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UMBILICAL SURFACES OF PRODUCTS OF SPACE FORMS

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Abstract. We give a complete classification of umbilical surfaces of arbitrary codimension of a product $\mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ of space forms whose curvatures satisfy $k_1 + k_2 \neq 0$.

1. Introduction. A submanifold of a Riemannian manifold is *umbilical* if it is equally curved in all tangent directions. More precisely, an isometric immersion $f\colon M^m \to \tilde{M}^n$ between Riemannian manifolds is umbilical if there exists a normal vector field H along f such that its second fundamental form $\alpha_f \in \mathrm{Hom}(TM \times TM, N_fM)$ with values in the normal bundle satisfies

$$\alpha_f(X, Y) = \langle X, Y\rangle H \quad \text{for all} \quad X, Y \in \mathfrak{X}(M)\,.$$

The classification of umbilical submanifolds of space forms is very well known. For a general symmetric space N, it was shown by Nikolayevsky (see Theorem 1 of [6]) that any umbilical submanifold of N is an umbilical submanifold of a product of space forms totally geodesically embedded in N. This makes the classification of umbilical submanifolds of a product of space forms an important problem. For submanifolds of dimension $m \geq 3$ of a product $\mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ of space forms whose curvatures satisfy $k_1 + k_2 \neq 0$, the problem was reduced in [3] to the classification of m-dimensional umbilical submanifolds with codimension two of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, where $\mathbb{S}^n$ and $\mathbb{H}^n$ stand for the sphere and hyperbolic space, respectively. The case of $\mathbb{S}^n \times \mathbb{R}$ (respectively, $\mathbb{H}^n \times \mathbb{R}$) was carried out in [4] (respectively, [5]), extending previous results in [7] and [8] (respectively, [1]) for hypersurfaces.

In this paper we extend the results of [3] to the surface case. In this case, the argument in one of the steps of the proof for the higher dimensional case (see Lemma 8.2 of [3]) does not apply, and requires more elaborate work. This is carried out in Lemma 4 below, which shows that the difficulty is due to the existence of new interesting families of examples in the surface case. Indeed, our main result (see Theorem 5 below) states that, in addition to the examples that appear already in higher dimensions, there are precisely two distinct two-parameter families of complete embedded flat umbilical surfaces that lie substantially in $\mathbb{H}^3_k \times \mathbb{R}^2$ and $\mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2}$, respectively. These are discussed in Section 3.

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2. Preliminaries. Let $f\colon M \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ be an isometric immersion of a Riemannian manifold. We always assume that M is connected. Denote by $\mathcal{R}$ and $\mathcal{R}^\perp$ the curvature tensors of the tangent and normal bundles TM and N_fM, respectively, by $\alpha = \alpha_f \in \Gamma(T^*M \otimes T^*M \otimes N_fM)$ the second fundamental form of f and by $A_\eta = A_\eta^f$ its shape operator in the normal direction η, given by

$$\langle A_\eta X, Y\rangle = \langle \alpha(X, Y), \eta\rangle$$

for all $X, Y \in \mathfrak{X}(M)$. Set

$$L = L^f := \pi_2 \circ f_* \in \Gamma(T^*M \otimes T\mathbb{Q}_{k_2}^{n_2}) \text{ and } K = K^f := \pi_2|_{N_fM} \in \Gamma((N_fM)^* \otimes T\mathbb{Q}_{k_2}^{n_2}),$$

where $\pi_i\colon \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2} \to \mathbb{Q}_{k_i}^{n_i}$ denotes the canonical projection, $1 \le i \le 2$, and by abuse of notation also its derivative, which we regard as a section of $T^*(\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}) \otimes T\mathbb{Q}_{k_i}^{n_i}$.

2.1. The fundamental equations. The tensors $R \in \Gamma(T^*M \otimes TM)$, $S \in \Gamma(T^*M \otimes N_fM)$ and $T \in \Gamma((N_fM)^* \otimes N_fM)$ given by

$$R = L^tL, \quad S = K^tL \quad \text{and} \quad T = K^tK, \tag{1}$$

or equivalently, by

$$L = f_*R + S \quad \text{and} \quad K = f_*S^t + T,$$

were introduced in [2] (see also [3]), where they were shown to satisfy the algebraic relations

$$S^tS = R(I - R), \quad TS = S(I - R) \quad \text{and} \quad SS^t = T(I - T), \tag{2}$$

as well as the differential equations

$$(\nabla_X R)Y = A_{SY}X + S^t\alpha(X, Y), \tag{3}$$

$$(\nabla_X S)Y = T\alpha(X, Y) - \alpha(X, RY) \tag{4}$$

and

$$(\nabla_X T)\xi = -SA_\xi X - \alpha(X, S^t\xi) \tag{5}$$

for all $X, Y \in \mathfrak{X}(M)$ and all $\xi \in \Gamma(N_fM)$. In particular, from the first and third equations of (1) and (2), respectively, it follows that R and T are nonnegative operators whose eigenvalues lie in $[0, 1]$.

The Gauss, Codazzi and Ricci equations of f are, respectively,

$$\mathcal{R}(X, Y)Z = (k_1(X\wedge Y - X\wedge RY - RX\wedge Y) + \kappa RX\wedge RY)Z + A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y, \tag{6}$$

$$(\nabla_X^\perp\alpha)(Y, Z) - (\nabla_Y^\perp\alpha)(X, Z) = \langle k_1X - \kappa RX, Z\rangle SY - \langle k_1Y - \kappa RY, Z\rangle SX \tag{7}$$

and

$$\mathcal{R}^\perp(X, Y)\eta = \alpha(X, A_\eta Y) - \alpha(A_\eta X, Y) + \kappa(SX \wedge SY)\eta, \tag{8}$$

where $\kappa = k_1 + k_2$.

2.2. The flat underlying space. In order to study isometric immersions $f\colon M \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$, it is useful to consider their compositions $F = h \circ f$ with the canonical isometric embedding

$$h\colon \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2} \to \mathbb{R}^{N_1}_{\sigma(k_1)} \times \mathbb{R}^{N_2}_{\sigma(k_2)} = \mathbb{R}^{N_1+N_2}_{\mu}.$$

Here, for $k \in \mathbb{R}$ we set $\sigma(k) = 1$ if $k < 0$ and $\sigma(k) = 0$ otherwise, and as a subscript of an Euclidean space it means the index of the corresponding flat metric. Also, we denote $\mu = \sigma(k_1) + \sigma(k_2)$, $N_i = n_i + 1$ if $k_i \neq 0$ and $N_i = n_i$ otherwise, in which case $\mathbb{Q}^{n_i}_{k_i}$ stands for $\mathbb{R}^{n_i}$.

Let $\tilde{\pi}_i\colon \mathbb{R}^{N_1+N_2}_{\mu} \to \mathbb{R}^{N_i}_{\sigma(k_i)}$, $1 \le i \le 2$, denote the canonical projection. Then, the normal space of h at each point $z \in \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ is spanned by $k_1\tilde{\pi}_1(h(z))$ and $k_2\tilde{\pi}_2(h(z))$, and its second fundamental form is given by

$$\alpha_h(X, Y) = -k_1\langle \pi_1 X, Y\rangle \tilde{\pi}_1 \circ h - k_2\langle \pi_2 X, Y\rangle \tilde{\pi}_2 \circ h. \tag{9}$$

Therefore, if $k_i \neq 0$, $1 \le i \le 2$, then, setting $r_i = |k_i|^{-1/2}$, the unit vector field $\nu_i = \nu_i^F = \frac{1}{r_i}\tilde{\pi}_i \circ F$ is normal to F and we have

$$\tilde{\nabla}_X \nu_1 = \frac{1}{r_1}\tilde{\pi}_1 F_* X = \frac{1}{r_1}(F_* X - h_* LX) = \frac{1}{r_1}(F_*(I - R)X - h_* SX)$$

and

$$\tilde{\nabla}_X \nu_2 = \frac{1}{r_2}\tilde{\pi}_2 F_* X = \frac{1}{r_2}h_* LX = \frac{1}{r_2}(F_* RX + h_* SX),$$

where $\tilde{\nabla}$ stands for the derivative in $\mathbb{R}^{N_1+N_2}_{\mu}$. Hence

$$^F\nabla^{\perp}_X \nu_1 = -\frac{1}{r_1}h_* SX, \qquad A^F_{\nu_1} = -\frac{1}{r_1}(I - R), \tag{10}$$

$$^F\nabla^{\perp}_X \nu_2 = \frac{1}{r_2}h_* SX \quad \text{and} \quad A^F_{\nu_2} = -\frac{1}{r_2}R. \tag{11}$$

2.3. Reduction of codimension. An isometric immersion $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ is said to *reduce codimension on the left by* ℓ if there exists a totally geodesic inclusion $j_1\colon \mathbb{Q}^{m_1}_{k_1} \to \mathbb{Q}^{n_1}_{k_1}$, with $n_1 - m_1 = \ell$, and an isometric immersion $\bar{f}\colon M^m \to \mathbb{Q}^{m_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ such that $f = (j_1 \times id) \circ \bar{f}$. Similarly one defines what it means by f reducing codimension *on the right*.

We will need the following result from [3] on reduction of codimension. In the statement, U and V stand for $\ker T$ and $\ker(I - T)$, respectively. Notice that the third equation in (2) implies that $S(TM)^{\perp}$ splits orthogonally as $S(TM)^{\perp} = U \oplus V$, with $U = (I - T)(S(TM)^{\perp})$ and $V = T(S(TM)^{\perp})$. Also, given an isometric immersion $f\colon M \to \tilde{M}$ between Riemannian manifolds, its *first normal space* at $x \in M$ is the subspace $N_1(x)$ of $N_f M(x)$ spanned by the image of its second fundamental form at x.

PROPOSITION 1. *Let $f\colon M^m \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ be an isometric immersion. If $U \cap N_1^\perp$ (respectively, $V \cap N_1^\perp$) is a vector subbundle of $N_f M$ with rank ℓ satisfying $\nabla^\perp(U \cap N_1^\perp) \subset N_1^\perp$ (respectively, $\nabla^\perp(V \cap N_1^\perp) \subset N_1^\perp$), then f reduces codimension on the left (respectively, on the right) by ℓ.*

2.4. Frenet formulae for space-like curves in $\mathbb{R}^4_1$. We briefly recall the definition of the Frenet curvatures and the Frenet frame of a unit-speed space-like curve $\gamma : I \to \mathbb{R}^4_1$ in the four dimensional Lorentz space, as well as the coresponding Frenet formulae, which will be needed in the sequel.

Thus, we assume that $t(s) = \gamma'(s)$ satisfies $\langle t(s), t(s)\rangle = 1$ for all $s \in I$. Assume first that $\langle \gamma''(s), \gamma''(s)\rangle \neq 0$ for all $s \in I$. Define $\hat{k}_1(s) = \|\gamma''(s)\| = |\langle \gamma''(s), \gamma''(s)\rangle|^{1/2}$ and $n_1(s) = \gamma''(s)/\hat{k}_1(s)$ for all $s \in I$. Denote $\epsilon_1 = \langle n_1, n_1\rangle$. Suppose that $v(s) = n_1'(s) + \epsilon_1 \hat{k}_1(s)t(s)$ satisfies $\langle v(s), v(s)\rangle \neq 0$ for all $s \in I$. Define $\hat{k}_2(s) = \|v(s)\|$ and $n_2(s) = v(s)/\hat{k}_2(s)$. Let $n_3(s)$ be chosen so that $\{t(s), n_1(s), n_2(s), n_3(s)\}$ is a positively-oriented orthonormal basis of $\mathbb{R}^4_1$ and set $\epsilon_3 = \langle n_3, n_3\rangle$. Then the following Frenet formulae hold, where $\hat{k}_3$ is defined by the third equation:

$$\begin{cases} t' = \hat{k}_1 n_1 \,, \\ n_1' = -\epsilon_1 \hat{k}_1 t + \hat{k}_2 n_2 \,, \\ n_2' = \epsilon_3 \hat{k}_2 n_1 + \hat{k}_3 n_3 \,, \\ n_3' = \epsilon_1 \hat{k}_3 n_2 \,. \end{cases}$$

Lesser known are the formulae in the case in which $\gamma''(s)$ is a nonzero light-like vector everywhere, i.e., $\tilde{n}_1(s) = \gamma''(s)$ satisfies $\langle \tilde{n}_1(s), \tilde{n}_1(s)\rangle = 0$ for all $s \in I$. We carry them out in more detail below.

First notice that $\langle t, \tilde{n}_1\rangle = 0$. Here, and in the next computations, we drop the "s" for simplicity of notation and understand that all equalities hold for all $s \in I$. Thus,

$$\langle \tilde{n}_1', t\rangle = -\langle t', \tilde{n}_1\rangle = -\langle \tilde{n}_1, \tilde{n}_1\rangle = 0\,.$$

Moreover, $\langle \tilde{n}_1', \tilde{n}_1\rangle = 0$, hence $\tilde{n}_1'$ is space-like. Define $\tilde{k}_1 = \|\tilde{n}_1'\|$ and $\tilde{n}_2$ by $\tilde{n}_1' = \tilde{k}_1 \tilde{n}_2$. Now let $\tilde{n}_3 \in \{t, \tilde{n}_2\}^\perp$ be the unique vector such that

$$\langle \tilde{n}_3, \tilde{n}_3\rangle = 0 \quad \text{and} \quad \langle \tilde{n}_1, \tilde{n}_3\rangle = 1\,,$$

that is, $\{\tilde{n}_1, \tilde{n}_3\}$ is a pseudo-othonormal basis of the time-like plane $\{t, \tilde{n}_2\}^\perp$. Since

$$\langle \tilde{n}_2', t\rangle = -\langle \tilde{n}_2, t'\rangle = -\langle \tilde{n}_2, \tilde{n}_1\rangle = 0$$

and

$$\langle \tilde{n}_2', \tilde{n}_1\rangle = -\langle \tilde{n}_2, \tilde{n}_1'\rangle = -\tilde{k}_1\,,$$

we have

$$\tilde{n}_2' = \langle \tilde{n}_2', \tilde{n}_1\rangle \tilde{n}_3 + \langle \tilde{n}_2', \tilde{n}_3\rangle \tilde{n}_1 = -\tilde{k}_1 \tilde{n}_3 - \tilde{k}_2 \tilde{n}_1\,,$$

where

$$\tilde{k}_2 = \langle \tilde{n}_3', \tilde{n}_2\rangle\,.$$

Finally, since

$$0 = \langle \tilde{n}_3', t\rangle = \langle \tilde{n}_3', \tilde{n}_1\rangle = \langle \tilde{n}_3', \tilde{n}_3\rangle\,,$$

we have

$$\tilde{n}_3' = \langle \tilde{n}_3', \tilde{n}_2\rangle \tilde{n}_2 = \tilde{k}_2 \tilde{n}_2\,.$$

In summary, for a unit-speed space-like curve $\gamma\colon I \to \mathbb{R}^4_1$ with light-like curvature vector γ'', one can define two Frenet curvatures $\tilde{k}_1$ and $\tilde{k}_2$ and a pseudo-orthonormal Frenet frame $\{t, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3\}$ with respect to which the Frenet formulae are

$$\begin{cases} t' = \tilde{n}_1\,, \\ \tilde{n}_1' = \tilde{k}_1\tilde{n}_2\,, \\ \tilde{n}_2' = -\tilde{k}_2\tilde{n}_1 - \tilde{k}_1\tilde{n}_3\,, \\ \tilde{n}_3' = \tilde{k}_2\tilde{n}_2\,. \end{cases}$$

In both cases, a unit-speed space-like curve $\gamma\colon I \to \mathbb{R}^4_1$ is completely determined by its Frenet curvatures, up to an isometry of $\mathbb{R}^4_1$.

3. Flat umbilical surfaces in $\mathbb{H}^3_k \times \mathbb{R}^2$ and $\mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2}$. We present below two families of complete flat properly embedded umbilical surfaces, the first one in $\mathbb{H}^3_k \times \mathbb{R}^2$ and the second in $\mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2}$, each of which depending on two parameters.

EXAMPLE 2. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^6_1 = \mathbb{R}^4_1 \times \mathbb{R}^2$, where $\mathbb{R}^4_1$ has signature $(-,+,+,+)$, be given by

$$F(s,t) = \left(a_1 \cosh\frac{s}{c}, a_1 \sinh\frac{s}{c}, a_2 \cos\frac{t}{c}, a_2 \sin\frac{t}{c}, b_1\frac{s}{c}, b_2\frac{t}{c}\right)\,, \tag{12}$$

with

$$a_1^2 - a_2^2 = r^2 \quad \text{and} \quad a_1^2 + b_1^2 = c^2 = a_2^2 + b_2^2\,. \tag{13}$$

Then $F(\mathbb{R}^2) \subset \mathbb{H}^3_k \times \mathbb{R}^2$, where $k = -1/r^2$, by the first relation in (13). If $\{e_1, \dots, e_6\}$ is the orthonormal basis of $\mathbb{R}^6_1$ with respect to which F is given by (12), then the subspaces V_1 and V_2 of $\mathbb{L}^6$ spanned by $\{e_1, e_2, e_5\}$ and $\{e_3, e_4, e_6\}$ can be identified with $\mathbb{R}^3_1$ and $\mathbb{R}^3$, respectively, and

$$F = \gamma_1 \times \gamma_2\colon \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \to V_1 \times V_2 = \mathbb{R}^3_1 \times \mathbb{R}^3 = \mathbb{R}^6_1\,,$$

where γ_1 and γ_2 are the helices in $\mathbb{R}^3_1$ and $\mathbb{R}^3$, respectively, parameterized by

$$\gamma_1(s) = \left(a_1 \cosh\frac{s}{c}, a_1 \sinh\frac{s}{c}, b_1\frac{s}{c}\right)$$

and

$$\gamma_2(t) = \left(a_2 \cos\frac{t}{c}, a_2 \sin\frac{t}{c}, b_2\frac{t}{c}\right)\,.$$

By the relations on the right in (13), both γ_1 and γ_2 are unit-speed curves, hence F is an isometric immersion. Since $F(\mathbb{R}^2) \subset \mathbb{H}^3_k \times \mathbb{R}^2$, there exists an isometric immersion $f\colon \mathbb{R}^2 \to$

$\mathbb{H}^3_k \times \mathbb{R}^2$ such that $F = h \circ f$, where $h\colon \mathbb{H}^3_k \times \mathbb{R}^2 \to \mathbb{R}^6_1$ denotes the inclusion. It is easily checked that the second fundamental form of f satisfies

$$\alpha_f\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = 0$$

and

$$\alpha_f\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = H(s,t) = \alpha_f\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right),$$

where

$$h_* H(s,t) = \frac{ka_1a_2}{c^2}\left(a_2 \cosh\frac{s}{c}, a_2 \sinh\frac{s}{c}, a_1 \cos\frac{t}{c}, a_1 \sin\frac{t}{c}, 0, 0\right).$$

Hence f is umbilical with mean curvature vector field H.

In view of (13), one can write

$$a_1^2 = r^2\frac{(1-\lambda_1)}{\lambda_2-\lambda_1},\quad a_2^2 = r^2\frac{(1-\lambda_2)}{\lambda_2-\lambda_1},\quad b_1^2 = r^2\frac{\lambda_1}{\lambda_2-\lambda_1},\quad b_2^2 = r^2\frac{\lambda_2}{\lambda_2-\lambda_1},\quad c^2 = \frac{r^2}{\lambda_2-\lambda_1},$$

with $0 < \lambda_1 < \lambda_2 < 1$. Then, one can check that the curvature vector γ_i'' of γ_i, $1 \le i \le 2$, satisfies

$$\langle \gamma_i'', \gamma_i''\rangle = k(\lambda_j - \lambda_i)(1-\lambda_i), \quad 1 \le i \ne j \le 2, \tag{14}$$

and that the second Frenet curvature (torsion) of γ_i satisfies

$$\tau_i^2 = -k\lambda_i|\lambda_j - \lambda_i|, \quad 1 \le i \ne j \le 2. \tag{15}$$

EXAMPLE 3. Let $\mathbb{R}^8_2 = \mathbb{R}^4_1 \times \mathbb{R}^4_1$ denote Euclidean space of dimension 8 endowed with an inner product of signature $(-,+,+,+,-,+,+,+)$, and let $F\colon \mathbb{R}^2 \to \mathbb{R}^8_2$ be given by

$$F(s,t) = \left(a_1 \cosh\frac{s}{c}, a_1 \sinh\frac{s}{c}, a_2 \cos\frac{t}{c}, a_2 \sin\frac{t}{c}, a_3 \cosh\frac{t}{d}, a_3 \sinh\frac{t}{d}, a_4 \cos\frac{s}{d}, a_4 \sin\frac{s}{d}\right), \tag{16}$$

with

$$a_1^2 - a_2^2 = r_1^2, \qquad a_3^2 - a_4^2 = r_2^2 \qquad \text{and} \qquad \frac{a_1^2}{c^2} + \frac{a_4^2}{d^2} = 1 = \frac{a_2^2}{c^2} + \frac{a_3^2}{d^2}. \tag{17}$$

The first pair of relations in (17) implies that $F(\mathbb{R}^2) \subset \mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2} \subset \mathbb{R}^4_1 \times \mathbb{R}^4_1$, with $k_i = -1/r_i^2$ for $1 \le i \le 2$. If $\{e_1, \dots, e_4, f_1, \dots, f_4\}$ is the orthonormal basis of $\mathbb{R}^8_2$ with respect to which F is given by (16), then the subspaces V_1 and V_2 of $\mathbb{R}^8_2$ spanned by $\{e_1, e_2, f_3, f_4\}$ and $\{f_1, f_2, e_3, e_4\}$ can also be identified with $\mathbb{R}^4_1$, and

$$F = \gamma_1 \times \gamma_2\colon \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \to V_1 \times V_2,$$

where γ_1 and γ_2 are the curves parameterized by

$$\gamma_1(s) = \left(a_1 \cosh\frac{s}{c}, a_1 \sinh\frac{s}{c}, a_4 \cos\frac{s}{d}, a_4 \sin\frac{s}{d}\right)$$

and

$$\gamma_2(t) = \left(a_3 \cosh\frac{t}{d}, a_3 \sinh\frac{t}{d}, a_2 \cos\frac{t}{c}, a_2 \sin\frac{t}{c}\right).$$

In view of the second pair of relations in (17), both γ_1 and γ_2 are unit-speed curves, hence F is an isometric immersion. Since $F(\mathbb{R}^2) \subset \mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2}$, there exists an isometric immersion $f\colon \mathbb{R}^2 \to \mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2}$ such that $F = h \circ f$, where $h\colon \mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2} \to \mathbb{R}^8_2$ denotes the inclusion. One can easily check that the second fundamental form of f satisfies

$$\alpha_f\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = 0$$

and

$$\alpha_f\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = H(s,t) = \alpha_f\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)$$

where

$$\begin{aligned} h_*H(s,t) =& \frac{k_1 a_1 a_2}{c^2}\left(a_2 \cosh\frac{s}{c}, a_2 \sinh\frac{s}{c}, a_1\cos\frac{t}{c}, a_1 \sin\frac{t}{c}, 0,0,0,0\right) \\ &+ \frac{k_2 a_3 a_4}{d^2}\left(0,0,0,0, a_4\cosh\frac{t}{d}, a_4 \sinh\frac{t}{d}, a_3\cos\frac{s}{d}, a_3\sin\frac{s}{d}\right). \end{aligned}$$

It follows that f is umbilical with mean curvature vector field H.

By the conditions in (17), one can write

$$a_1^2 = r_1^2\frac{(1-\lambda_1)}{\lambda_2-\lambda_1}, \quad a_2^2 = r_1^2\frac{(1-\lambda_2)}{\lambda_2-\lambda_1}, \quad a_3^2 = r_2^2\frac{\lambda_2}{\lambda_2-\lambda_1}, \quad a_4^2 = r_2^2\frac{\lambda_1}{\lambda_2-\lambda_1},$$

$$c^2 = \frac{r_1^2}{\lambda_2-\lambda_1} \quad\text{and}\quad d^2 = \frac{r_2^2}{\lambda_2-\lambda_1},$$

with $0 < \lambda_1 < \lambda_2 < 1$. Then, the curvature vector γ_i'' of the curve γ_i, $1 \le i \le 2$, satisfies

$$\langle \gamma_i'', \gamma_i''\rangle = (\lambda_i - \lambda_j)(\kappa\lambda_i - k_1), \quad 1 \le i \ne j \le 2, \quad \kappa = k_1 + k_2. \tag{18}$$

If $\kappa\lambda_i - k_1 \ne 0$, one can check that γ_i, $1 \le i \le 2$, has constant Frenet curvatures $\hat{k}^i_\ell$, $1 \le \ell \le 3$, given by

$$(\hat{k}^i_1)^2 = |(\lambda_i - \lambda_j)(\kappa\lambda_i - k_1)|, \tag{19}$$

$$(\hat{k}^i_2)^2 = \frac{\kappa^2|\lambda_i - \lambda_j|\lambda_i(1-\lambda_i)}{|\kappa\lambda_i - k_1|} \tag{20}$$

and

$$(\hat{k}^i_3)^2 = \frac{k_1k_2|\lambda_i - \lambda_j|}{|\kappa\lambda_i - k_1|}, \quad 1 \le j \ne i \le 2. \tag{21}$$

If $\kappa\lambda_i - k_1 = 0$, that is, the curvature vector of γ_i is light-like, then one can check that γ_i has constant Frenet curvatures $\tilde{k}^i_1$ and $\tilde{k}^i_2$, $1 \le i \le 2$ (see Subsection 2.4), given by

$$(\tilde{k}^i_1)^2 = \frac{k_1k_2(\kappa\lambda_j - k_1)^2}{\kappa^2}, \quad 1 \le j \ne i \le 2, \tag{22}$$

and

$$(\tilde{k}^i_2)^2 = \frac{(k_1-k_2)^2}{4k_1k_2}, \quad 1 \le i \le 2. \tag{23}$$

It is also easily checked that the isometric immersions in both of the preceding examples have the frame of coordinate vector fields $\{\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\}$ as a frame of principal directions for the associated tensor R, with corresponding eigenvalues λ_1 and λ_2, respectively. Moreover, they are clearly injective and proper, hence embeddings. Therefore, all surfaces in both families are properly embedded and isometric to the plane.

4. The main step. Umbilical submanifolds of $\mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ were studied in [3] according to the possible structures of the tensor S. When $\ker S = \{0\}$, it was shown that R must be a constant multiple of the identity tensor whenever the dimension of the submanifold is at least three (see [3], Lemma 8.2), which corresponds to case (i) in the statement of Lemma 4 below. We now show that in the surface case the only exceptions are the surfaces of the two families in the preceding section.

LEMMA 4. *Let $f\colon M^2 \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$, $k_1 + k_2 \neq 0$, be an umbilical isometric immersion. Assume that* $\ker S = \{0\}$ *at some point $x \in M^2$. Then one of the following holds:*

(i) *there exist umbilical isometric immersions $f_i\colon M^2 \to \mathbb{Q}_{\tilde{k}_i}^{n_i}$, $1 \leq i \leq 2$, with $\tilde{k}_1 = k_1 \cos^2\theta$ and $\tilde{k}_2 = k_2 \sin^2\theta$ for some $\theta \in (0, \pi/2)$, such that $f = (\cos\theta f_1, \sin\theta f_2)$;*

(ii) *after interchanging the factors, if necessary, we have $k_2 = 0$, $n_1 \geq 3$, $n_2 \geq 2$ and $f = j \circ \tilde{f}$, where $j\colon \mathbb{Q}_{k_1}^3 \times \mathbb{R}^2 \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{R}^{n_2}$ and $\tilde{f}\colon M^2 \to \mathbb{Q}_{k_1}^3 \times \mathbb{R}^2$ are isometric immersions such that j is totally geodesic and $\tilde{f}(M^2)$ is an open subset of a surface as in Example* 2*;*

(iii) *$k_i < 0$ and $n_i \geq 3$, $1 \leq i \leq 2$, and $f = j \circ \tilde{f}$, where $j\colon \mathbb{Q}_{k_1}^3 \times \mathbb{Q}_{k_2}^3 \to \mathbb{Q}_{k_1}^{n_1} \times \mathbb{Q}_{k_2}^{n_2}$ and $\tilde{f}\colon M^2 \to \mathbb{Q}_{k_1}^3 \times \mathbb{Q}_{k_2}^3$ are isometric immersions such that j is totally geodesic and $\tilde{f}(M^2)$ is an open subset of a surface as in Example* 3.

PROOF. Let λ_1 and λ_2 be the eigenvalues of R. If $\lambda_1 = \lambda_2$ on M then f is as in (i) by Proposition 5.2 of [3]. Now assume that $\lambda_1 \neq \lambda_2$ at x and let $\mathcal{U} \subset \mathcal{M}$ be the maximal connected open neighborhood of x where $\ker S = \{0\}$ and $\lambda_1 \neq \lambda_2$. In particular, λ_1 and λ_2 are differentiable on $\mathcal{U}$.

Fix an orthonormal frame $\{X_1, X_2\}$ of eigenvectors of R, with X_i associated to λ_i, and define $\xi_i := SX_i$ for $i = 1, 2$. Thus, from (2) we have

$$\langle \xi_i, \xi_j \rangle = \langle S^t S X_i, X_j \rangle = \delta_{ij} \lambda_i (1 - \lambda_i) \tag{24}$$

and

$$T\xi_i = TSX_i = (1 - \lambda_i)\xi_i \tag{25}$$

for $i, j = 1, 2$. We can write equations (6)–(8) in the frames $\{X_1, X_2\}$ and $\{\xi_1, \xi_2\}$, in terms of the Gaussian curvature K of M^2 and the mean curvature vector H of f, as

$$K = k_1(1 - \lambda_1)(1 - \lambda_2) + k_2 \lambda_1 \lambda_2 + |H|^2\,, \tag{26}$$

$$\nabla_{X_i}^{\perp} H = (\kappa \lambda_j - k_1)\xi_i\,,\ \ 1 \leq i \neq j \leq 2\,, \tag{27}$$

and

$$\mathcal{R}^{\perp}(X_1, X_2) = \kappa(\xi_1 \wedge \xi_2)\,, \tag{28}$$

whereas equations (3)–(5) become

$$\left(\nabla_{X_i} R\right) X_j = \langle \xi_j, H\rangle X_i + \delta_{ij} S^t H\,, \tag{29}$$

$$\left(\nabla_{X_i} S\right) X_j = \delta_{ij}(T - \lambda_j I)H \tag{30}$$

and

$$\left(\nabla_{X_i} T\right) \xi = -\langle \xi, H\rangle \xi_i - \langle \xi, \xi_i\rangle H \tag{31}$$

for $i, j = 1, 2$ and all $\xi \in \Gamma(N_f M^2)$. Define the Christoffel symbols Γ_{11}^2 and Γ_{22}^1 by

$$\nabla_{X_1} X_1 = \Gamma_{11}^2 X_2 \quad \text{and} \quad \nabla_{X_2} X_2 = \Gamma_{22}^1 X_1\,. \tag{32}$$

Substituting

$$\begin{aligned}\left(\nabla_{X_i} R\right) X_j &= \nabla_{X_i} R X_j - R \nabla_{X_i} X_j \\ &= X_i(\lambda_j) X_j + (\lambda_j I - R)\nabla_{X_i} X_j\end{aligned}$$

into (29) yields

$$X_i(\lambda_j) = \delta_{ij} 2\langle \xi_i, H\rangle \tag{33}$$

and

$$\langle \xi_i, H\rangle = (\lambda_j - \lambda_i)\Gamma_{jj}^i, \quad 1 \le i \ne j \le 2\,. \tag{34}$$

On the other hand, from (30) we get

$$\nabla_{X_i}^{\perp} \xi_j = -\Gamma_{ii}^j \xi_i, \quad 1 \le i \ne j \le 2\,. \tag{35}$$

Using (27), (32), (33) and (35) we obtain

$$\begin{aligned}\mathcal{R}^{\perp}(X_1, X_2)H =& \nabla_{X_1}^{\perp}\nabla_{X_2}^{\perp} H - \nabla_{X_2}^{\perp}\nabla_{X_1}^{\perp} H - \nabla_{[X_1, X_2]}^{\perp} H \\ =& \nabla_{X_1}^{\perp}(\kappa\lambda_1 - k_1)\xi_2 - \nabla_{X_2}^{\perp}(\kappa\lambda_2 - k_1)\xi_1 + (\kappa\lambda_2 - k_1)\Gamma_{11}^2\xi_1 \\ &- (\kappa\lambda_1 - k_1)\Gamma_{22}^1\xi_2 \\ =& \kappa X_1(\lambda_1)\xi_2 + (\kappa\lambda_1 - k_1)\nabla_{X_1}^{\perp}\xi_2 - \kappa X_2(\lambda_2)\xi_1 - (\kappa\lambda_2 - k_1)\nabla_{X_2}^{\perp}\xi_1 \\ &+ (\kappa\lambda_2 - k_1)\Gamma_{11}^2\xi_1 - (\kappa\lambda_1 - k_1)\Gamma_{22}^1\xi_2 \\ =& 2\kappa\langle \xi_1, H\rangle\xi_2 - (\kappa\lambda_1 - k_1)\Gamma_{11}^2\xi_1 - 2\kappa\langle \xi_2, H\rangle\xi_1 + (\kappa\lambda_2 - k_1)\Gamma_{22}^1\xi_2 \\ &+ (\kappa\lambda_2 - k_1)\Gamma_{11}^2\xi_1 - (\kappa\lambda_1 - k_1)\Gamma_{22}^1\xi_2 \\ =& -\kappa(2\langle \xi_2, H\rangle + (\lambda_1 - \lambda_2)\Gamma_{11}^2)\xi_1 + \kappa(2\langle \xi_1, H\rangle + (\lambda_2 - \lambda_1)\Gamma_{22}^1)\xi_2\,.\end{aligned}$$

In view of (34), the above equation becomes

$$\mathcal{R}^{\perp}(X_1, X_2)H = -3\kappa(\langle \xi_2, H\rangle\xi_1 - \langle \xi_1, H\rangle\xi_2)\,.$$

Comparing the preceding equation with

$$\mathcal{R}^{\perp}(X_1, X_2)H = \kappa(\langle\xi_2, H\rangle\xi_1 - \langle\xi_1, H\rangle\xi_2)\,,$$

which follows from (28), and using that $\kappa \neq 0$, we get $\langle\xi_1, H\rangle = 0 = \langle\xi_2, H\rangle$, i.e.,

$$H \in \Gamma(S(TM)^{\perp})\,. \tag{36}$$

In particular, we obtain from (33) that λ_1 and λ_2 assume constant values in $(0, 1)$ everywhere on $\mathcal{U}$. If $\mathcal{U}$ were a proper subset of M^2, then λ_1 and λ_2 would assume the same values on the boundary of $\mathcal{U}$, hence $\lambda_i(1-\lambda_i) \neq 0$ on an open connected neighborhood of $\bar{\mathcal{U}}$, $1 \leq i \leq 2$, contradicting the maximality of $\mathcal{U}$ as an open connected neighborhood of x where $\ker S = \{0\}$ and $\lambda_1 \neq \lambda_2$. It follows that $\mathcal{U} = M^2$.

We obtain from (34) and (36) that $\Gamma^2_{11} = 0 = \Gamma^1_{22}$. In particular, we have $K = 0$ everywhere, and then (26) gives

$$|H|^2 = -k_1(1-\lambda_1)(1-\lambda_2) - k_2\lambda_1\lambda_2\,. \tag{37}$$

Set $\xi = H$ in (31). By using (25), (27) and (37), we obtain

$$\nabla^{\perp}_{X_i} TH = k_2\lambda_j\xi_i, \quad 1 \leq i \neq j \leq 2\,, \tag{38}$$

and similarly

$$\nabla^{\perp}_{X_i}(I-T)H = -(1-\lambda_j)k_1\xi_i, \quad 1 \leq i \neq j \leq 2\,. \tag{39}$$

In particular, bearing in mind (36) and the fact that T leaves $S(TM)$ invariant, as follows from the second equation in (2), we obtain that both TH and $(I-T)H$ have constant length on M^2, hence either $TH = 0$, $TH = H$ or both TH and $(I-T)H$ are nonzero everywhere. Therefore $L_1 = U \cap \{H\}^{\perp} = U \cap N_1^{\perp}$ and $L_2 = V \cap \{H\}^{\perp} = V \cap N_1^{\perp}$ have constant dimensions on M^2, which are, accordingly, $(\operatorname{rank} U - 1, \operatorname{rank} V)$, $(\operatorname{rank} U, \operatorname{rank} V - 1)$ or $(\operatorname{rank} U - 1, \operatorname{rank} V - 1)$. Moreover, equations (27) and (36) imply that $\nabla^{\perp}_{TM} L_i \subset \{H\}^{\perp}$ for $i = 1, 2$. Hence, the assumptions of Proposition 1 are satisfied, and we conclude that there are three corresponding possibilities for the pairs (n_1, n_2) of *substantial* values of n_1 and n_2: $(3, 2)$, $(2, 3)$ and $(3, 3)$.

We first consider the case $(n_1, n_2) = (3, 2)$. This is the case in which $TH = 0$, and hence $k_2 = 0$ by (38). Thus we have $k_1 < 0$ from (37), and we may assume that f takes values in $\mathbb{H}^3_k \times \mathbb{R}^2$, with $k = k_1 < 0$.

Set $F = h \circ f$, where $h\colon \mathbb{H}^3_k \times \mathbb{R}^2 \to \mathbb{R}^6_1$ denotes the inclusion. By (9), the second fundamental form of F is given by

$$\alpha_F(X, Y) = \langle X, Y\rangle h_* H + \frac{1}{r}\langle (I-R)X, Y\rangle \nu\,,$$

where $r = (-k)^{-1/2}$ and $\nu = \frac{1}{r}\tilde{\pi}_1 \circ F$. Therefore

$$\alpha_F(X_i, X_j) = \delta_{ij}\Big(h_* H + \frac{1}{r}(1-\lambda_i)\nu\Big) := \delta_{ij} Z_i = \tilde{\nabla}_{X_j} F_* X_i, \quad 1 \leq i, j \leq 2\,. \tag{40}$$

Notice that

$$\langle Z_1, Z_2\rangle = |H|^2 + k(1-\lambda_1)(1-\lambda_2) = 0$$

by (37), and that

$$\langle Z_i, Z_i\rangle = k(\lambda_j - \lambda_i)(1-\lambda_i), \quad 1 \le i \ne j \le 2. \tag{41}$$

Moreover, since

$$\tilde{\pi}_2(h_* H) = h_*\pi_2 H = h_*(f_* S^t H + TH) = 0,$$

it follows that

$$\tilde{\pi}_2 Z_i = 0, \;\; 1 \le i \le 2.$$

Using (27), we have

$$\begin{aligned}\tilde{\nabla}_{X_i} h_* H &= h_*\hat{\nabla}_{X_i} H + \alpha_h(f_* X_i, H)\\ &= -F_* A^f_H X_i + h_*\nabla^\perp_{X_i} H + \frac{1}{r}\langle \pi_1 f_* X_i, H\rangle \nu\\ &= -|H|^2 F_* X_i - k(1-\lambda_j) h_*\xi_i + \frac{1}{r}\langle f_*(I-R)X_i - SX_i, H\rangle\nu\\ &= k(1-\lambda_j)\left((1-\lambda_i)F_* X_i - h_*\xi_i\right), \quad 1 \le i \ne j \le 2.\end{aligned}$$

On the other hand, by (10) we have

$$\tilde{\nabla}_{X_i}\nu = \frac{1}{r}(F_*(I-R)X_i - h_* S X_i) = \frac{1}{r}((1-\lambda_i)F_* X_i - h_*\xi_i).$$

Therefore

$$\tilde{\nabla}_{X_i} Z_j = 0, \;\; \text{if } i \ne j, \tag{42}$$

and

$$\tilde{\nabla}_{X_i} Z_i = k(\lambda_i - \lambda_j)((1-\lambda_i)F_* X_i - h_*\xi_i), \;\; 1 \le i \ne j \le 2. \tag{43}$$

Also, using that

$$\nabla^\perp_{X_i}\xi_i = -\frac{1}{|H|^2}\langle \nabla_{X_i} H, \xi_i\rangle H = -\lambda_i H,$$

as follows from (24), (27) and (37), we obtain that

$$\begin{aligned}\tilde{\nabla}_{X_i} h_*\xi_j &= h_*\hat{\nabla}_{X_i}\xi_j + \alpha_h(f_* X_i, \xi_j)\\ &= -F_* A^f_{\xi_j} X_i + h_*\nabla^\perp_{X_i}\xi_j + \frac{1}{r}\langle \pi_1 f_* X_i, \xi_j\rangle\nu\\ &= -\delta_{ij}\lambda_i Z_i, \quad 1 \le i, j \le 2.\end{aligned} \tag{44}$$

It follows from (40), (42), (43) and (44) that the subspaces $V_i = \text{span}\{F_* X_i, Z_i, h_*\xi_i\}$, $1 \le i \le 2$, are constant. Moreover, they are orthogonal to each other, hence $\mathbb{R}^6_1$ splits orthogonally as $\mathbb{R}^6_1 = V_1 \oplus V_2$.

Since $\Gamma^2_{11} = \Gamma^1_{22} = 0$, for each $x \in M^2$ there exists an isometry $\psi\colon\ W = I_1 \times I_2 \to U_x$ of a product of open intervals $I_j \subset \mathbb{R}$, $1 \le j \le 2$, onto an open neighborhood of x, such that $\psi_*\frac{\partial}{\partial s} = X_1$ and $\psi_*\frac{\partial}{\partial t} = X_2$, where s and t are the standard coordinates on I_1 and I_2, respectively. Write $g = F \circ \psi$. In terms of the coordinates (s, t), the fact that $\alpha_F(X_1, X_2) = 0$ translates into

$$\frac{\partial^2 g}{\partial s\partial t} = 0,$$

which implies that there exist smooth curves $\gamma_1 \colon I_1 \to V_1$ and $\gamma_2 \colon I_2 \to V_2$ such that $g = \gamma_1 \times \gamma_2$. By (40), (43) and (44), each γ_i is a helix in V_i with curvature vector $\gamma_i'' = Z_i$ and binormal vector $h_*(\xi_i/|\xi_i|)$, $1 \le i \le 2$. It follows from (41) that (14) holds for γ_i, whereas (41) and (44) imply that the second Frenet curvature of γ_i satisfies (15), i.e.,

$$\tau_i^2 = \frac{\lambda_i^2 |\langle Z_i, Z_i\rangle|}{\langle \xi_i, \xi_i\rangle} = -k\lambda_i|\lambda_j - \lambda_i|, \quad 1 \le i \ne j \le 2 .$$

Therefore, the helices γ_1 and γ_2 are precisely, up to congruence, those given in Example 2. Moreover, since the curvature vector Z_i along γ_i spans a two-dimensional subspace of V_i orthogonal to the axis of γ_i and $\tilde{\pi}_2 Z_i = Z_i$, $1 \le i \le 2$, it follows that the subspace $\mathbb{R}^2$ in the orthogonal decomposition $\mathbb{R}^6_1 = \mathbb{R}^4_1 \oplus \mathbb{R}^2$ adapted to $\mathbb{H}^3_k \times \mathbb{R}^2$ is spanned by the axes of γ_1 and γ_2. We conclude that g is (the restriction to W of) an isometric immersion as in Example 2.

We have shown that, for each $x \in M^2$, there exists an open neighborhood U_x of x such that $f(U_x)$ is contained in a surface as in Example 2 in a totally geodesic $\mathbb{Q}^3_{k_1} \times \mathbb{R}^2 \subset \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$. A standard connectedness argument now shows that f is as in the statement.

The case $(n_1, n_2) = (2, 3)$ is entirely similar and leads to the same conclusion with the factors interchanged. Let us consider the case $(n_1, n_2) = (3, 3)$, so we may now assume that f takes values in $\mathbb{Q}^3_{k_1} \times \mathbb{Q}^3_{k_2}$. Here both TH and $(I - T)H$ are nonzero everywhere, and we can choose unit vector fields $\xi_3 \in \ker T$, $\xi_4 \in \ker(I - T)$ and write $H = \rho_3\xi_3 + \rho_4\xi_4$, where $\rho_k = \langle \xi_k, H\rangle \ne 0$ for $k = 3, 4$. Applying (31) to $\xi = \xi_k$, with $k = 3, 4$, we get

$$T\nabla^\perp_{X_i}\xi_3 = \rho_3\xi_i \quad \text{and} \quad (I - T)\nabla^\perp_{X_i}\xi_4 = -\rho_4\xi_i$$

for $i = 1, 2$. Therefore, for $i = 1, 2$ we obtain

$$\nabla^\perp_{X_i}\xi_3 = \frac{\rho_3}{1 - \lambda_i}\xi_i \quad \text{and} \quad \nabla^\perp_{X_i}\xi_4 = -\frac{\rho_4}{\lambda_i}\xi_i . \tag{45}$$

Using (24), the preceding equations yield

$$\nabla^\perp_{X_i}\xi_i = -\lambda_i\rho_3\xi_3 + (1 - \lambda_i)\rho_4\xi_4, \quad 1 \le i \le 2 . \tag{46}$$

On the other hand, we have

$$(I - T)H = \rho_3\xi_3 \quad \text{and} \quad TH = \rho_4\xi_4 . \tag{47}$$

Thus, combining (38), (39) and (47) we get

$$\rho_3^2 = -k_1(1 - \lambda_1)(1 - \lambda_2) \quad \text{and} \quad \rho_4^2 = -k_2\lambda_1\lambda_2 . \tag{48}$$

In particular, we must have $k_1, k_2 < 0$, so f takes values in $\mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2}$.

Set $F = h \circ f$, where $h \colon \mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2} \to \mathbb{R}^4_1 \times \mathbb{R}^4_1 = \mathbb{R}^8_2$ denotes the inclusion. By (9), the second fundamental form of F is given by

$$\alpha_F(X, Y) = \langle X, Y\rangle h_* H + \frac{1}{r_1}\langle (I - R)X, Y\rangle \nu_1 + \frac{1}{r_2}\langle RX, Y\rangle \nu_2 ,$$

where $r_i = (-k_i)^{-1/2}$ and $\nu_i = \frac{1}{r_i}\tilde{\pi}_i \circ F$, $1 \le i \le 2$. Therefore

$$\alpha_F(X_i, X_j) = \delta_{ij}\Big(h_* H + \frac{1}{r_1}(1 - \lambda_i)\nu_1 + \frac{1}{r_2}\lambda_i\nu_2\Big) := \delta_{ij} Z_i = \tilde{\nabla}_{X_j} F_* X_i, \quad 1 \le i \le 2 . \tag{49}$$

Notice that

$$\langle Z_i, Z_j\rangle = |H|^2 + k_1(1-\lambda_i)(1-\lambda_j) + k_2\lambda_i\lambda_j, \quad 1 \le i, j \le 2\,.$$

It follows from (37) that

$$\langle Z_1, Z_2\rangle = 0$$

and

$$\langle Z_i, Z_i\rangle = (\lambda_i - \lambda_j)(\kappa\lambda_i - k_1), \quad 1 \le i \ne j \le 2\,. \tag{50}$$

Using (27), we obtain

$$\begin{aligned}\tilde\nabla_{X_i} h_* H &= h_*\hat\nabla_{X_i} H + \alpha_h(f_* X_i, H)\\ &= -F_* A^f_H X_i + h_*\nabla^\perp_{X_i} H\\ &= -|H|^2 F_* X_i + (\kappa\lambda_j - k_1)h_*\xi_i\,.\end{aligned}$$

On the other hand, by (10) and (11) we have

$$\tilde\nabla_{X_i}\nu_1 = \frac{1}{r_1}(F_*(I-R)X_i - h_*SX_i) = \frac{1}{r_1}((1-\lambda_i)F_*X_i - h_*\xi_i)$$

and

$$\tilde\nabla_{X_i}\nu_2 = \frac{1}{r_2}(F_*RX_i + h_*SX_i) = \frac{1}{r_2}(\lambda_i F_* X_i + h_*\xi_i)\,.$$

Using (37), it follows that

$$\tilde\nabla_{X_i} Z_j = 0, \ \text{ if } i \ne j\,,$$

whereas

$$\tilde\nabla_{X_i} Z_i = -\langle Z_i, Z_i\rangle F_* X_i + \kappa(\lambda_j - \lambda_i)h_*\xi_i, \ \ 1 \le i \ne j \le 2\,. \tag{51}$$

Also,

$$\begin{aligned}\tilde\nabla_{X_i} h_*\xi_j &= h_*\hat\nabla_{X_i}\xi_j + \alpha_h(f_*X_i, \xi_j)\\ &= -F_*A^f_{\xi_j}X_i + h_*\nabla^\perp_{X_i}\xi_j + \frac{1}{r_1}\langle\pi_1 f_* X_i, \xi_j\rangle\nu_1 + \frac{1}{r_2}\langle\pi_2 f_* X_i, \xi_j\rangle\nu_2\\ &= \delta_{ij}\Big(-\lambda_i Z_i + \rho_4 h_*\xi_4 + \frac{\lambda_i}{r_2}\nu_2\Big)\,, \qquad (52)\end{aligned}$$

where we have used (46).

If $\kappa\lambda_i - k_1 \ne 0$, that is, $\langle Z_i, Z_i\rangle \ne 0$, define

$$W_i = \tilde\nabla_{X_i} h_*\xi_i - \frac{\langle\tilde\nabla_{X_i} h_*\xi_i, Z_i\rangle}{\langle Z_i, Z_i\rangle} Z_i = \frac{-k_2\lambda_i}{\kappa\lambda_i - k_1} Z_i + \rho_4 h_*\xi_4 + \frac{\lambda_i}{r_2}\nu_2, \ \ 1 \le i \le 2\,.$$

Then the vectors $F_*X_i, Z_i, h_*\xi_i$ and W_i are pairwise orthogonal and the subspaces $V_i = \operatorname{span}\{F_*X_i, Z_i, h_*\xi_i, W_i\}$, $1 \le i \le 2$, are orthogonal to each other. Using the second equations in (48) and (45), we obtain that $\tilde\nabla_{X_i} W_j = 0$ and

$$\tilde\nabla_{X_i} W_i = \frac{k_1 k_2(\lambda_i - \lambda_j)}{\kappa\lambda_i - k_1} h_*\xi_i, \quad 1 \le i \ne j \le 2\,. \tag{53}$$

It follows that the subspaces V_1 and V_2 are constant, and that $\mathbb{R}^8_2$ also splits orthogonally as $\mathbb{R}^8_2 = V_1 \oplus V_2$.

If $\kappa\lambda_i - k_1 = 0$, define

$$\zeta_i = \frac{\kappa}{2k_1k_2(\kappa\lambda_j - k_1)}(-2\kappa\tilde{\nabla}_{X_i}h_*\xi_i + (k_2 - k_1)Z_i), \quad 1 \le i \ne j \le 2\,.$$

Then $\langle\zeta_i, \zeta_i\rangle = 0$, $\langle\zeta_i, Z_i\rangle = 1$ and $\zeta_i \in \operatorname{span}\{F_*X_i, h_*\xi_i\}^\perp$. Moreover, the subspaces $V_i = \operatorname{span}\{F_*X_i, Z_i, h_*\xi_i, \zeta_i\}$, $1 \le i \le 2$, are orthogonal to each other. Furthermore, since

$$\tilde{\nabla}_{X_i}\zeta_i = \frac{k_1^2 - k_2^2}{2k_1k_2}h_*\xi_i\,, \tag{54}$$

it follows that V_1 and V_2 are constant and that $\mathbb{R}^8_2$ also splits orthogonally as $\mathbb{R}^8_2 = V_1 \oplus V_2$.

Since $\Gamma^2_{11} = \Gamma^1_{22} = 0$, for each $x \in M^2$ there exists an isometry $\psi\colon W = I_1 \times I_2 \to U_x$ of a product of open intervals $I_j \subset \mathbb{R}$, $1 \le j \le 2$, onto a neighborhood of x, such that $\psi_*\frac{\partial}{\partial s} = X_1$ and $\psi_*\frac{\partial}{\partial t} = X_2$, where s and t are the standard coordinates on I_1 and I_2, respectively. Write $g = F \circ \psi$. In terms of the coordinates (s, t), the fact that $\alpha_F(X_1, X_2) = 0$ translates into

$$\frac{\partial^2 g}{\partial s \partial t} = 0\,,$$

which implies that there exist smooth curves $\gamma_1\colon I_1 \to V_1$ and $\gamma_2\colon I_2 \to V_2$ such that $g = \gamma_1 \times \gamma_2$.

If $\kappa\lambda_i - k_1 \ne 0$, it follows from (49), (51), (52) and (53) that γ_i is a unit-speed space like curve in V_i with constant Frenet curvatures $\hat{k}^i_\ell$, $1 \le \ell \le 3$, and Frenet frame $\{F_*X_i, \hat{Z}_i, h_*\hat{\xi}_i, \hat{W}_i\}$, where $\hat{Z}_i$, $\hat{\xi}_i$ and $\hat{W}_i$ denote the unit vectors in the direction of Z_i, ξ_i and W_i, respectively. Moreover, by (49) and (50) we have

$$(\hat{k}^i_1)^2 = |\langle Z_i, Z_i\rangle| = |(\lambda_i - \lambda_j)(\kappa\lambda_i - k_1)|\,,$$

whereas from (43) and (53) we obtain, respectively, that

$$(\hat{k}^i_2)^2 = \frac{\kappa^2(\lambda_j - \lambda_i)^2\langle\xi_i, \xi_i\rangle}{|\langle Z_i, Z_i\rangle|} = \frac{\kappa^2|\lambda_j - \lambda_i|\lambda_i(1 - \lambda_i)}{|\kappa\lambda_i - k_1|}$$

and

$$(\hat{k}^i_3)^2 = \frac{k_1^2k_2^2(\lambda_i - \lambda_j)^2\langle\xi_i, \xi_i\rangle}{(\kappa\lambda_i - k_1)^2|\langle W_i, W_i\rangle|} = \frac{k_1k_2|\lambda_i - \lambda_j|}{|\kappa\lambda_i - k_1|}, \quad 1 \le j \ne i \le 2\,.$$

If $\kappa\lambda_i - k_1 = 0$, it follows from (49), (51), (52) and (54) that γ_i is a unit-speed space like curve in V_i with light-like curvature vector, constant Frenet curvatures $\tilde{k}^i_\ell$, $1 \le \ell \le 2$, and Frenet frame $\{F_*X_i, Z_i, h_*\hat{\xi}_i, \zeta_i\}$, where $\hat{\xi}_i$ is the unit vector in the direction of ξ_i. Moreover, from (51) we obtain that

$$(\tilde{k}^i_1)^2 = \frac{k_1k_2(\kappa\lambda_j - k_1)^2}{\kappa^2}\,,$$

whereas from (54) it follows that

$$(\tilde{k}^i_2)^2 = \langle\xi_i, \xi_i\rangle\frac{(k_1^2 - k_2^2)^2}{4k_1^2k_2^2} = \frac{(k_1 - k_2)^2}{4k_1k_2}\,.$$

Comparying with (19), (20) and (21) in the first case, and with (22) and (23) in the second, we see that γ_1 and γ_2 are precisely, up to congruence, the curves given in Example 3.

Now observe that

$$\tilde{\pi}_2 F_* \xi_i = h_* \pi_2 f_* \xi_i = h_*(f_* \xi_i + S X_i) = \lambda_i F_* X_i + h_* \xi_i ,$$

whereas

$$\tilde{\pi}_2 h_* \xi_i = h_* \pi_2 \xi_i = h_*(f_* S^t \xi_i + T \xi_i) = (1 - \lambda_i)(\lambda_i F_* X_i + h_* \xi_i) ,$$

where we have used that

$$S^t \xi_i = S^t S X_i = R(I - R) X_i = \lambda_i (1 - \lambda_i) X_i$$

and

$$T \xi_i = T S X_i = S(I - R) X_i = (1 - \lambda_i) S X_i = (1 - \lambda_i) \xi_i .$$

On the other hand,

$$\tilde{\pi}_2 Z_i = h_* \pi_2 H + \frac{\lambda_i}{r_2} \pi_2 \nu_2 = \rho_4 h_* \xi_4 + \frac{\lambda_i}{r_2} \nu_2 .$$

Since

$$\tilde{\pi}_2 h_* \xi_4 = h_* \pi_2 \xi_4 = h_*(f_* S^t \xi_4 + T \xi_4) = h_* \xi_4 ,$$

we obtain that

$$\tilde{\pi}_2 \Big(\rho_4 h_* \xi_4 + \frac{\lambda_i}{r_2} \nu_2 \Big) = \rho_4 h_* \xi_4 + \frac{\lambda_i}{r_2} \nu_2 .$$

If $\langle Z_i, Z_i \rangle \neq 0$, it follows that $\tilde{\pi}_2 W_i$ and $\tilde{\pi}_2 Z_i$ are colinear. Similarly, $\tilde{\pi}_2 \zeta_i$ and $\tilde{\pi}_2 Z_i$ are colinear if $\langle Z_i, Z_i \rangle = 0$. It follows that $\tilde{\pi}_2(V_i)$ is spanned by

$$\lambda_i F_* X_i + h_* \xi_i \quad \text{and} \quad \rho_4 h_* \xi_4 + \frac{\lambda_i}{r_2} \nu_2 .$$

Therefore, the subspaces $\tilde{\pi}_2(V_1)$ and $\tilde{\pi}_2(V_2)$ (and hence also $\tilde{\pi}_1(V_1)$ and $\tilde{\pi}_1(V_2)$) are mutually orthogonal, thus the first (respectively, second) factor $\mathbb{R}^4_1$ in the decomposition $\mathbb{R}^4_1 \times \mathbb{R}^4_1$ adapted to the product $\mathbb{H}^3_{k_1} \times \mathbb{H}^3_{k_2}$ splits orthogonally as $\mathbb{R}^4_1 = \tilde{\pi}_1(V_1) \oplus \tilde{\pi}_1(V_2)$ (respectively, $\mathbb{R}^4_1 = \tilde{\pi}_2(V_1) \oplus \tilde{\pi}_2(V_2)$). We conclude that g is (the restriction to W of) an isometric immersion as in Example 3, and the conclusion follows as in the preceding case. □

5. The main result.

We are now in a position to state and prove our main result.

THEOREM 5. *Let $f \colon M^2 \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$, $k_1 + k_2 \neq 0$, be an umbilical non totally geodesic isometric immersion. Then one of the following possibilities holds:*

(i) *f is an umbilical isometric immersion into a slice of $\mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$;*

(ii) *there exist umbilical isometric immersions $f_i \colon M^2 \to \mathbb{Q}^{n_i}_{\tilde{k}_i}$, $1 \leq i \leq 2$, with $\tilde{k}_1 = k_1 \cos^2\theta$ and $\tilde{k}_2 = k_2 \sin^2\theta$ for some $\theta \in (0, \pi/2)$, such that $f = (\cos\theta f_1, \sin\theta f_2)$;*

(iii) *after interchanging the factors, if necessary, we have* $k_2 = 0$, $n_1 \geq 3$, $n_2 \geq 2$ *and* $f = j \circ \tilde{f}$, *where* $j \colon \mathbb{Q}^3_{k_1} \times \mathbb{R}^2 \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{R}^{n_2}$ *and* $\tilde{f} \colon M^2 \to \mathbb{Q}^3_{k_1} \times \mathbb{R}^2$ *are isometric immersions such that* j *is totally geodesic and* $\tilde{f}(M^2)$ *is an open subset of a surface as in Example* 2*;*

(iv) $k_i < 0$ *and* $n_i \geq 3$, $1 \leq i \leq 2$, *and* $f = j \circ \tilde{f}$, *where* $j \colon \mathbb{Q}^3_{k_1} \times \mathbb{Q}^3_{k_2} \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ *and* $\tilde{f} \colon M^2 \to \mathbb{Q}^3_{k_1} \times \mathbb{Q}^3_{k_2}$ *are isometric immersions such that* j *is totally geodesic and* $\tilde{f}(M^2)$ *is an open subset of a surface as in Example* 3*;*

(v) *after possibly reordering the factors, we have* $k_1 > 0$ (*respectively,* $k_1 \leq 0$) *and* $f \circ \tilde{\Pi} = j \circ \Pi \circ \tilde{f}$ (*respectively,* $f = j \circ \Pi \circ \tilde{f}$), *where* $\tilde{\Pi} \colon \tilde{M}^2 \to M^2$ *is the universal covering of* M^2, $\tilde{f} \colon \tilde{M}^2 \to \mathbb{R} \times \mathbb{Q}^{2+\delta}_{k_2}$ (*respectively,* $\tilde{f} \colon M^2 \to \mathbb{R} \times \mathbb{Q}^{2+\delta}_{k_2}$) *is an umbilical isometric immersion with* $\delta \in \{0, 1\}$, $j \colon \mathbb{Q}^1_{k_1} \times \mathbb{Q}^{2+\delta}_{k_2} \to \mathbb{Q}^{n_1}_{k_1} \times \mathbb{Q}^{n_2}_{k_2}$ *is totally geodesic and* $\Pi \colon \mathbb{R} \times \mathbb{Q}^{2+\delta}_{k_2} \to \mathbb{Q}^1_{k_1} \times \mathbb{Q}^{2+\delta}_{k_2}$ *is a locally isometric covering map* (*respectively, isometry*).

PROOF. If S vanishes everywhere on M^2, then f is as in (i) by Lemma 8.1 in [3]. If $\ker S = \{0\}$ at some point $x \in M^2$, then f is as in (ii), (iii) or (iv) by Lemma 4. Then, we are left with the case in which there is an open subset $\mathcal{U} \subset M^2$ where $\ker S$ has rank one. In this case, the argument in the proof of Theorem 1.4 of [3] applies and shows that f is as in (v). □

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