t_2)^2 \\ &\geq 2r_1^2t_1^2 + 2s_1^2t_2^2 \geq \varepsilon_1^2\|(t_1,t_2)\|^2\,. \end{aligned}$$

Assertion 2 then follows for $t_1 \geq 0$ and $t_2 \geq 0$. We set $\tilde{t}_i = \pm t_i$ as appropriate to reparametrize Σ and establish Assertion 2 in the remaining quadrants. $\square$

By Lemma 3.1, $g > 0$. This implies Σ is an immersion. We show that Σ has infinite volume by estimating

$$\begin{aligned}\mathrm{vol}(\Sigma) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t_1, t_2) dt_1 dt_2 \\ &\geq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_1(s_1 - c_1)e^{2r_1t_1+(s_1+c_1)t_2} dt_1 dt_2 = \infty\,.\end{aligned}$$

Let C be a compact subset of $\mathbb{R}^n$. Since Σ is continuous and C is closed, $\Sigma^{-1}(C)$ is closed. Since C is compact, C is bounded so we can find R so that $\|C\| \leq R$. Thus if $(t_1, t_2) \in \Sigma^{-1}(C)$, then $\varepsilon_1\|(t_1, t_2)\| \leq \|\Sigma(t_1, t_2)\| \leq R$. This shows that $\Sigma^{-1}(C)$ is bounded and hence, being closed, is compact. Since the inverse image of a compact set is compact, Σ is a proper map.

Let $\sigma(u) = \Sigma(t_1(u), t_2(u))$ be a unit speed geodesic in Σ. Then $\|\dot{\sigma}\| = 1$ and $\ddot{\sigma}(u) \perp T_{\sigma(u)}\Sigma$. Choose a maximal domain $[0, u_0)$ for σ. Suppose $u_0 < \infty$. As σ is a unit speed curve in $\mathbb{R}^n$,

$$\|\sigma(0) - \sigma(u)\| \leq u_0 \quad \text{so} \quad \|\sigma(u)\| \leq \|\sigma(0)\| + u_0$$

for $u < u_0$. We use Lemma 3.1 to see that

$$\varepsilon_1\|(t_1(u), t_2(u))\| \leq \|\sigma(u)\| \leq u_0 + \|\sigma(0)\|\,.$$

Since $(t_1(u), t_2(u))$ is uniformly bounded, we may choose a sequence of values u_n which converge to u_0 so that $\{t_1(u_n)\}$ and $\{t_2(u_n)\}$ are convergent sequences, i.e. so that for some (t_1^0, t_2^0) we have that:

$$\lim_{n\to\infty}(t_1(u_n), t_2(u_n)) = (t_1^0, t_2^0)\,.$$

Since Σ is continuous, this implies $\lim_{n\to\infty}\sigma(u_n)$ exists and belongs to Σ. This implies that σ can be extended smoothly beyond the limiting value of u_0; this contradiction shows Σ is geodesically complete.

Let $\Psi_1(t_1, t_2) := e^{r_1t_1+s_1t_2}$ and $\Psi_2(t_1, t_2) := e^{r_kt_1+s_1t_2}$ be two of the coordinate functions of Σ. Suppose that $\Sigma(t_1, t_2) = \Sigma(\tilde{t}_1, \tilde{t}_2)$. Then $\Psi_1(t_1, t_2) = \Psi_1(\tilde{t}_1, \tilde{t}_2)$ and $\Psi_2(t_1, t_2) = \Psi_2(\tilde{t}_1, \tilde{t}_2)$. Consequently:

$$e^{(r_1-r_k)t_1} = \Psi_1(t_1, t_2)\Psi_2(t_1, t_2)^{-1} = \Psi_1(\tilde{t}_1, \tilde{t}_2)\Psi_2(\tilde{t}_1, \tilde{t}_2)^{-1} = e^{(r_1-r_k)\tilde{t}_1}\,.$$

Since $r_1 - r_k > 0$, we conclude $t_1 = \tilde{t}_1$. A similar argument shows $t_2 = \tilde{t}_2$ so Σ is 1-1. This completes the proof of Theorem 1.7. $\square$

REMARK 3.2. It is possible to prove Theorem 1.7 under somewhat weaker assumptions. If we assume there exist roots $\lambda_1, \lambda_2 \in \mathcal{R}_1$ and $\lambda_3, \lambda_4 \in \mathcal{R}_2$ so that $\Re(\lambda_1) > 0 > \Re(\lambda_2)$ and $\Re(\lambda_3) > 0 > \Re(\lambda_4)$, then Theorem 1.7 continues to hold. We omit details in the interests of brevity.

4. The proof of Theorem 1.8. We now examine the Gauss curvature K. We suppose $t_1 \geq 0$ and $t_2 \geq 0$ as the remaing 3 quadrants can be handled similarly. We begin with the following estimate:

LEMMA 4.1. *There exists a constant $C = C(\Sigma)$ so that if $t_1 \geq 0$ and $t_2 \geq 0$ then:*
(1) $(\tilde{L}_{11}, \tilde{L}_{22}) \leq Ce^{(4r_1+2a_1)t_1+(4s_1+2c_1)t_2}$.
(2) $(\tilde{L}_{12}, \tilde{L}_{12}) \leq Ce^{(4r_1+2a_1)t_1+(4s_1+2c_1)t_2}$.

PROOF. Expand

$$\sigma_1(t_1) = \sum_{i=0}^{n_1-1} \phi_i(t_1)e_i \text{ and } \sigma_2(t_2) = \sum_{j=0}^{n_2-1} \psi_j(t_2)f_j\,.$$

We assume $\phi_0(t_1) = e^{r_1t_1}$ and $\psi_0(t_2) = e^{s_1t_2}$. We also assume that $\phi_i(t_1)$ and $\psi_j(t_2)$ for $i \geq 1$ and $j \geq 1$ have one of the following 3 forms:

$$\phi_i(t_1) = e^{a_it_1}\,, \quad \text{or } \phi_i(t_1) = e^{a_it_1}\cos(b_it_1)\,, \quad \text{or } \phi_i(t_1) = e^{a_it_1}\sin(b_it_1)\,, \text{ and}$$
$$\psi_j(t_2) = e^{c_jt_2}\,, \text{ or } \psi_j(t_2) = e^{c_jt_2}\cos(d_jt_2)\,, \text{ or } \psi_j(t_2) = e^{c_jt_2}\sin(d_jt_2)\,.$$

We order the roots so $r_1 = a_0 > a_1 \geq a_2 \geq \cdots$ and $s_1 = c_0 > c_1 \geq c_2 \geq \cdots$; thus the remaining exponentials grow less rapidly. Exponential growth of the form

$$e^{(a_{i_1}+a_{i_2}+a_{i_3})t_1+(c_{j_1}+c_{j_2}+c_{j_3})t_2}$$

in $\Sigma_{/1} \wedge \Sigma_{/2} \wedge \Sigma_{/ij}$ arises from terms of the form

$$(e_{m_1} \otimes f_{k_1}) \wedge (e_{m_2} \otimes f_{k_2}) \wedge (e_{m_3} \otimes f_{k_3})$$

where $\{i_1, i_2, i_3\}$ is a permutation of $\{m_1, m_2, m_3\}$ and $\{j_1, j_2, j_3\}$ is a permutation of $\{k_1, k_2, k_3\}$. Possible terms of maximal growth can be enumerated as follows:

Type 1. Terms involving $e^{3r_1t_1}$. This corresponds to $i_1 = i_2 = i_3 = 0$ and hence $\{k_1, k_2, k_3\}$ are distinct. Thus these grow at most like $e^{(s_1+2c_1)t_2}$ in t_2.

Type 2. Terms involving $e^{3s_1t_2}$. This corresponds to $j_1 = j_2 = j_3 = 0$ and hence $\{m_1, m_2, m_3\}$ are distinct. Thus these grow at most like $e^{(r_1+2a_1)t_1}$ in t_1.

Type 3. Terms involving at least 2 different exponentials in t_1 and at least 2 different exponentials in t_2. Here at least one of the $\{i_1, i_2, i_3\}$ involves an index which is not 0 and at least one of the $\{j_1, j_2, j_3\}$ involves an index which is not 0. Thus those grow at most like $e^{(2r_1+a_1)t_1+(2s_1+c_1)t_2}$.

When considering $(\tilde{L}_{ij}, \tilde{L}_{kl})$, terms must be paired against like terms. Let

$$\xi_{M,K} := (e_{m_1} \otimes f_{k_1}) \wedge (e_{m_2} \otimes f_{k_2}) \wedge (e_{m_3} \otimes f_{k_3})\,.$$

Then $(\xi_{M,K}, \xi_{\tilde{M},\tilde{K}}) = 0$ if $\{m_1, m_2, m_3\}$ is not a permutation of $\{\tilde{m}_1, \tilde{m}_2, \tilde{m}_3\}$ or if $\{k_1, k_2, k_3\}$ is not a permutation of $\{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3\}$. Thus terms of Type 1 must be paired against terms of Type 1, of Type 2 against Type 2, and of Type 3 against Type 3. We consider $\tilde{L}_{11} = \Sigma_{/1} \wedge \Sigma_{/2} \wedge \Sigma_{/11}$. We have

$$\Sigma_{/1} \wedge \Sigma_{/11} = ((\partial_{t_1}\sigma_1 \otimes \sigma_2) \wedge (\partial_{t_1}\partial_{t_1}\sigma_1 \otimes \sigma_2))\,.$$

In this expression, σ_2 can be treated as a constant vector and essentially ignored for the moment. Since at least 2 different terms must occur in any non-zero wedge product, there are no $e^{2r_1t_1}$ exponentials appearing. Thus there are no terms of Type 1 in $\tilde{L}_{11}$. Similarly there are no terms of Type 2 in $\tilde{L}_{22}$. Thus $(\tilde{L}_{11}, \tilde{L}_{22})$ contains only terms of Type 3 so Assertion 1 follows. Next, we shall consider $\tilde{L}_{12} = \Sigma_{/1} \wedge \Sigma_{/2} \wedge \Sigma_{/12}$. We have

$$\Sigma_{/1} \wedge \Sigma_{/12} = (\partial_{t_1}\sigma_1 \otimes \sigma_2) \wedge (\partial_{t_1}\sigma_1 \otimes \partial_{t_2}\sigma_2)\,.$$

In this expression, $\partial_{t_1}\sigma_1$ can be treated as a constant vector and essentially ignored for the moment. Since at least 2 different terms must occur in any non-zero term, there are no $e^{2s_1t_2}$ exponentials appearing. Thus there are no terms of Type 2 to be considered and, similarly no terms of Type 1 to be considered and Assertion 2 follows. □

We apply Lemma 2.1 and Lemma 4.1 to estimate therefore that:

$$\begin{aligned} &|K| \le Cg^{-4}e^{(4r_1+2a_1)t_1+(4s_1+2c_1)t_2}\,, \\ &g|K| \le Cg^{-3}e^{(4r_1+2a_1)t_1+(4s_1+2c_1)t_2}\,. \end{aligned} \tag{4.a}$$

We use Lemma 3.1 to estimate $g^2 \ge \varepsilon^2 e^{(3r_1+a_1)t_1+(3s_1+c_1)t_2}$. Raising this to the third and fourth power yields

$$\begin{aligned} &g^4 \ge \varepsilon^4 e^{(6r_1+2a_1)t_1+(6s_1+2c_1)t_2}\,, \\ &g^3 \ge \varepsilon^3 e^{(\frac{9}{2}r_1+\frac{3}{2}a_1)t_1+(\frac{9}{2}s_1+\frac{3}{2}c_1)t_2}\,. \end{aligned} \tag{4.b}$$

Let ε_1 be as in Equation (3.a). Choose $\varepsilon_2 = \varepsilon_2(\Sigma) > 0$ to measure the spectral gap, i.e. so:

$$\begin{aligned} &r_1 - \varepsilon_2 \ge \Re(\lambda) \ge r_k + \varepsilon_2 \text{ for all } \lambda \in \mathcal{R}_1 - \{r_1, r_k\}\,, \\ &s_1 - \varepsilon_2 \ge \Re(\mu) \ge s_\ell + \varepsilon_2 \text{ for all } \mu \in \mathcal{R}_2 - \{s_1, s_\ell\}\,. \end{aligned}$$

Combining Equation (4.a) with Equation (4.b) then yields the estimates:

$$\begin{aligned} |K| &\le Ce^{(-2r_1t_1-2s_1t_2)} \le Ce^{-2\varepsilon_1(t_1+t_2)} \le Ce^{-2\varepsilon_1\|(t_1,t_2)\|}\,, \\ g|K| &\le Ce^{((4-\frac{9}{2})r_1+(2-\frac{3}{2})a_1)t_1+((4-\frac{9}{2})s_1+(2-\frac{3}{2})c_1)t_2} \\ &= Ce^{-\frac{1}{2}(r_1-a_1)t_1-\frac{1}{2}(s_1-c_1)t_2} \le Ce^{-\frac{1}{2}(\varepsilon_2t_1+\varepsilon_2t_2)} \le Ce^{-\frac{1}{2}\varepsilon_2\|(t_1,t_2)\|}\,. \end{aligned}$$

This establishes Assertion 1 and Assertion 2 on the first quadrant $t_1 \ge 0$ and $t_2 \ge 0$; we use similar arguments to establish these estimates in the remaining quadrants. Integrating the estimate for $g|K|$ in polar coordinates then shows $|K|[\Sigma] \le C\varepsilon_2^{-1}$ which completes the proof of Theorem 1.8. □

REMARK 4.2. It is not necessary to assume that roots μ of $\mathcal{P}_1$ with $r_1 > \Re(\mu) > r_k$ are simple; multiple roots can appear in this range as the exponential estimates swamp any powers of t_1. Similarly, it is not necessary to assume that the remaining roots μ of $\mathcal{P}_2$ with $s_1 > \Re(\mu) > s_\ell$ are simple; the arguments go through unchanged. More care must be taken, however, if the dominant roots r_1 or r_k of $\mathcal{P}_1$ or the dominant roots s_1 or s_ℓ of $\mathcal{P}_2$ are not simple and a further investigation by the second author into this case is planned.

5. The proof of Theorem 1.11. Adopt the notation of Definition 1.10.

LEMMA 5.1. *Let $\gamma_{\pm r}(t) := \Sigma(t, \pm r) = \sigma_1(t) \otimes \sigma_2(\pm r)$. If all the roots of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are simple and if the real roots are dominant, then:*

$$\lim_{r\to\infty} \int_{-r}^{r} \kappa_g(\gamma_{\pm r})(t)ds = -\Theta[\sigma_1]\,.$$

PROOF. We will use the inward unit normal to apply the Gauss–Bonnet theorem. This points in the direction of $\mp\Sigma_{/1} \wedge \Sigma_{/2}(t, \pm r)$. Lemma 2.1 shows that:

$$\kappa_g(t, \pm r)ds = \mp(\Sigma_{/1} \wedge \Sigma_{/2}, \Sigma_{/1} \wedge \Sigma_{/11}) \cdot g^{-1}\|\Sigma_{/1}\|^{-2}(t, \pm r)dt\,.$$

First let $t_2 = r$. We express $\sigma_2(t_2) = e^{s_1 t_2}(f_1 + \mathcal{E}(t_2))$ where the remainder $\mathcal{E}(t_2)$ is exponentially suppressed, i.e. satisfies an estimate of the form $\|\mathcal{E}(t_2)\| \le e^{-\varepsilon t_2}$ for some $\varepsilon > 0$ if $t_2 >> 0$. In this setting, to simplify the notation, we shall simply write $\sigma_2(t_2) \sim e^{s_1 t_2} f_1$. We compute:

$$\begin{aligned}
&\Sigma_{/1} \sim \dot\sigma_1 \otimes e^{s_1 r} f_1\,, && \Sigma_{/2} \sim \sigma_1 \otimes s_1 e^{s_1 r} f_1\,,\\
&g = \|\Sigma_{/1} \wedge \Sigma_{/2}\| \sim |s_1| e^{2s_1 r} f_1^2 \|\dot\sigma_1 \wedge \sigma_1\|\,, && \Sigma_{/11} \sim \ddot\sigma_1 \otimes e^{s_1 r} f_1\,,
\end{aligned}$$

$$\begin{aligned}
\kappa_g(\gamma_r)ds &= -(\Sigma_{/1} \wedge \Sigma_{/2}, \Sigma_{/1} \wedge \Sigma_{/11}) g^{-1}\|\Sigma_{/1}\|^{-2}dt\\
&\sim -\frac{s_1 e^{4s_1 r}}{|s_1| e^{4s_1 r}} \frac{(\dot\sigma_1(t_1) \wedge \sigma_1(t_1), \dot\sigma_1(t_1) \wedge \ddot\sigma_1(t_1))}{\|\sigma_1(t_1) \wedge \dot\sigma_1(t_1)\| \cdot \|\dot\sigma_1(t_1)\|^2} dt\,.
\end{aligned}$$

This gives $-\Theta(\sigma_1)dt$ in the limit since $s_1 > 0$. We do not need to change the sign of the normal but again get a negative sign if $s_k < 0$ since $+\frac{s_k}{|s_k|} = -1$. □

We apply the Gauss–Bonnet theorem to the square $\Sigma([-r, r] \times [-r, r])$. Let α_i be the interior angles. We then have:

$$\begin{aligned}
2\pi = &\int_{-r}^{r}\int_{-r}^{r} K(t_1, t_2) g dt_1 dt_2 + \sum_{i=1}^{4}(\pi - \alpha_i) + \int_{-r}^{r} \kappa_g(\Sigma(t, r))ds\\
&+ \int_{-r}^{r} \kappa_g(\Sigma(t, -r))ds + \int_{-r}^{r} \kappa_g(\Sigma(r, t))ds + \int_{-r}^{r} \kappa_g(\Sigma(-r, t))ds\,.
\end{aligned}$$

We examine the angle α_1 at $\Sigma(r, r)$. Because $\Sigma_{/1}(r, r) \sim r_1\Sigma(r, r)$ and because $\Sigma_{/2}(r, r) \sim s_1\Sigma(r, r)$, $\Sigma_{/1}$ and $\Sigma_{/2}$ point in approximately the same direction. Consequently, $\cos(\alpha_1) \sim 1$ and $\alpha_1 \sim 0$. Keeping careful track of the signs shows the other angles also are close to 0. Theorem 1.11 then follows from Lemma 5.1. □

6. The proof of Theorem 1.12. We apply Theorem 1.11 to the setting $n_1 = n_2$. Let $\{\xi_1, \xi_2\}$ be the standard orthonormal basis for $\mathbb{R}^2$. Suppose $\sigma(t) = e^{at}e_1 + e^{bt}e_2$ for $a > 0 > b$. We use Equation (1.d) to see that:

$$\Theta[\sigma] = \int_{-\infty}^{\infty} \frac{|(a-b)ab|e^{(a+b)t}}{a^2e^{2at} + b^2e^{2bt}}dt = \int_{-\infty}^{\infty} \frac{|(a-b)ab|e^{(a-b)t}}{a^2e^{2(a-b)t} + b^2}dt\,.$$

We have $a - b > 0$. We change variables setting $x := e^{(a-b)t}$ to express

$$\Theta[\sigma] = \int_0^\infty \frac{|ab|}{a^2x^2 + b^2}dx = \int_0^\infty \frac{|a|}{|b|}\frac{1}{\frac{a^2}{b^2}x^2 + 1}dx\,.$$

We again change variables setting $y = \frac{|a|}{|b|}x$ to express

$$\Theta[\sigma] = \int_0^\infty \frac{1}{y^2 + 1}dy = \frac{\pi}{2}\,.$$

Theorem 1.12 now follows from Theorem 1.11. □

7. The proof of Theorem 1.15. Let $\Sigma(t_1, t_2) = \sigma_1(t_1) \otimes \sigma_2(t_2)$ where

$$\begin{aligned}\sigma_1(t_1) &= (e^{r_1t_1}, \dots, e^{r_kt_1}) \quad \text{for } r_1 > r_2 > 0 > r_{k-1} > r_k,\\ \sigma_2(t_2) &= (e^{s_1t_2}, \dots, e^{s_\ell t_2}) \quad \text{for } s_1 > s_2 > 0 > s_{\ell-1} > s_\ell\,.\end{aligned}$$

We focus on the first quadrant and assume $t_1 \geq 0$ and $t_2 \geq 0$; the other quadrants are handled similarly. By Lemma 3.1, g is growing exponentially at ∞ and the growth rate is controlled by the function $\mathcal{G}$ of Equation (3.b); this need not be the case if $s_2 << 0$ and $t_2 << 0$. Let $\{e_i\}$ (resp. $\{f_a\}$ and $\{e_i \otimes f_a\}$) be an orthonormal basis for $\mathbb{R}^{n_1}$ (resp. $\mathbb{R}^{n_2}$ and $\mathbb{R}^{n_1n_2}$) so that summing over i, a, and (i, a), yields:

$$\sigma_{P_1}(t_1) = e^{r_it_1}e_i\,, \quad \sigma_{P_2}(t_2) = e^{s_at_2}f_a\,, \quad \Sigma(t_1, t_2) = e^{r_it_1+s_at_2}e_i \otimes f_a\,.$$

We express $\|H\|$ in terms of wedge products and establish its asymptotic growth rate at infinity as follows:

LEMMA 7.1. *Let Σ satisfy the hypotheses of Theorem 1.15. Adopt the notation established above.*

(1) *Let $\mathfrak{H} := \Sigma_{/1} \wedge \Sigma_{/2} \wedge (g_{11}\Sigma_{/22} + g_{22}\Sigma_{/11} - 2g_{12}\Sigma_{/12}) \in \Lambda^3(\mathbb{R}^{n_1n_2})$. Then:*
$$\|H\| = g^{-3}\|\mathfrak{H}\|\,.$$
(2) *Let $\mathcal{H} := e^{5r_1t_1+(3s_1+s_2+s_3)t_2} + e^{(3r_1+r_2+r_3)t_1+5s_1t_2} + e^{(4r_1+r_2)t_1+(4s_1+s_2)t_2}$. There exist constants $C_i = C_i(\Sigma) > 0$ so that if $t_1 \geq 0$ and if $t_2 \geq 0$, then*
$$C_1g^{-3}\mathcal{H} \leq \|H\| \leq C_2g^{-3}\mathcal{H}\,.$$

PROOF. The unnormalized mean curvature vector is given by $H = g^{ij}L_{ij} \in T_P\Sigma^\perp$. Let $\{\xi_1, \xi_2\}$ be an orthonormal frame for $T\Sigma$ so $\Sigma_{/1} \wedge \Sigma_{/2} = g\xi_1 \wedge \xi_2$. By Lemma 2.1,

$$\Sigma_{/1} \wedge \Sigma_{/2} \wedge \Sigma_{/ij} = g\xi_1 \wedge \xi_2 \wedge L_{ij}\,.$$

Since $g^{11} = g^{-2}g_{22}$, $g^{22} = g^{-2}g_{11}$, $g_{12} = -g^{-2}g_{12}$, and since $\{\xi_1, \xi_2, L_{ij}\}$ form an orthogonal set, we prove Assertion 1 by computing:

$$\begin{aligned}\|H\| &= g^{-2}\|g_{22}L_{11} + g_{11}L_{22} - 2g_{12}L_{12}\|\\ &= g^{-3}\|\Sigma_{/1} \wedge \Sigma_{/2} \wedge \{g_{22}\Sigma_{/11} + g_{11}\Sigma_{/22} - 2g_{12}\Sigma_{/12}\}\|\\ &= g^{-3}\|\mathfrak{H}\|\,.\end{aligned}$$

If $\{ua, vb, wc\}$ are distinct pairs of indices, set

$$\xi_{ua,vb,wc} := (e_u \otimes f_a) \wedge (e_v \otimes f_b) \wedge (e_w \otimes f_c)\,.$$

If $\omega \in \Lambda^3(\mathbb{R}^{n_1 n_2})$, let $c(\xi_{ua,vb,wc}, \omega)$ denote the coefficient of $\xi_{ua,vb,wc}$ in ω. Since $\omega = \sum_\xi c(\xi, \omega)\omega$, there exist constants $C_i = C_i(n_1, n_2)$ so that

$$C_1 \sum_\xi |c(\xi, \omega)| \le \|\omega\| \le C_2 \sum_\xi |c(\xi, \omega)|\,.$$

We wish to show that $\mathcal{H}$ controls the growth rate of $\|H\|$ at infinity. Thus we must estimate each coefficient $c(\xi, \mathfrak{H})$ from above by $\mathcal{H}$ and exhibit 3 different ξ which we will use to estimate $\|\mathfrak{H}\|$ from below in terms of the 3 terms comprising $\mathcal{H}$. We shall use the same argument given to establish Lemma 4.1. We may express

$$c(\xi_{ua,vb,wc}, \Sigma_{/1} \wedge \Sigma_2 \wedge \Sigma_{/\mu\nu})(t_1, t_2) = e^{(r_u+r_v+r_w)t_1+(s_a+s_b+s_c)t_2} c_{\mu\nu,ua,vb,wc}$$

where

$$c_{11,ua,vb,wc} = \det\begin{pmatrix} r_u & r_v & r_w \\ s_a & s_b & s_c \\ r_u^2 & r_v^2 & r_w^2 \end{pmatrix}, \quad c_{22,ua,vb,wc} = \det\begin{pmatrix} r_u & r_v & r_w \\ s_a & s_b & s_c \\ s_a^2 & s_b^2 & s_c^2 \end{pmatrix},$$

$$c_{12,ua,vb,wc} = \det\begin{pmatrix} r_u & r_v & r_w \\ s_a & s_b & s_c \\ r_u s_a & r_v s_b & r_w s_c \end{pmatrix}.$$

Terms of Type 1. Suppose $u = v = w = 1$. Then $c(\xi_{1a,1b,1c}, \tilde{L}_{11}) = 0$ and $c(\xi_{1a,1b,1c}, \tilde{L}_{12}) = 0$. Since $\{a, b, c\}$ are distinct, we may bound

(7.a) $$|c(\xi_{1a,1b,1c}, \mathfrak{H})(t_1, t_2)| \le |c_{22,1a,1b,1c}| e^{5r_1t_1+(3s_1+s_2+s_3)t_2} \le C\mathcal{H}(t_1, t_2)\,.$$

Let $\xi = \xi_{11,12,13}$. As $g_{22} \ge Ce^{2r_1t_1+2s_1t_2}$ and as $c_{\mu\nu,1a,1b,1c} = 0$ for $(\mu, \nu) \ne (2, 2)$,

(7.b) $$\begin{aligned}\|H(t_1, t_2)\| &\ge |c_{22,11,12,13}| e^{2r_1t_1+2s_1t_2} e^{3r_1t_1+(s_1+s_2+s_3)t_2} \\ &= r_1(s_1 - s_2)(s_1 - s_3)(s_2 - s_3) e^{5r_1t_1+(3s_1+s_2+s_3)t_2}\,.\end{aligned}$$

Terms of Type 2. Suppose $a = b = c = 1$. We argue similarly to conclude:

(7.c) $$\begin{aligned}&|c(\xi_{u1,v1,w1}, \mathfrak{H})(t_1, t_2)| \le C\mathcal{H}(t_1, t_2)\,, \\ &\|H(t_1, t_2)\| \ge s_1(r_1 - r_2)(r_1 - r_3)(r_2 - r_3) e^{(3r_1+r_2+r_3)t_1+5s_1t_2}\,.\end{aligned}$$

Terms of Type 3. We suppose $(u, v, w) \ne (1, 1, 1)$ and $(a, b, c) \ne (1, 1, 1)$. The following upper bound is then immediate:

(7.d) $$|c(\xi_{ua,vb,wc}, \mathfrak{H})(t_1, t_2)| \le Ce^{(4r_1+r_2)t_1+(4s_1+s_2)t_2}\,.$$

Let $\xi = \xi_{11,12,21}$. We expand, modulo lower order terms,

$$\begin{aligned}&g_{11} = r_1^2 e^{2r_1t_1+2s_1t_2} + \cdots\,, \quad g_{12} = r_1 s_1 e^{2r_1t_1+2s_1t_2} + \cdots\,, \\ &g_{22} = s_1^2 e^{2r_1t_1+2s_1t_2} + \cdots\,.\end{aligned}$$

We compute, again modulo lower order terms, that:

$$\begin{aligned}
&|c(\xi_{11,12,21},\mathfrak{H})(t_1,t_2)| = \left|c(\xi_{11,12,21}, g_{22}\tilde L_{11} + g_{11}\tilde L_{22} - 2g_{12}\tilde L_{12})\right|\\
&= e^{(4r_1+r_2)t_1+(4s_1+s_2)t_2}\left|s_1^2 c_{11,11,12,21} + r_1^2 c_{22,11,12,21} - 2r_1s_1c_{12,11,12,21}\right| + \cdots\\
&= e^{(4r_1+r_2)t_1+(4s_1+s_2)t_2}\{r_1(r_1-r_2)s_1(s_1-s_2)(2r_1s_1-r_2s_1-r_1s_2)\} + \cdots .
\end{aligned}$$

Since $\{r_1(r_1-r_2)s_1(s_1-s_2)(2r_1s_1-r_2s_1-r_1s_2)\} > 0$, we have

$$|c(\xi_{11,12,21},\mathfrak{H})(t_1,t_2)| \ge Ce^{(4r_1+r_2)t_1+(4s_1+s_2)t_2}\,. \tag{7.e}$$

Assertion 2 now follows from Equation (7.a)–Equation (7.e). □

We restrict to the first quadrant $t_1 \ge 0$ and $t_2 \ge 0$. By Lemma 3.1,

$$g \ge Ce^{2r_1t_1+(s_1+s_2)t_2} \quad \text{and} \quad g \ge Ce^{(r_1+r_2)t_1+2s_1t_2} \tag{7.f}$$

for some $C > 0$. We use Equation (7.f) to see if $\delta \in [0,1]$, then:

$$g^k \ge Ce^{k\delta\{2r_1t_1+(s_1+s_2)t_2\}+k(1-\delta)\{(r_1+r_2)t_1+2s_1t_2\}}\,. \tag{7.g}$$

We apply Lemma 7.1.

7.1. The proof that $\|H\|$ is exponentially decaying. Let $\delta \in [0,1]$. We use Equation (7.g) and Lemma 7.1. We bound terms of Type I by:

$$\begin{aligned}
&g^{-3}e^{5r_1t_1+(3s_1+2s_2)t_2} \le Ce^{a_1(\delta)t_1+a_2(\delta)t_2} \quad \text{for}\\
&a_1(\delta) = 5r_1 - 3\{\delta 2r_1 + (1-\delta)(r_1+r_2)\} \quad \text{and}\\
&a_2(\delta) = 3s_1 + 2s_2 - 3\{\delta(s_1+s_2) + (1-\delta)2s_1\}\,.
\end{aligned}$$

We show such terms exhibit exponential decay by estimating:

$$\begin{aligned}
&a_1(\tfrac{2}{3}) = 5r_1 - 4r_1 - r_1 - r_2 = -r_2 < 0\,,\\
&a_2(\tfrac{2}{3}) = 3s_1 + 2s_2 - 2s_1 - 2s_2 - 2s_1 = -s_1 < 0\,.
\end{aligned}$$

The terms of Type 2 are estimated similarly. We estimate the terms of Type 3:

$$\begin{aligned}
&g^{-3}e^{(4r_1+r_2)t_1+(4s_1+s_2)t_2} \le Ce^{a_1(\delta)t_1+a_2(\delta)t_2} \quad \text{for}\\
&a_1(\delta) = 4r_1 + r_2 - 3\{\delta(2r_1) - (1-\delta)(r_1+r_2)\} \quad \text{and}\\
&a_2(\delta) = 4s_1 + s_2 - 3\{\delta(s_1+s_2) - 3(1-\delta)(2s_1)\}\,.
\end{aligned}$$

We take $\delta = \frac{1}{2}$ and show such terms exponential decay by computing:

$$\begin{aligned}
&a_1(\tfrac{1}{2}) = 4r_1 + r_2 - 3r_1 - \tfrac{3}{2}r_1 - \tfrac{3}{2}r_2 = -\tfrac{1}{2}r_1 - \tfrac{1}{2}r_2 < 0\,,\\
&a_2(\tfrac{1}{2}) = 4s_1 + s_2 - \tfrac{3}{2}s_1 - \tfrac{3}{2}s_2 - 3s_1 = -\tfrac{1}{2}s_1 - \tfrac{1}{2}s_2 < 0\,.
\end{aligned}$$

This completes the proof that $\|H\|$ decays exponentially. □

7.2. The proof that $\|H\| \in L^3(g dt_1 dt_2)$. We examine

$$g\|H\|^3 = g^{-8}\|\Sigma_{/1} \wedge \Sigma_{/2} \wedge \{g_{22}\Sigma_{/11} + g_{11}\Sigma_{/22} - 2g_{12}\Sigma_{/12}\}\|^3 .$$

We estimate the terms of type 1. Set

$$\begin{aligned} a_1(\delta) &:= 3(5r_1) - 8\{\delta 2r_1 + (1-\delta)(r_1+r_2)\} , \\ a_2(\delta) &:= 3(3s_1+2s_2) - 8\{\delta(s_1+s_2) + (1-\delta)2s_1\} . \end{aligned}$$

We take $\delta = \frac{7}{8}$ to compute:

$$\begin{aligned} a_1(\tfrac{7}{8}) &= 15r_1 - 14r_1 - r_1 - r_2 = -r_2 < 0 , \\ a_2(\tfrac{7}{8}) &= 9s_1 + 6s_2 - 7s_1 - 7s_2 - 2s_1 = -s_2 < 0 . \end{aligned}$$

The terms of Type 2 are estimated similarly. To estimate the terms of Type 3, we take $\delta = \frac{1}{2}$ and compute:

$$\begin{aligned} a_1(\tfrac{1}{2}) &= 3\{4r_1 + r_2\} - 8\{\delta(2r_1) + (1-\delta)(r_1+r_2)\} \\ &= 12r_1 + 3r_2 - 8\{\tfrac{3}{2}r_1 + \tfrac{1}{2}r_2\} = -r_2 < 0 , \\ a_2(\tfrac{1}{2}) &= 3\{4s_1 + s_2\} - 8\{\delta(2s_1) + (1-\delta)(s_1+s_2)\} \\ &= 12s_1 + 3s_2 - 8\{\tfrac{3}{2}s_1 + \tfrac{1}{2}s_2\} = -s_2 < 0 . \end{aligned}$$

This estimates all the terms comprising $g\|H\|^3$; thus $g\|H\|^3 dt_1 dt_2$ is integrable. □

8. Examples. In this section, we present a number of examples to illustrate various points; many of them were Mathematica assisted and used a program developed by M. Brozos-Vazquez [2].

8.1. Finite total first curvature. Theorem 1.2 shows the total first curvature of σ is finite if all the roots of $\mathcal{P}$ are simple and if the real roots of $\mathcal{P}$ are dominant. This can fail if a dominant root is complex.

EXAMPLE 8.1. Let $\sigma(t) := (e^t\cos(t), e^t\sin(t), e^{-t})$. The dominant root here is complex. We show that κds is not in L^1 by computing:

$$\begin{aligned} \dot\sigma &= (e^t(\cos(t) - \sin(t)), e^t(\cos(t) + \sin(t)), -e^{-t}) , \\ \ddot\sigma &= (-2e^t\sin(t), 2e^t\cos(t), e^{-t}) , \\ \|\dot\sigma \wedge \ddot\sigma\| &= \{4e^{4t} + 10\}^{1/2} , \\ \kappa ds &= \{4e^{4t} + 10\}^{1/2}\{2e^{2t} + e^{-t}\}^{-1} dt . \end{aligned}$$

8.2. Finite total Gauss curvature. Theorem 1.8 shows that if all the roots of $\mathcal{P}_1$ and $\mathcal{P}_2$ are simple and if the real roots are dominant, then the total Gauss curvature is finite. This can fail if one of the dominant roots is complex.

EXAMPLE 8.2. Let $\sigma_1(t_1) = (e^{t_1}\cos(t_1), e^{t_1}\sin(t_1), e^{-t_1})$ and $\sigma_2(t_2) = (e^{t_2}, e^{-t_2})$. Set $\mathcal{E}_1 := (e^{4t_1-4t_2} + e^{4(t_1+t_2)} + 4e^{-4t_1} + 6e^{4t_1} + 5e^{-4t_2} + 5e^{4t_2} + 2)$. We use a Mathematica notebook [2] to see that:

$$gK = -\frac{16(e^{4t_1}+2)(e^{4t_2}+1)e^{10t_1+6t_2}\mathcal{E}_1^{0.5}}{(2e^{4(t_1+t_2)} + e^{8(t_1+t_2)} + 6e^{8t_1+4t_2} + 5e^{4t_1+8t_2} + 5e^{4t_1} + e^{8t_1} + 4e^{4t_2})^2}\,.$$

This permits to estimate for $t_1 \geq 0$ and $t_2 \geq 0$ that:

$$gK \leq -\frac{16e^{4t_1+4t_2+10t_1+16t_2+(4t_1+4t_2)/2}}{\big((2+1+6+5+5+1+4)e^{8t_1+8t_2}\big)^2} = -\frac{16}{24}e^{0t_1+6t_2}\,.$$

Thus $gK\,dt_1dt_2$ is not integrable for $0 \leq t_1 < \infty$ and $0 \leq t_2 < \infty$.

8.3. The total Gauss curvature if $n_1 = n_2 = 2$. In Theorem 1.13, we showed that if the roots of $\mathcal{P}_1$ and $\mathcal{P}_2$ are real and simple and if $n_1 = n_2 = 2$, then $K[\Sigma] = 0$. This result is non-trivial; there are examples where $|K|[\Sigma] \neq 0$ in this setting.

EXAMPLE 8.3. Let $\sigma_1(t_1) = (e^{t_1}, e^{-2t_1})$ and $\sigma_2(t_2) = (e^{t_2}, e^{-2t_2})$. We use Mathematica [2] to compute:

$$gK = \frac{9.e^{-6(t_1+t_2)}(e^{6(t_1+t_2)}-4)}{\big(e^{-8(t_1+t_2)}(e^{6(t_1+t_2)} + e^{6(2t_1+t_2)} + e^{6(t_1+2t_2)} + 4e^{6t_1} + 4e^{6t_2})\big)^{1.5}}\,,$$

$$\int_{\mathbb{R}^2} gK\,dt_1dt_2 = 0\,, \quad \text{and} \quad \int_{\mathbb{R}^2} |gK|\,dt_1dt_2 \approx .811319\,.$$

The Gauss curvature changes sign; it is positive for $6t_1 + 6t_2 > \ln(4)$ and negative for $6t_1 + 6t_2 < \ln(4)$. It does not vanish identically and Theorem 1.12 is non-trivial.

8.4. Uniform estimates on the first curvature. Let σ be defined by an ODE where the dominant roots of $\mathcal{P}$ are real. If all the roots are real, then Theorem 1.4 gives a uniform estimate for the total first curvature which depends only on the dimension. If sub-dominant complex roots are permitted, then no such uniform upper bound exists.

EXAMPLE 8.4. Let $\sigma_k(t) = (e^t, \cos(kt), \sin(kt), e^{-t})$ for $k \geq 1$. We have

$$\begin{aligned}
&\dot\sigma_k(t) = (e^t, -k\sin(kt), k\cos(kt), -e^{-t})\,,\\
&\ddot\sigma_k(t) = (e^t, -k^2\cos(kt), -k^2\sin(kt), e^{-t})\,,\\
&\|\dot\sigma_k(t)\|^2 = e^{2t} + k^2 + e^{-2t}\,, \quad \|\dot\sigma_k(t) \wedge \ddot\sigma_k(t)\| \geq k^3,\\
&\lim_{k\to\infty} \kappa[\sigma_k] \geq \lim_{k\to\infty}\int_{-\infty}^{\infty} \frac{k^3}{e^{2t}+k^2+e^{-2t}}dt \geq \lim_{k\to\infty}\int_0^1 \frac{k^3}{e^2+k^2+1}dt\\
&\qquad = \lim_{k\to\infty}\frac{k^3}{e^2+k^2+1} = \infty\,.
\end{aligned}$$

8.5. Applying the Gauss–Bonnet Theorem with all real roots. Because $K[\Sigma]$ can be non-zero if $n_1 > 2$ and $n_2 > 2$, Theorem 1.13 is non-trivial.

EXAMPLE 8.5. We examine the identity $\mathcal{E} := K[\Sigma] - 2\Theta[\sigma_1] - 2\Theta[\sigma_2] + 2\pi = 0$. We take $\sigma_1(t_1) = (e^{t_1}, e^{a_1 t_1}, e^{a_2 t_1})$ and $\sigma_2(t_2) = (e^{t_2}, e^{b_1 t_2}, e^{b_2 t_2})$ where $0 \geq a_1 > a_2$ and $0 \geq b_1 > b_2$. We computed [2] that:

a_1	a_2	b_1	b_2	$K[\Sigma]$	$\Theta[\sigma_1]$	$\Theta[\sigma_2]$	$\mathcal{E}$	$\|K\|[\Sigma]$
0	−1	0	−1	−1.8649	1.10423	1.10423	$1*10^{-3}$	1.866
0	−2	0	−3	−2.0356	1.07859	1.04485	$7*10^{-4}$	2.26658
−1	−1.1	−1	−1.2	−1.51466	1.26238	1.09344	.06	1.73122
−1	−2	−1	−2	−1.96762	1.07875	1.07875	$5*10^{-4}$	2.27566
−1	−5	−1	−5	−1.96884	1.07859	1.07859	$8*10^{-7}$	2.3783
−2	−4	−1	−3	−1.88447	1.09513	1.10423	$6*10^{-7}$	2.56669
−5	−6	−1	−2	−2.17533	.975259	1.07861	$1*10^{-4}$	3.33547
−5	−6	−7	−8	−2.43838	.975259	.947119	$5*10^{-5}$	4.32915

These calculations show that K takes on both positive and negative values since $|K|[\Sigma] \neq K[\Sigma]$. If, for example, $(a_1, a_2, b_1, b_2) = (-5, -6, -7, -8)$, then

$$|\Theta|[\sigma_1] \approx 2.03662 \neq \Theta[\sigma_1] \approx .975259\,,$$
$$|\Theta|[\sigma_2] \approx 2.10877 \neq \Theta[\sigma_2] \approx .947119$$

Θ takes on both positive and negative values.

8.6. A uniform estimate on the Gauss curvature does not exist if complex roots are allowed. If we allow complex roots, no uniform upper bound is possible. We extend Example 8.4 as follows.

EXAMPLE 8.6. Let $\Sigma_k(t_1, t_2) := (e^{t_1}, \cos(kt_1), \sin(kt_1), e^{-t_1}) \otimes (e^{t_2}, e^{-t_2})$. We use Mathematica [2] to express $gK := -\mathcal{E}_1 \cdot \mathcal{E}_2 \cdot \mathcal{E}_3^{\frac{1}{2}} \cdot \mathcal{E}_4^{-2}$ where:

$$\mathcal{E}_1 := 4(e^{4t_2} + 1)e^{4t_1+6t_2},$$

$$\begin{aligned}\mathcal{E}_2 := {}& 2(k^2+1)k^2e^{2t_1} + 2(k^2+1)k^2e^{6t_1} + (k^2+1)e^{8t_1} + k^2 \\ & +2(k^4+3k^2-1)e^{4t_1} + 1\,,\end{aligned}$$

$$\begin{aligned}\mathcal{E}_3 := {}& e^{-4(t_1+t_2)}\big(2k^2e^{4(t_1+t_2)} + 2(k^2+1)e^{2t_1+4t_2} + 2(k^2+1)e^{6t_1+4t_2} \\ & +(k^2+1)e^{2t_1+8t_2} + (k^2+1)e^{6t_1+8t_2} + (k^2+4)e^{4t_1+8t_2} \\ & +(k^2+1)e^{2t_1} + (k^2+1)e^{6t_1} + (k^2+4)e^{4t_1} + 4e^{8t_1+4t_2} + 4e^{4t_2}\big)\,,\end{aligned}$$

$$\begin{aligned}\mathcal{E}_4 := {}& 2k^2e^{4(t_1+t_2)} + 2(k^2+1)e^{2t_1+4t_2} + 2(k^2+1)e^{6t_1+4t_2} \\ & +(k^2+1)e^{2t_1+8t_2} + (k^2+1)e^{6t_1+8t_2} + (k^2+4)e^{4t_1+8t_2} \\ & +(k^2+1)e^{2t_1} + (k^2+1)e^{6t_1} + (k^2+4)e^{4t_1} + 4e^{8t_1+4t_2} + 4e^{4t_2}\,.\end{aligned}$$

As a function of k, this is behaving like $1 \cdot k^4 \cdot k \cdot k^{-4}$ and thus the integral goes to infinity as $k \to \infty$. We also examine $\Theta(\sigma_{1,k})$ computing

$$\Theta(\sigma_{1,k}) = \left\{ \frac{4-k^4}{(k^2+e^{-2t}+e^{2t})(k^2e^{-2t}+k^2e^{2t}+k^2+e^{-2t}+e^{2t}+4)^{0.5}} \right\} .$$

This is growing linearly in k as $k \to \infty$ and hence $\lim_{k\to\infty} \Theta(\sigma_{1,k}) = \infty$. We examine the identity $\mathcal{E}_k := K[\Sigma_k] - 2\Theta[\sigma_{1,k}] - 2\Theta[\sigma_2] + 2\pi = 0$ numerically:

k	$K[\Sigma_k]$	$\Theta[\sigma_{1,k}]$	$\Theta[\sigma_2]$	$\mathcal{E}_k$
0	−0.933127	1.10423	$\frac{\pi}{2}$	$2.96624 * 10^{-9}$
1	−2.15652	0.49253	$\frac{\pi}{2}$	$-7.07655 * 10^{-10}$
2	−4.74826	−0.803332	$\frac{\pi}{2}$	$-4.54394 * 10^{-9}$
3	−7.77242	−2.31541	$\frac{\pi}{2}$	$6.01698 * 10^{-9}$
4	−10.9544	−3.90643	$\frac{\pi}{2}$	$-6.68724 * 10^{-8}$
10	−30.8223	−13.8403	$\frac{\pi}{2}$	$8.18756 * 10^{-7}$
50	−165.483	−81.1709	$\frac{\pi}{2}$	$1.25382 * 10^{-7}$
200	−671.171	−334.015	$\frac{\pi}{2}$	.000010552
2000	−6739.86	−3368.36	$\frac{\pi}{2}$	$-6.08458 * 10^{-6}$
20000	−67426.9	−33711.9	$\frac{\pi}{2}$	−0.000982013
200000	−674297	−337147	$\frac{\pi}{2}$	−0.0211308

8.7. The norm of the mean curvature vector. Let

$$\sigma_1(t_1) = (e^{r_1t_1}, \dots, e^{r_kt_1}) \text{ and } \sigma_2(t_2) = (e^{s_1t_2}, \dots, e^{s_\ell t_2})$$

for $r_1 > \dots > r_k$ and $s_1 > \dots > s_\ell$. By Lemma 3.1 and Lemma 7.1, there exist constants $C_i > 0$ so

$$C_1 \le \frac{g}{e^{2r_1t_1+(s_1+s_2)t_2} + e^{(r_1+r_2)t_1+2s_1t_2}} \le C_2$$
$$\|H\| \ge C_3 g^{-3}\{e^{5r_1t_1+(3s_1+s_2+s_3)t_2} + e^{(3r_1+r_2+r_3)t_1+5s_1t_2} + e^{(4r_1+r_2)t_1+(4s_1+s_2)t_2}\} .$$

EXAMPLE 8.7. If we set $t_1 = t_2 = t$, $r_1 = s_1$, $r_2 = s_2$, and $r_3 = s_3$, then we get

$$\|H\|(t) \ge C\frac{e^{(8r_1+r_2+r_3)t}}{e^{3(3r_1+r_2)t}} = Ce^{(-r_1-2r_2+r_3)t} .$$

If we take $r_1 = 1$, $r_2 = -3$, and $r_3 = -4$, then $\|H\| \ge Ce^{(-1+6-4)t}$ and this tends to infinity as t becomes large. Thus Assertion 1 of Theorem 1.15 can fail if we permit r_2 or s_2 to be negative.

EXAMPLE 8.8. We set $r_1 = s_1$, $r_2 = s_2$, and $r_3 = s_3$. We restrict to $t_1 \le t_2$ and estimate

$$\|H\|^2 g \ge C\frac{e^{2\{(3r_1+r_2+r_3)t_1+5s_1t_2\}}}{e^{5((r_1+r_2)t_1+2r_1t_2)}} = Ce^{(r_1-3r_2+2r_3)t_1} ,$$
$$\int_\Sigma \|H\|^2 \,\text{dvol} \ge \int_0^\infty \int_0^{t_2} Ce^{(r_1-3r_2+2r_3)t_1} dt_1 dt_2$$

$$= \frac{C}{r_1 - 3r_2 + 2r_3} \int_0^\infty \{(e^{(r_1-3r_2+2r_3)t_2} - 1\} dt_2 \,.$$

This is infinite provided $r_1 - 3r_2 + 2r_2 > 0$. We could, for example, take $r_1 = 10$, $r_2 = 2$, and $r_3 = 1$. So in general $\|H\|$ is not in L^2. More generally, let $p > 2$. We may estimate:

$$\|H\|^p g \geq C \frac{e^{p\{(3r_1+r_2+r_3)t_1+5r_1t_2\}}}{e^{(3p-1)\{(r_1+r_2)t_1+2r_1t_2\}}}$$

$$\int_\Sigma \|H\|^p \operatorname{dvol} \geq C \int_0^\infty \int_0^{t_2} e^{(r_1+(1-2p)r_2+pr_3)t_1+(2-p)r_1t_2} dt_1 dt_2$$

$$= \frac{C}{(r_1+(1-2p)r_2+pr_3)} \int_0^\infty \{e^{((3-p)r_1+(1-2p)r_2+pr_3)t_2} - e^{(2-p)r_1t_2}\} dt_2 \,.$$

This will be divergent if $r_1 + (1-2p)r_2 + pr_3 > 0$ and $(3-p)r_1 + (1-2p)r_2 + pr_3 > 0$. Given $2 < p < 3$, we can take $r_1 = 1$ and r_2 and r_3 very close to zero to see these inequalities are satisfied and the integral is divergent. Thus $p = 3$ is the best that can be established in general although in specific cases, better convergence can be obtained.

Acknowledgements. Research of P. Gilkey was partially supported by MTM2013-41335-P with FEDER funds (Spain). Research of J. H. Park was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2014053413). We are grateful for the helpful comments and assistance of M. Brozos-Vázquez of the Universidade de Coruña (Spain).

REFERENCES

[1] D. BLEECKER, The Gauss–Bonnet Inequality and almost geodesic loops, Adv. in Math. 14 (1974), 183–193.

[2] M. BROZOS-VÁZQUEZ, Universidade da Coruña, Spain – (Mathematica notebook).

[3] G. CARRON, P. EXNER AND D. KREJCINIK, Topologically nontrivial quantum layers, J. Math. Phys. 45 (2004), 774–784.

[4] Q. CHEN AND Y. CHENG, Chern-Osserman inequality for minimal surfaces in H^n, Proc. Amer. Math. Soc. 128 (2000), 2445–2450.

[5] S. CHERN AND R. OSSERMAN, Complete minimal surfaces in euclidean n-space, J. Analyse Math. 19 (1967), 15–34.

[6] B. CHOW, P. LU AND B. YANG, A necessary and sufficient condition for Ricci shrinkers to have positive AVR, Proc. Amer. Math. Soc. 140 (2012), 2179–2181.

[7] S. COHN-VOSSEN, Kürzeste Wege und Totalkrümmung auf Flächen, Compositio Math. 2 (1935), 69–133.

[8] S. COHN-VOSSEN, Totalkrümmung und geodätische linien auf einfach zusammenhän genden offenen vollständigen flächenstücken, Recueil Math. Moscow 43 (1936), 139–163.

[9] F. DILLEN AND W. KÜHNEL, Total curvature of complete submanifolds of Euclidean space, Tohoku Math. J. 57 (2005), 171–200.

[10] A. ESTEVE AND V. PALMER, The Chern-Osserman inequality for minimal surfaces in a Cartan–Hadamard manifold with strictly negative sectional curvatures, Ark. Mat. 52 (2014), 61–92.

[11] P. GILKEY, C. Y. KIM, H. MATSUDA, J. H. PARK AND S. YOROZU, Non-closed curves in $\mathbb{R}^n$ with finite total first curvature arising from the solutions of an ODE, Hokkaido Math. J. 46 (2017).

[12] A. HUBER, On the subharmonic functions and differential geometry in the large, Comment Math Helv 32 (1957), 13–72.

[13] S. Hwang, J. Chang and G. Yun, Variational characterizations of the total scalar curvature and eigenvalues of the Laplacian, Pacific J. Math. 261 (2013), 395–415.

[14] L. Jorge and W. Meeks, The topology of complete minimal surfaces of finite total Gaussian curvature, Topology 22 (1983), 203–221.

[15] M. Kokubu, M. Umehara and K. Yamada, Minimal surfaces that attain equality in the Chern–Osserman inequality, Differential geometry and integrable systems (Tokyo, 2000), 223–228, Contemp. Math., 308, Amer. Math. Soc., Providence, RI, 2002.

[16] J. Li, Evolution of eigenvalues along rescaled Ricci flow, Canad. Math. Bull. 56 (2013), 127–135.

[17] Y. Li, Y. Luo and H. Tang, On the moving frame of a conformal map from 2-disk into $\mathbb{R}^n$, Calc. Var. Partial Differential Equations 46 (2013), 31–37.

[18] X. Ma, Complete stationary surfaces in $\mathbb{R}^4_1$ with total Gaussian curvature $K[M] = -6\pi$, Differential Geom. Appl. 31 (2013),611–622.

[19] X. Ma, C. Wang and P. Wang, Global geometry and topology of spacelike stationary surfaces in the 4-dimensional Lorentz space, Adv. Math. 249 (2013), 311–347.

[20] A. Mafra, Finitely curved orbits of complex polynomial vector fields, Anais da Academia Brasileira de Cincias 79 (2007), 13–16.

[21] K. Seo, Rigidity of minimal submanifolds in hyperbolic space, Arch. Math. (Basel) 94 (2010), 173–181.

[22] K. Shiohama, Cut locus and parallel circles of a closed curve on a Riemannian plane admitting total curvature, Comment. Math. Helv. 60 (1985), 125–138.

[23] K. Shiohama, T. Shioya and M. Tanaka, The Geometry of Total Curvature on Complete Open Surfaces, Cambridge Tracts in Mathematics, 159. Cambridge University Press, Cambridge, 2003.

[24] K. Shiohama, T. Shioya and M. Tanaka, Mass of rays on complete open surfaces, Pacific J. Math 143 (1990), 349–359.

[25] T. Shioya, Behavior of distant maximal geodesics in finitely connected complete 2-dimensional Riemannian manifolds, Mem. Amer. Math. Soc. 108 (1994), no. 517.

[26] T. Shioya, Behavior of distant maximal geodesics in finitely connected complete two-dimensional Riemannian manifolds. II, Geom. Dedicata 103 (2004), 1–32.

[27] S. Stepanov, I. Tsyganok and J. Mikes, On scalar and total scalar curvatures of Riemann-Cartan manifolds, Kragujevac J. Math. 35 (2011), 291–301.

[28] S. Willerton, On the magnitude of spheres, surfaces and other homogeneous spaces, Geom. Dedicata 168 (2014), 291–310.

Mathematics Department
University of Oregon
Eugene OR 97403
USA

E-mail address: gilkey@uoregon.edu

Department of Mathematics
Sungkyunkwan University
Suwon-Si, 16419
Korea

E-mail addresses: intcomplex@skku.edu
parkj@skku.edu