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ISOMETRIC DEFORMATIONS OF CUSPIDAL EDGES

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Abstract. Along cuspidal edge singularities on a given surface in Euclidean 3-space $\boldsymbol{R}^3$, which can be parametrized by a regular space curve $\hat{\gamma}(t)$, a unit normal vector field ν is well-defined as a smooth vector field of the surface. A cuspidal edge singular point is called *generic* if the osculating plane of $\hat{\gamma}(t)$ is not orthogonal to ν. This genericity is equivalent to the condition that its limiting normal curvature κ_ν takes a non-zero value. In this paper, we show that a given generic (real analytic) cuspidal edge f can be isometrically deformed preserving κ_ν into a cuspidal edge whose singular set lies in a plane. Such a limiting cuspidal edge is uniquely determined from the initial germ of the cuspidal edge.

Introduction. Let Σ^2 be a 2-manifold. A singular point $p \in \Sigma^2$ of a C^∞-map germ $f : (\Sigma^2, p) \to \boldsymbol{R}^3$ is a *cuspidal edge* if f is right-left equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at the origin. Recently, the differential geometry of co-rank one singularities (including cuspidal edges) on surfaces was discussed by several geometers ([1, 3, 4, 5, 9, 13, 15, 16]). In particular, in [5], isometric deformations of a special class of cross caps were discussed. Relating to this, Martins-Saji [10] defined several differential geometric invariants on cuspidal edges, and gave geometric meanings for them. Moreover, it was shown in [11] that the limiting normal curvature κ_ν defined in [15] (cf. (1.1)) is closely related to the behavior of Gauss maps around cuspidal edges.

On the other hand, the proof of the classical Janet-Cartan theorem on the local existence of isometric embeddings of real analytic Riemannian n-manifolds into the Euclidean space $\boldsymbol{R}^{n(n+1)/2}$ yields that any generic regular surface in $\boldsymbol{R}^3$ has a non-trivial family of isometric deformations. So it is natural to expect the existence of such non-trivial isometric deformations for surfaces with singularities. As shown in [5] and [11], certain classes of ruled cross caps and cuspidal edges admit non-trivial isometric deformations, respectively. However, general cases have not been discussed yet.

Along cuspidal edge singularities of a given C^∞-map germ[1] $f : (\Sigma^2, p) \to \boldsymbol{R}^3$, the unit normal vector field ν is well-defined as a smooth vector field of the surface. Let $\gamma(t)$ be a regular curve in Σ^2 satisfying $\gamma(0) = p$ as a parametrization of cuspidal edge singularities of

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[1] Though the definitions here are for C^∞-maps, we consider only real analytic map germs in this paper since we shall apply the Cauchy-Kowalevski theorem.

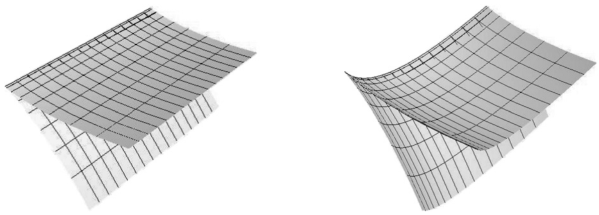

FIGURE 1. Cuspidal edges with $\kappa_\nu = 0$ (left) and $\kappa_\nu \neq 0$ (right).

the map f. We call $\gamma(t)$ the *singular curve* of f. We set

$$\hat{\gamma}(t) := f \circ \gamma(t),$$

which is a regular space curve. Let $\kappa_s(t)$ be the singular curvature function along the curve $\gamma(t)$ (cf. [15, (1.7)]), and $\kappa_\nu(t)$ the limiting normal curvature along $\gamma(t)$ (cf. [15, (3.11)] and (1.1)). Then the curvature function of $\hat{\gamma}(t)$ as a space curve is given by (cf. [11])

$$\kappa(t) = \sqrt{\kappa_s(t)^2 + \kappa_\nu(t)^2}. \tag{0.1}$$

In [15], [10] and [11], the singular curvature function $\kappa_s(t)$ and the limiting normal curvature function $\kappa_\nu(t)$ are considered as geometric invariants along cuspidal edge singularities, as well as the curvature function $\kappa(t)$ and the torsion function $\tau(t)$ of $\hat{\gamma}(t)$. The relationships amongst κ_s, κ_ν and τ are given in [10].

An invariant I of map germs at $p \in \Sigma^2$ is called *intrinsic* if it is determined only by the first fundamental form (cf. [6]). The singular curvature κ_s is a typical intrinsic invariant of a cuspidal edge singularity (cf. [15, 6]). In [11], the *cuspidal curvature* κ_c at a given cuspidal edge singular point p is defined. Let Π be the plane in $\boldsymbol{R}^3$ passing through $f(p)$ perpendicular to the vector $d\hat{\gamma}(0)/dt$. Then the intersection of the image of the singular set of f by Π gives a $3/2$-cusp in the plane Π. The value $\kappa_c(p)$ coincides with the cuspidal curvature of this $3/2$-cusp (cf. [11]). The following assertion holds:

FACT 1 ([11]). *The value $|\kappa_c\kappa_\nu|$ is an intrinsic invariant.*

To prove our main result (cf. Theorem A), this fact plays a crucial role. A cuspidal edge singular point p is called *generic* if the osculating plane of $\hat{\gamma}$ is not orthogonal to ν at p, that is, the limiting tangent plane does not coincide with the osculating plane of $\hat{\gamma}$.

The Gaussian curvature at a generic cuspidal edge is unbounded (cf. [11]). Moreover, as shown in [11, Theorem A], the following four conditions are equivalent:

(a) A cuspidal edge singular point is generic (cf. Figure 1).
(b) The limiting normal curvature κ_ν does not vanish at the singular point.
(c) The inequality $\kappa > |\kappa_s|$ holds (cf. (0.1)).
(d) Let K be the Gaussian curvature and $d\hat{A} = \det(f_u, f_v, \nu)\, du \wedge dv$ the signed area element of f, where $(U; u, v)$ is a local coordinate system near the singular point p. Then $K\, d\hat{A}$ is well-defined on U and does not vanish at p.

We denote by $\mathcal{C}$ the set of *real analytic* map germs of cuspidal edges

$$f\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$$

which is defined on a neighborhood of the origin in $\boldsymbol{R}^2$. Moreover, let $\mathcal{C}^*(\subset \mathcal{C})$ be the set of map germs of generic cuspidal edges. In this paper, we show the following:

THEOREM A. *Let $\kappa_s(t)$ be the singular curvature function along the singular curve $\gamma(t)$ of $f \in \mathcal{C}^*$ such that $\gamma(0) = 0$. Let $\sigma(t)$ be a real analytic regular space curve whose curvature function $\tilde{\kappa}(t)$ satisfies*

$$\tilde{\kappa}(t) > |\kappa_s(t)| \tag{0.2}$$

for all sufficiently small t. Then there exist at most two map germs $g \in \mathcal{C}^*$ such that

(1) *the first fundamental form of g coincides with that of f, in particular, the singular curve $\gamma(t)$ in the domain of f is the same as that of g,*
(2) *$g(\gamma(t)) = \sigma(t)$ holds for each t.*

It is classically known that for a given planar curve $\gamma(t)$ having curvature function $\kappa(t)$, there are at most two developable surfaces having "origami-singularities" corresponding to γ as a space curve whose curvature function $\tilde{\kappa}(t)$ satisfies $\tilde{\kappa}(t) > \kappa(t)$ (see [2]). The above theorem can be considered as an analogue of this classical phenomenon.

Kossowski [7] is the first geometer who considered the realizing problem of given first fundamental forms as generic wave fronts. However, in [7], the isometric deformations of singularities were not discussed, and the above theorem can be considered as a refinement of [7, Theorem 1] in the case of cuspidal edge singularities. Since the intrinsic formulation of wave fronts are rather technical, the statement of Theorem A as a refinement of Kossowski's realization theorem will be given and proved in the final section (Section 3). Since Kossowski applied the Cauchy-Kowalevski theorem to construct suitable second fundamental forms, our approach is completely different from his, and can be beneficial for the applications to isometric deformations. However, it should be also remarked that Kossowski's appoach will work for not only cuspidal edges but also other wave front singularities, (for example, swallowtail singular points).

We get the following consequences of Theorem A:

COROLLARY B. *Each map germ $f \in \mathcal{C}^*$ admits an isometric deformation (in $\mathcal{C}^*$) which moves the limiting normal curvature κ_ν. In particular, κ_ν and κ_c are not intrinsic invariants*[2].

A given map germ $f \in \mathcal{C}$ is said to be *planar* (resp. *non-planar*) if the image $\hat{\gamma}$ of the singular curve of f is contained in a plane (resp. the torsion of $\hat{\gamma}$ does not equal to zero). We show the following normalization theorem of generic cuspidal edges:

[2]In [11], it has been shown that κ_c is an extrinsic invariant, by construction the isometric deformation of ruled cuspidal edges satisfying $\kappa_\nu = 0$.

COROLLARY C. *For each $f \in \mathcal{C}^*$, there exists a unique map germ $g \in \mathcal{C}^*$ of planar cuspidal edge singularities up to congruence such that*

- *f and g induce the same first fundamental form, and*
- *the curvature function of $\hat{\gamma}(t) := f \circ \gamma(t)$ coincides with that of $g \circ \gamma(t)$, where $\gamma(t)$ is the singular curve of f.*

Moreover, there exists a real analytic isometric deformation of f into g, preserving the curvature function along the image of the cuspidal edge singularities.

In the statement of Corollary C, the condition $f \in \mathcal{C}^*$ cannot be weakened to $f \in \mathcal{C}$ (cf. Remark 8 and Proposition 9). As a consequence, we also cannot weaken the condition (0.2) to $\tilde{\kappa}(t) \geq |\kappa_s(t)|$ in the statement of Theorem A.

Two map germs $f, g \in \mathcal{C}$ are said to be *congruent* if there exist an (orientation preserving or reversing) local isometry $T : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ and a local analytic diffeomorphism $\varphi : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ such that $T \circ f \circ \varphi = g$. On the other hand, two map germs $f, g \in \mathcal{C}$ are said to be *strongly isometric* if there exist an isometry $T : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ and a local analytic diffeomorphism $\varphi : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ satisfying the following properties:

- $f \circ \varphi$ and g induce the same first fundamental form, and
- the restriction of $T \circ f \circ \varphi$ to the singular curve of f coincides with that of g.

A regular space curve σ passing through a point $x_0 \in \boldsymbol{R}^3$ is called *symmetric* if there exists an isometry T of $\boldsymbol{R}^3$ which is not the identity map such that $T(x_0) = x_0$ and the image of σ is invariant under the action of T. For example, the image of the singular curve of a generic planar cuspidal edge is symmetric. We get the following duality theorem for generic cuspidal edges:

COROLLARY D. *There exists an involution $\mathcal{C}^* \ni f \mapsto \check{f} \in \mathcal{C}^*$ such that*

(1) *$\check{f}$ is strongly isometric to f,*

(2) *if $g \in \mathcal{C}^*$ is strongly isometric to f, then g is congruent to f or $\check{f}$.*

Moreover, f is not congruent to $\check{f}$ if the image of the singular set of f is non-symmetric and non-planar.

We call $\check{f}$ the *isomer* of f. For non-generic cuspidal edges, the existence of isomers cannot be expected in general (see Proposition 10). The proofs of these results are accomplished by an appropriate modification of the proof of the 2-dimensional case of the Janet-Cartan theorem. In [5], we defined 'normal cross caps', expecting a similar normalization theorem as in Corollary C. However, it seems difficult to apply the same technique, because cross cap singularities are isolated.

1. Preliminaries. The fundamental tool to prove our results is the following fact (cf. [14, pages 37–38]):

FACT 2 (Cauchy-Kowalevski theorem). *Let*

$$x^i_v(u, v) = \Phi^i(u, v, x^1, \ldots, x^k, x^1_u, \ldots, x^k_u) \quad (i = 1, \ldots, k)$$

be a partial differential equation having $x^i := x^i(u,v)$ $(i = 1, 2, \dots, k)$ *as unknown functions, where* $\Phi := (\Phi^1, \dots, \Phi^k)$ *is a real analytic map and*

$$x^i_u := \frac{\partial x^i}{\partial u}\,, \quad x^i_v := \frac{\partial x^i}{\partial v} \quad (i = 1, \dots, k)\,.$$

This equation has a unique real analytic solution $x = (x^1, \dots, x^k)$ *with an initial condition*

$$x^i(u, 0) = w^i(u) \quad (i = 1, \dots, k)\,,$$

where w^i $(i = 1, \dots, k)$ *are given real analytic functions.*

The classical Janet-Cartan theorem for the existence of local isometric embeddings of real analytic Riemannian 2-manifolds can be proved as an application of this fact (cf. Chapter 11 of [14]). Our main results are also proved by applying Fact 2 using the following special coordinate system along cuspidal edges:

DEFINITION 3. Let $f : \Sigma^2 \to \boldsymbol{R}^3$ be a real analytic map and $p \in \Sigma^2$ be a cuspidal edge singular point of f. (We can take a real analytic unit normal vector field ν defined on a neighborhood of p.) A real analytic local coordinate system (u, v) at p is called *adapted* if it satisfies the following properties along the u-axis:

(1) $|f_u| = 1$,
(2) $f_v = 0$, in particular, the singular set is contained in the u-axis,
(3) $\{f_u, f_{vv}, \nu\}$ is an orthonormal basis which is compatible with respect to the orientation of $\boldsymbol{R}^3$.

The existence of an adapted coordinate system was shown in [15, Lemma 3.2]. Throughout this paper, we fix a real analytic map

$$f : (U; u, v) \longrightarrow \boldsymbol{R}^3$$

defined on a domain $U \subset \boldsymbol{R}^2$ of the uv-plane which has a generic cuspidal edge singular point at the origin $(0, 0)$, and assume that (u, v) is an adapted coordinate system. Since (u, v) is an adapted coordinate system, the *limiting normal curvature* κ_ν of f is given by (cf. Equation (3.11) in [15])

$$\kappa_\nu(u) := f_{uu}(u, 0) \cdot \nu(u, 0) = \det(f_{uu}(u, 0), f_u(u, 0), f_{vv}(u, 0))\,, \tag{1.1}$$

where the dot '$\cdot$' is the inner product in $\boldsymbol{R}^3$. Since $f_v = 0$ along the u-axis, there exists a real analytic map germ φ such that

$$f_v(u, v) = v\varphi(u, v)\,, \quad f_{vv}(u, 0) = \varphi(u, 0) \tag{1.2}$$

hold on a neighborhood of the u-axis.

On the other hand, let $g : (U; u, v) \to \boldsymbol{R}^3$ be another real analytic map which has a generic cuspidal edge singular point at $(0, 0)$, and (u, v) an adapted coordinate system. Similarly, there exists a real analytic map germ ψ such that

$$g_v(u, v) = v\psi(u, v)\,, \quad g_{vv}(u, 0) = \psi(u, 0) \tag{1.3}$$

hold on a neighborhood of the u-axis.

PROPOSITION 4. *Let f and g be as above, and φ and ψ as in* (1.2) *and* (1.3). *Suppose that the first fundamental form of g coincides with that of f, then there exists a real analytic map $\mathcal{F} : U \times \boldsymbol{R}^3 \times \mathrm{GL}_3(\boldsymbol{R}) \to \mathrm{M}_3(\boldsymbol{R})$ such that*

$$(g_v, r_v, \psi_v) = \mathcal{F}(u, v;\ \psi_u, (\psi, g_u, r_u)),$$

where $r(u,v) := g_u(u,v)$ and $\mathrm{M}_3(\boldsymbol{R})$ (*resp.* $\mathrm{GL}_3(\boldsymbol{R})$) *is the set of 3×3-matrices* (*resp. the set of regular 3×3-matrices*).

PROOF. Since f and g have the same first fundamental form on the same local coordinate system, we have

$$f_u \cdot f_u = g_u \cdot g_u, \quad f_u \cdot f_v = g_u \cdot g_v, \quad f_v \cdot f_v = g_v \cdot g_v, \tag{1.4}$$

that reduce to

$$f_u \cdot f_u = g_u \cdot g_u, \quad f_u \cdot \varphi = g_u \cdot \psi, \quad \varphi \cdot \varphi = \psi \cdot \psi. \tag{1.5}$$

We define $\mathcal{F} := (\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3)$ by

$$\begin{aligned}
&\mathcal{F}^1(u, v;\ x, (y_1, y_2, y_3)) := v y_1, \\
&\mathcal{F}^2(u, v;\ x, (y_1, y_2, y_3)) := v x, \\
&\mathcal{F}^3(u, v;\ x, (y_1, y_2, y_3)) \\
&\qquad := ((y_1, y_2, y_3)^T)^{-1} \begin{pmatrix} \varphi_v \cdot \varphi \\ \varphi_v \cdot f_u \\ (\varphi \cdot f_{uu})_v - \dfrac{v}{2}(\varphi \cdot \varphi)_{uu} + v(x \cdot x) \end{pmatrix},
\end{aligned} \tag{1.6}$$

where $x \in \boldsymbol{R}^3$, $(y_1, y_2, y_3) \in \mathrm{GL}_3(\boldsymbol{R})$ and $(y_1, y_2, y_3)^T$ is the transpose of the regular matrix (y_1, y_2, y_3). Since

$$g_v = v\psi, \quad r_v = g_{uv} = (v\psi)_u = v\psi_u,$$

it holds that

$$\mathcal{F}^1(u, v;\ \psi_u, (\psi, g_u, r_u)) = v\psi = g_v, \quad \mathcal{F}^2(u, v;\ \psi_u, (\psi, g_u, r_u)) = v\psi_u = r_v.$$

The relation (cf. (1.2) and (1.3))

$$v(g_u \cdot \psi) = g_u \cdot g_v = f_u \cdot f_v = v(f_u \cdot \varphi)$$

reduces to

$$g_u \cdot \psi = f_u \cdot \varphi. \tag{1.7}$$

Since (cf. (1.2) and (1.5))

$$\begin{aligned}
v(\psi \cdot g_{uu}) &= (g_v \cdot g_{uu}) = (g_v \cdot g_u)_u - g_{uv} \cdot g_u \\
&= (g_v \cdot g_u)_u - \frac{1}{2}(g_u \cdot g_u)_v = (f_v \cdot f_u)_u - \frac{1}{2}(f_u \cdot f_u)_v = v(\varphi \cdot f_{uu}),
\end{aligned}$$

it holds that

$$g_{uu} \cdot \psi = f_{uu} \cdot \varphi. \tag{1.8}$$

On the other hand, by (1.7) and (1.5), we have that

$$
\begin{aligned}
(1.9)\qquad \psi_v \cdot g_u &= (\psi \cdot g_u)_v - (\psi \cdot g_{uv}) = (\varphi \cdot f_u)_v - \frac{1}{v} g_v \cdot g_{uv} \\
&= (\varphi \cdot f_u)_v - \frac{1}{2v}(g_v \cdot g_v)_u = (\varphi \cdot f_u)_v - \frac{1}{2v}(f_v \cdot f_v)_u \\
&= (\varphi \cdot f_u)_v - \frac{1}{v} f_v \cdot f_{uv} = \varphi_v \cdot f_u .
\end{aligned}
$$

Since

$$
\begin{aligned}
v(\psi \cdot g_{uuv}) = g_v \cdot g_{uuv} &= (g_v \cdot g_{uv})_u - g_{uv} \cdot g_{uv} = \frac{(g_v \cdot g_v)_{uu}}{2} - g_{uv} \cdot g_{uv} \\
&= \frac{(f_v \cdot f_v)_{uu}}{2} - g_{uv} \cdot g_{uv} = v^2 \frac{(\varphi \cdot \varphi)_{uu}}{2} - v^2 \psi_u \cdot \psi_u ,
\end{aligned}
$$

we have

$$
(1.10)\qquad \psi \cdot g_{uuv} = \frac{1}{2} v(\varphi \cdot \varphi)_{uu} - v(\psi_u \cdot \psi_u) .
$$

Here, (1.8) yields

$$
(1.11)\qquad \psi \cdot g_{uuv} = (\psi \cdot g_{uu})_v - \psi_v \cdot g_{uu} = (\psi \cdot f_{uu})_v - \psi_v \cdot r_u .
$$

By (1.10) and (1.11), we have

$$
(1.12)\qquad \psi_v \cdot r_u = (\varphi \cdot f_{uu})_v - \frac{v}{2}(\varphi \cdot \varphi)_{uu} + v(\psi_u \cdot \psi_u) .
$$

The space curve $\sigma(u) := g(u, 0)$ parametrizes the image of the singular set of g. Since the cuspidal edge singularities of g are generic, the osculating plane Π of the space curve $\sigma(u)$ is independent of the tangential direction $\psi(u, 0) = g_{vv}(u, 0)$ of g (cf. (1.3), Definition 3). Since Π is spanned by $\{g_u(u, 0), g_{uu}(u, 0)\}$, the matrix

$$
(\psi(u, 0), g_u(u, 0), r_u(u, 0)) = (g_{vv}(u, 0), g_u(u, 0), g_{uu}(u, 0))
$$

is regular. Hence the fact $\psi_v \cdot \psi = \varphi_v \cdot \varphi$, (1.9) and (1.12) yield that

$$
\mathcal{F}^3(u, v;\ \psi_u, (\psi, g_u, r_u)) = \psi_v ,
$$

in particular, $\mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3)$ attains the desired real analytic map. □

2. Proof of the main results. Let $f : (U; u, v) \to \boldsymbol{R}^3$ be as in the previous section. Then the space curve defined by

$$
\hat{\gamma}(t) := f(t, 0)
$$

gives a parametrization of the image of the singular curve $\gamma(t) = (t, 0)$ of f. To prove Theorem A in the introduction, we prepare the following assertion:

PROPOSITION 5. *Let $\sigma(t)$ be a regular space curve satisfying* (0.2) *such that t is the arclength parameter. Then there exists a unique $\boldsymbol{R}^3$-valued vector field $X^+_\sigma(t)$* (*resp.* $X^-_\sigma(t)$) *along σ satisfying the following properties:*

(1) $|X^+_\sigma(t)| = 1$ (*resp.* $|X^-_\sigma(t)| = 1$),
(2) $X^+_\sigma(t) \cdot \dot{\sigma}(t) = 0$ (*resp.* $X^-_\sigma(t) \cdot \dot{\sigma}(t) = 0$),

(3) $X^+_\sigma(t) \cdot \ddot{\sigma}(t) = \varphi(t, 0) \cdot \ddot{\gamma}(t)$ (*resp.* $X^-_\sigma(t) \cdot \ddot{\sigma}(t) = \varphi(t, 0) \cdot \ddot{\gamma}(t)$), *where* φ *is given in* (1.2),

(4) $\det(\dot{\sigma}(t), X^+_\sigma(t), \ddot{\sigma}(t)) > 0$ (*resp.* $\det(\dot{\sigma}(t), X^-_\sigma(t), \ddot{\sigma}(t)) < 0$).

PROOF. Since f has non-zero limiting normal curvature, the curvature function $\kappa(t)$ of the space curve $\hat{\gamma}(t) = f(t, 0)$ is positive (cf. (0.1)). Since $\varphi(t, 0) = f_{vv}(t, 0)$ is a unit vector field (cf. Definition 3) orthogonal to $\nu(t, 0)$ and $\dot{\hat{\gamma}}(t)$, we have that

$$\left|\varphi(t, 0) \cdot \ddot{\hat{\gamma}}(t)\right| = |\kappa_s(t)| \ .$$

Let $\tilde{\kappa}(t)$ be the curvature function of $\sigma(t)$. If we set

$$c(t) := \varphi(t, 0) \cdot \frac{\ddot{\hat{\gamma}}(t)}{\tilde{\kappa}(t)}\,, \tag{2.1}$$

the inequality (0.2) yields

$$|c(t)| = \frac{|\kappa_s(t)|}{\tilde{\kappa}(t)} < 1 \,.$$

On the other hand, since t is the arclength parameter of σ,

$$n(t) := \frac{\ddot{\sigma}(t)}{|\ddot{\sigma}(t)|} = \frac{\ddot{\sigma}(t)}{\tilde{\kappa}(t)}$$

gives the principal unit normal vector field of the space curve $\sigma(t)$. Then applying the lemma in the appendix for $a = n(t)$, $b = \dot{\sigma}(t)$ and $\mu = c(t)$ as in (2.1), we can verify that $X^+_\sigma(t) := w$ (resp. $X^-_\sigma(t) := w$) satisfies (1)–(4). The uniqueness of $X^+_\sigma(t)$ and $X^-_\sigma(t)$ also follows from the lemma in the appendix. □

PROOF OF THEOREM A. We consider the partial differential equation (cf. Proposition 4)

$$(g_v, r_v, \psi_v) = \mathcal{F}(u, v;\ \psi_u, (\psi, g_u, r_u)) \tag{2.2}$$

for $\mathcal{F}$ as in (1.6) with the following initial conditions:

$$\begin{aligned} &g(u, 0) = \sigma(u)\,, \quad r(u, 0) = \dot{\sigma}(u)\,, \\ &\psi(u, 0) = X^+_\sigma(t) \quad (\text{resp. } \psi(u, 0) = X^-_\sigma(t))\,. \end{aligned} \tag{2.3}$$

By Fact 2, there exists a real analytic map

$$g^+ : U \to \boldsymbol{R}^3 \quad (\text{resp. } g^- : U \to \boldsymbol{R}^3)$$

on a sufficiently small neighborhood U of the origin satisfying (2.2) and (2.3). Since (2.2) is equivalent to the conditions

$$g^+_v = v\psi\,, \quad (\text{resp. } g^-_v = v\psi)\,, \tag{2.4}$$

$$r_v = v\psi_u\,, \tag{2.5}$$

$$\psi_v \cdot \psi = \varphi_v \cdot \varphi\,, \tag{2.6}$$

$$\psi_v \cdot g^+_u = \varphi_v \cdot f_u \quad (\text{resp. } \psi_v \cdot g^-_u = \varphi_v \cdot f_u)\,, \tag{2.7}$$

$$\psi_v \cdot r_u = (\varphi \cdot f_{uu})_v - \frac{v}{2}(\varphi \cdot \varphi)_{uu} + v(\psi_u \cdot \psi_u)\,, \tag{2.8}$$

one can deduce the relation (1.5) (and also (1.4) as a consequence) directly from the relations (2.3) and (2.4)–(2.8). Consequently, the map g^+ (resp. g^-) has the same first fundamental form as f and satisfies the properties (2) and (3) of Theorem A. The ambiguity of the construction of g as in the statement of Theorem A depends on the choice of the vector field $X := \psi(u,0)$ along σ satisfying (1), (2) and (3) in Proposition 5. By the lemma in the appendix, X coincides with either X^+_σ or X^-_σ, which yields at most two possibilities for g. The proof of Theorem A is now reduced to the following Proposition 6. □

PROPOSITION 6. *The real analytic map* $(g :=)g^\pm$ *has generic cuspidal edge singularities along the* u*-axis.*

PROOF. Since (u,v) is an adapted coordinate system of g, the vector field ψ in (1.3) is perpendicular to g_u on the singular set. Moreover, by the definition (1.3) of ψ,

$$\tilde{\nu}(u,v) := \frac{g_u(u,v)\times\psi(u,v)}{|g_u(u,v)\times\psi(u,v)|}$$

is a unit normal vector field to g which is well-defined on the singular set, where $\times$ denotes the vector product in $\boldsymbol{R}^3$. Moreover, the function

$$\lambda := \det(g_u, g_v, \tilde{\nu}) = v\det(g_u, \psi, \tilde{\nu})$$

satisfies $\lambda_v \neq 0$ on the u-axis, because (u,v) is an adapted coordinate system. Hence the singular points are *non-degenerate* (cf. [8, Proposition 2.3] or [15, Definition 1.1]). Moreover, since $g_v = v\psi = 0$ on the u-axis, the *null direction* at a point on the u-axis (cf. [8, Page 306] or [15, Page 495]) is $\partial/\partial v$, which is linearly independent to the *singular direction* $\partial/\partial u$. Thus, to show that g is a cuspidal edge, it is sufficient to show that g is a front (cf. [8] or [15]), which is equivalent to $\tilde{\nu}_v \neq 0$ on the u-axis: Let κ_c and κ_ν be the cuspidal curvature (cf. [11, (2.4)]) and the limiting normal curvature of f along γ, respectively. Since the singularity of f consists of cuspidal edges, $\kappa_c \neq 0$ holds (cf. [11, Lemma 2.8]), and since f is generic (i.e. the singularities of f cossists only of generic cuspidal edges), $\kappa_\nu \neq 0$. By Fact 1, $|\kappa_c\kappa_\nu|$ depends only on the first fundamental from. Thus, we have

$$\tilde{\kappa}_c(t)\tilde{\kappa}_\nu(t) = \kappa_c(t)\kappa_\nu(t) \neq 0\,, \tag{2.9}$$

where $\tilde{\kappa}_c$ and $\tilde{\kappa}_\nu$ are the cuspidal curvature and the limiting normal curvature of g, respectively. Then by [11, (2.4)],

$$\tilde{\kappa}_c(t) = \det(g_u, g_{vv}, g_{vvv})|_{(u,v)=(t,0)} = 2\det(g_u(t,0), \psi(t,0), \psi_v(t,0)) \neq 0. \tag{2.10}$$

So it holds on the u-axis that

$$\begin{aligned}
\tilde{\nu}_v \cdot \psi &= \left(\frac{g_u\times\psi}{|g_u\times\psi|}\right)_v \cdot \psi \\
&= \left(\frac{g_{uv}\times\psi + g_u\times\psi_v}{|g_u\times\psi|}\right)\cdot\psi + \big((g_u\times\psi)\cdot\psi\big)\left(\frac{1}{|g_u\times\psi|}\right)_v \\
&= -\frac{\det(g_u,\psi,\psi_v)}{|g_u\times\psi|} \neq 0\,.
\end{aligned}$$

Hence the singular points of g consist of cuspidal edge singularities. Moreover, by (2.10), the limiting normal curvature $\tilde{\kappa}_\nu$ does not vanish, which implies that g is generic. □

PROOF OF COROLLARY B. Since f has non-zero limiting normal curvature, the curvature function $\kappa(t)$ of the space curve $\hat{\gamma}(t) = f(t, 0)$ is greater than the absolute value $|\kappa_s(t)|$ of the singular curvature (cf. (0.1)). Let $\tau(t)$ be the torsion function of $\hat{\gamma}(t)$. For sufficiently small $\varepsilon > 0$, there exists a regular space curve $\sigma^s(t)$ ($|s| < \varepsilon$) satisfying the following properties by the fundamental theorem of space curves

- $\sigma^s(0) = 0$,
- $\sigma^0(t) = \hat{\gamma}(t)$,
- the curvature function of $\sigma^s(t)$ is equal to $\kappa(t) + s$,
- the torsion function of $\sigma^s(t)$ is equal to $\tau(t)$.

Since ε is sufficiently small, we may assume that $\kappa(t) + s > |\kappa_s(t)|$. By Theorem A, there exists $g^s \in \mathcal{C}^*$ ($|s| < \varepsilon$) such that

(1) the vector field $\psi^s(u, v)$ satisfying $g^s_v = v\psi^s$ is equal to $X^+_{\sigma^s}(u)$ along $v = 0$, where $X^+_{\sigma^s}$ is a vector field along σ^s defined in Proposition 5,
(2) the first fundamental form of g^s is equal to that of f,
(3) the singular curve $\gamma(t)$ of f is the same as that of g^s, and
(4) $g^s(\gamma(t)) = \sigma^s(t)$ holds for each t.

Since the geodesic curvature κ_s is intrinsic, (0.1) yields that $\sqrt{(\kappa(t) + s)^2 - \kappa_s(t)^2}$ is equal to the absolute value of the limiting normal curvature of g^s, which proves Corollary B. □

PROOF OF COROLLARY C. Let $\kappa(t)$ and $\tau(t)$ be the curvature function and the torsion function of the space curve $\hat{\gamma}(t)$, respectively. For each $s \in [0, 1]$, there exists a regular space curve $\sigma^s(t)$ satisfying the following properties by the fundamental theorem of space curves

- $\sigma^s(0) = 0$,
- $\sigma^0(t) = \hat{\gamma}(t)$,
- the curvature function of $\sigma^s(t)$ is equal to $\kappa(t)$,
- the torsion function of $\sigma^s(t)$ is equal to $(1 - s)\tau(t)$.

Since f is generic, $\kappa(t) > |\kappa_s(t)|$ holds. Then by Theorem A, there exists $g^{s,+} \in \mathcal{C}^*$ (resp. $g^{s,-} \in \mathcal{C}^*$) for $s \in [0, 1]$ such that

(1) the vector field $\psi^{s,+}(u, v)$ (resp. $\psi^{s,-}(u, v)$) satisfying $g^{s,+}_v = v\psi^{s,+}$ (resp. $g^{s,-}_v = v\psi^{s,-}$) is equal to $X^+_{\sigma^s}(u)$ (resp. $X^-_{\sigma^s}(u)$) along $v = 0$, where $X^\pm_{\sigma^s}$ are as in Proposition 5,
(2) the first fundamental form of $g^{s,+}$ (resp. $g^{s,-}$) is equal to that of f,
(3) the singular curve $\gamma(t)$ of f is the same as that of $g^{s,+}$ (resp. $g^{s,-}$), and
(4) $g^{s,+}(\gamma(t)) = \sigma^s(t)$ (resp. $g^{s,-}(\gamma(t)) = \sigma^s(t)$) holds for each t.

By this construction, the torsion function of the curve σ^1 vanishes identically. So we can conclude that $g^{1,+}$ (resp. $g^{1,-}$) is a germ of a planar cuspidal edge. In particular, σ^1 lies in a plane Π, and the curve is invariant under the reflection with respect to the plane Π. Let T be the reflection with respect to the plane Π. Then we have

$$T \circ \sigma^1 = \sigma^1, \quad dT(\dot{\sigma}^1) = \dot{\sigma}^1, \quad dT(\ddot{\sigma}^1) = \ddot{\sigma}^1, \quad dT(X^+_{\sigma^1}) = X^-_{\sigma^1}.$$

In fact, the lemma in the appendix implies the fourth equality. Thus, by the uniqueness of Fact 2, we have $g^{1,-} = T \circ g^{1,+}$. This implies the uniqueness of g as in Corollary C. □

PROOF OF COROLLARY D. By replacing ν with $-\nu$, we may assume that the limiting normal curvature κ_ν of f takes positive values without loss of generality. By Theorem A, there exists $g^+ \in \mathcal{C}^*$ (resp. $g^- \in \mathcal{C}^*$) for $s \in [0, 1]$ such that

(1) the vector field $\psi^+(u, v)$ (resp. $\psi^-(u, v)$) satisfying $g^+_v = v\psi^+$ (resp. $g^-_v = v\psi^-$) is equal to $X^+_{\hat{\gamma}}(u)$ (resp. $X^-_{\hat{\gamma}}(u)$) along $v = 0$, where $X^\pm_{\hat{\gamma}}$ are as in Proposition 5,
(2) the first fundamental form of g^+ (resp. g^-) is equal to that of f,
(3) the singular curve $\gamma(t)$ of f is the same as that of g^+ (resp. g^-), and
(4) $g^+(\gamma(t)) = \hat{\gamma}(t)$ (resp. $g^-(\gamma(t)) = \hat{\gamma}(t)$) holds for each t.

Since (u, v) is an adapted coordinate system of f, it holds that

$$|\varphi(t, 0)| = |f_{vv}(t, 0)| = 1, \qquad \varphi(t, 0) \cdot \dot{\hat{\gamma}}(t) = f_{vv}(t, 0) \cdot f_u(t, 0) = 0. \tag{2.11}$$

Since $\ddot{\hat{\gamma}} \cdot \nu = \kappa_\nu > 0$ and $\nu(t, 0) = f_u(t, 0) \times f_{vv}(t, 0)$, we have

$$\det(\dot{\hat{\gamma}}(t), \varphi(t, 0), \ddot{\hat{\gamma}}(t)) > 0. \tag{2.12}$$

So the uniqueness of $X^+_{\hat{\gamma}}$ implies that $X^+_{\hat{\gamma}}(t) = \varphi(t, 0)$ holds. Thus we have that

$$g^+(u, v) = f(u, v).$$

We now define an involution

$$\mathcal{C}^* \ni f \longmapsto \check{f} := g^- \in \mathcal{C}^*.$$

It can be easily checked that $\check{f}(u, v) := g^-(u, v)$ is strongly isometric to $f(u, v)(= g^+(u, v))$. From now on, we suppose that the image of the singular curve of f is non-symmetric and non-planar. To prove Corollary D, it is sufficient to show that $\check{f}$ is not congruent to f. Suppose that there exists an isometry T in $\boldsymbol{R}^3$ such that

$$f(u, v) = T \circ \check{f}(\xi(u, v), \eta(u, v)), \tag{2.13}$$

where $(u, v) \mapsto (\xi(u, v), \eta(u, v))$ is a local analytic diffeomorphism such that

$$\big(\xi(0, 0), \eta(0, 0)\big) = (0, 0).$$

LEMMA 7. *Under the situation above, we have*

$$\xi(u, 0) = \varepsilon u, \quad \xi_v(u, 0) = 0, \quad \eta(u, 0) = 0,$$

where $\varepsilon = \pm 1$.

PROOF. Since T is an isometry and (u, v) (resp. (ξ, η)) is an adapted coordinate system for f (resp. $\check{f}$), the singular set $\{v = 0\}$ of f coincides with the singular set $\{\eta = 0\}$ of $\check{f}$. Hence we have

$$\eta(u, 0) = 0.$$

Let $\tilde{f}(u,v) = \check{f}\big(\xi(u,v), \eta(u,v)\big)$. Since $f = T \circ \tilde{f}$,

$$\begin{aligned} 1 &= f_u(u,0) \cdot f_u(u,0) = \tilde{f}_u(u,0) \cdot \tilde{f}_u(u,0) \\ &= \left|\xi_u(u,0)\check{f}_\xi\big(\xi(u,0), \eta(u,0)\big) + \eta_u(u,0)\check{f}_\eta\big(\xi(u,0), \eta(u,0)\big)\right|^2 \\ &= |\xi_u(u,0)|^2 . \end{aligned}$$

Here, we used the fact that (ξ, η) is an adapted coordinate system for $\check{f}$. Hence we have $\xi_u(u,0) = \varepsilon$ ($\varepsilon = \pm 1$). Since $\xi(0,0) = 0$, we have the first conclusion.

On the other hand, $\partial/\partial v$ (resp. $\partial/\partial\eta$) is the null direction of f (resp. $\check{f}$) along the singular curve, so it holds that

$$\begin{aligned} 0 &= f_v(u,0) = \tilde{f}_v(u,0) \\ &= \xi_v(u,0) f_\xi\big(\xi(u,0), \eta(u,0)\big) + \eta_v(u,0) f_\eta\big(\xi(u,0), \eta(u,0)\big) \\ &= \xi_v(u,0) f_\xi\big(\xi(u,0), 0\big) . \end{aligned}$$

Since $|f_\xi| = 1$ on the singular set, we have $\xi_v(u,0) = 0$. □

Since $\hat{\gamma}(t)$ is non-planar, its torsion function does not vanish. Recall that the torsion function of a regular space curve does not depend on the choice of orientation of the curve, but changes sign by orientation reversing isometries of $\boldsymbol{R}^3$. Hence the isometry T as in (2.13) must be orientation preserving. Since

$$\tilde{f}_{uu} = \xi_{uu}\check{f}_\xi + \eta_{uu}\check{f}_\eta + (\xi_u)^2\check{f}_{\xi\xi} + 2\xi_u\eta_u\check{f}_{\xi\eta} + (\eta_u)^2\check{f}_{\eta\eta} ,$$

where $\tilde{f}(u,v) = \check{f}(\xi(u,v), \eta(u,v))$, Lemma 7 implies that $\tilde{f}_{uu}(u,0) = \check{f}_{\xi\xi}(\xi(u,0),0)$. Since $f = T \circ \tilde{f}$ and $\xi_u(u,0)^2 = 1$, it holds that,

$$\begin{aligned} 0 < \kappa_\nu(u) &= \det(f_u(u,0), f_{vv}(u,0), f_{uu}(u,0)) \\ &= \det(T \circ \tilde{f}_u(u,0), T \circ \tilde{f}_{vv}(u,0), T \circ \tilde{f}_{uu}(u,0)) \\ &= \det(\tilde{f}_u(u,0), \tilde{f}_{vv}(u,0), \tilde{f}_{uu}(u,0)) \\ &= \xi_u(u,0)\det(\check{f}_\xi(\xi(u,0),0), \check{f}_{\eta\eta}(\xi(u,0),0), \check{f}_{\xi\xi}(\xi(u,0),0)) . \end{aligned}$$

By definition of $\check{f}(= g^-)$, it holds that

$$\det(\check{f}_\xi(0,0), \check{f}_{\eta\eta}(0,0), \check{f}_{\xi\xi}(0,0)) = \det(\dot{\hat{\gamma}}(0), X^-_{\hat{\gamma}}(0), \ddot{\hat{\gamma}}(0)) < 0 .$$

So we can conclude that $\xi_u(u,0) = -1$, and by Lemma 7, it holds that

$$\xi(u,0) = -u . \tag{2.14}$$

Then t is a common arclength parameter of $\hat{\gamma}(t)$ and $\check{\gamma}(t) := \check{f}(\xi(t,0),0)$. Hence we have

$$\check{\gamma}(-u) = \check{\gamma}(\xi(u,0)) = T \circ \check{f}(\xi(u,0), \eta(u,0)) = f(u,0) = \hat{\gamma}(u) ,$$

that is, the curve $\hat{\gamma}$ is symmetric at the origin ($= \hat{\gamma}(0)$), which contradicts our assumption. Hence $\check{f}$ cannot be congruent to f. By the definition of strongly isometric equivalence, it is obvious that $\check{f}$ is strongly isometric to f, and (2) of Corollary D holds. □

REMARK 8. For a real analytic map germ f of a non-generic cuspidal edge singularity, the partial differential equation (2.2) cannot be solved, since f_u, f_{uu}, φ are not linearly independent. Cuspidal edges on surfaces of constant Gaussian curvature are all non-generic (cf. [11]). Moreover, as shown in Proposition 9 below, Theorem A does not hold when f is not generic. Although our method is not effective for such surfaces, examples of isometric deformations of flat cuspidal edges with vanishing κ_ν are given in [5] and [11].

PROPOSITION 9. *Let $\gamma(t)$ be the singular curve of the cuspidal edge singularities of a C^∞-map f having vanishing Gaussian curvature on its regular set. Suppose that $\hat{\gamma}(t) := f \circ \gamma(t)$ lies in a plane in $\boldsymbol{R}^3$. Then the image of the curve $\hat{\gamma}$ lies in a straight line*[3].

PROOF. The map f is a flat front in the sense of [12][4]. By [12, Proposition 1.10], the singular points of f are not umbilical points. Then by [12, Proposition 2.2], f is developable. Since κ_ν vanishes along the singular curve γ, the asymptotic direction at $\hat{\gamma}(t)$ is $\dot{\hat{\gamma}}(t)$. In particular, f is a tangential developable surface, that is, we may set $f(u,v) = \hat{\gamma}(u) + v\dot{\hat{\gamma}}(u)$. Then the unit normal vector field ν of f is equal to the binormal vector of $\hat{\gamma}(u)$. Suppose that $\hat{\gamma}(t)$ is a regular curve with non-zero curvature function which lies in a plane. Since f is a front, the fact $\nu_v = 0$ implies that ν_u does not vanish, that is, the torsion function of $\hat{\gamma}$ does not vanish, which contradicts the fact that $\hat{\gamma}(t)$ is a planar. Thus, the curvature function of $\hat{\gamma}(t)$ vanishes identically, namely, its image lies in a straight line. □

In Corollary D, we have shown the existence of isomers of given cuspidal edges. However, for the case of developable surfaces (they are non-generic), there are no such isomers:

PROPOSITION 10. *Let $\sigma(t)$ be a regular space curve whose curvature function $\kappa(t)$ and torsion function $\tau(t)$ have no zeros. Then there exists a unique flat front germ which has cuspidal edge singularities along σ.*

PROOF. Let f be a flat front which has cuspidal edge singularities along σ. By [12, Proposition 1.10], the singular set of f cannot be umbilical points. So [12, Proposition 2.2] implies that f is a developable surface. In paticular, f must be a tangential developable of σ, which proves the assertion. □

Finally, we shall give a concrete example: We set

$$f(u,v) := \left(u,\ -\frac{v^2}{2} + \frac{u^3}{6},\ \frac{u^2}{2} + \frac{u^3}{6} + \frac{v^3}{6}\right),$$

whose singular set consists of generic cuspidal edges, and is parametrized by

$$\hat{\gamma}(t) := f(t,0) = \left(t,\ \frac{t^3}{6},\ \frac{t^2}{2} + \frac{t^3}{6}\right).$$

[3] Let C be a 3/2-cusp on xy-plane in $\boldsymbol{R}^3$. By considering a cylinder or a cone over C, one can actually get a flat cuspidal edge whose image of singular set is contained in a line.

[4] A front whose Gaussan caurvature vanishes is called a flat front. The precise definition is given in [12].

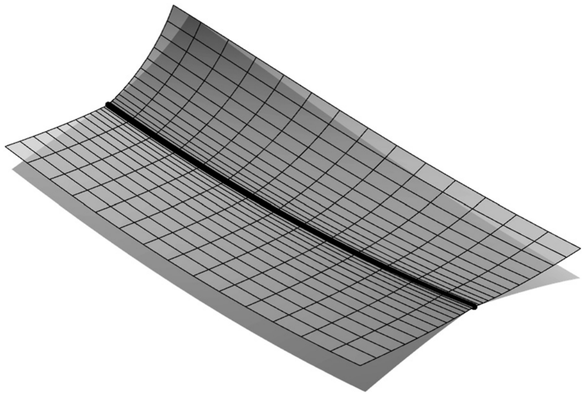

FIGURE 2. The images of $f(u,v)$ (bottom-left) and $g(u,v)$ (top-right) respectively, which have the common image of the singular set (the range of the map is $|u| < 1/8$ and $|v| < 1/4$).

We remark that the curvature function κ and torsion function τ of $\hat{\gamma}$ is given by

$$\kappa(t) = \sqrt{\frac{2\delta}{(2+2t^2+2t^3+t^4)^3}}\,, \quad \tau(t) = -\frac{4}{\delta} \qquad (\delta := 4+8t^2+8t^3+t^4)\,.$$

In particular, $\hat{\gamma}$ is non-symmetric and non-planar as the singular curve of f. We set $g := (g^1, g^2, g^3)$, where

$$\begin{aligned}
g^1(u,v) :=& u + \frac{u^2v^2}{2} - \frac{u^3v^2}{2} - \frac{uv^4}{2} + \frac{v^5}{30} + \frac{u^3v^3}{6} + \frac{9u^2v^4}{4} + \frac{v^6}{6}\,, \\
g^2(u,v) :=& \frac{v^2}{2} + \frac{u^3}{6} - u^2v^2 + \frac{uv^3}{3} + 2u^3v^2 - \frac{u^2v^3}{3} + uv^4 - \frac{v^5}{5} - \frac{9u^4v^2}{4} - 6u^2v^4 \\
&+ \frac{13uv^5}{15} - \frac{11v^6}{36}\,, \\
g^3(u,v) :=& \frac{u^2}{2} + \frac{u^3}{6} - uv^2 + \frac{v^3}{6} + u^2v^2 + \frac{v^4}{2} - \frac{u^2v^3}{3} - \frac{5uv^4}{2} - \frac{v^5}{15} - 2u^4v^2 \\
&+ \frac{2u^3v^3}{3} + 6u^2v^4 + \frac{2uv^5}{5} + \frac{25v^6}{18}\,.
\end{aligned}$$

The singular set of g consists of generic cuspidal edges, and coincides with that of f, namely $g(t,0) = \hat{\gamma}(t)$ holds. This g gives an approximation of $\check{f}$. In fact, one can easily check that the coefficients of the first fundamental form of g coincide with those of f up to the fifth-order terms of u, v near the origin (see Figure 2).

3. Realization of generic intrinsic cuspidal edges into $\boldsymbol{R}^3$. Let $d\sigma^2$ be a positive semi-definite *real analytic* metric defined on a neighborhood of the origin in the uv-plane U. Then $d\sigma^2$ can be written in the following form:

$$d\sigma^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2\,.$$

The metric is called a (germ of) *Kossowski metric* (cf. [6]) if it satisfies the following conditions:

(1) the u-axis consists of a singular set of $d\sigma^2$,
(2) $E_v = G_v = 0$ along the u-axis,
(3) there exists a real analytic function λ on U such that $EG - F^2 = \lambda^2$, and
(4) the gradient vector field $\nabla\lambda = (\lambda_u, \lambda_v)$ does not vanish along the u-axis.

Further systematic treatments of Kossowski metrics are given in [6]. Let K be the Gaussian curvature of $d\sigma^2$ on $U \setminus \{v = 0\}$. Kossowski showed in [7] that

$$K\, d\hat{A} \quad (d\hat{A} := \lambda\, du \wedge dv)$$

can be smoothly extended on U (cf. the condition (d) in the introduction), and proved the following assertion.

FACT 11 (Kossowski). *Suppose that $K\, d\hat{A}$ does not have zeros on the u-axis. Then there exist a neighborhood $V (\subset U)$ and a real analytic wave front $f : V \to \boldsymbol{R}^3$ such that the pull-back metric of the canonical metric of $\boldsymbol{R}^3$ by f coincides with $d\sigma^2$.*

See [7] and [6] for detailed discussions. If the null-direction of the metric $d\sigma^2$ is transversal to the u-axis, the singular points of $d\sigma^2$ are called of *A_2-singularities* or *intrinsic cuspidal edges*. The induced metrics of wave fronts in $\boldsymbol{R}^3$ are all considered as Kossowski metrics and cuspidal edge singular points correspond to A_2-singularities (cf. [6]). Moreover, in [6], the following expression of the singular curvature is given;

$$\kappa_s := \frac{-F_v E_u + 2EF_{uv} - EE_{vv}}{2E^{3/2}\lambda_v}$$

under the assumption that $\lambda_v > 0$. Since this expression of κ_s does not depend on a choice of such a local coordinate (u, v), it can be considered as an invariant of the metric $d\sigma^2$ at A_2-singularities. We can prove the following assertion as an modification of the proof of Theorem A:

THEOREM 12. *Let $d\sigma^2$ be a real analytic Kossowski metric given as above and $\kappa_s(t)$ the singular curvature function along the u-axis. Let $\sigma(t)$ be a real analytic regular space curve whose curvature function $\tilde{\kappa}(t)$ satisfies*

$$\tilde{\kappa}(t) > |\kappa_s(t)| \tag{3.1}$$

for all sufficiently small t. Suppose that $K\, d\hat{A}$ does not have zeros on the u-axis. Then there exist a neighborhood $V (\subset U)$ and a real analytic cuspidal edge $f : V \to \boldsymbol{R}^3$ such that

(1) *the first fundamental form of f coincides with $d\sigma^2$,*
(2) *$f(t, 0) = \sigma(t)$ holds for each t.*

In [7], the isometric deformations of singularities are not discussed, and Theorem 12 can be considered as a refinement of [7, Theorem 1] in the case of cuspidal edge singularities. As pointed out in the introduction, the following proof is different from Kossowski's original approach.

PROOF. Let $d\sigma^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2$ be a Kossowski's metric such that the singular set $\{v = 0\}$ consists of A_2-singularities. Then, without loss of generality, we may assume that $d\sigma^2$ satisfies the following expressions (cf. [6]);

$$E = 1 + v^2 E_0\,, \quad F = 0\,, \quad G = v^2 G_0\,, \tag{3.2}$$

where $E_0 = E_0(u, v)$ and $G_0 = G_0(u, v)$ are certain real analytic functions. We now suppose that there exists a real analytic wave front $f : U \to \boldsymbol{R}^3$ so that the first fundamental form of f is equal to $d\sigma^2$. We can define a real analytic map $\varphi = \varphi(u, v)$ so that $f_v = v\varphi$. Then, it holds that $G_0 = \varphi \cdot \varphi$. Keeping (2.6) in mind, we have that

$$\varphi_v \cdot \varphi = \frac{(\varphi \cdot \varphi)_v}{2} = \frac{1}{2}(G_0)_v\,. \tag{3.3}$$

On the other hand, since

$$v f_u \cdot \varphi = f_u \cdot f_v = F = 0$$

we have $f_u \cdot \varphi = 0$. In particular, we get (cf. (2.7))

$$\varphi_v \cdot f_u = (\varphi \cdot f_u)_v - \varphi \cdot f_{uv} = -\varphi \cdot f_{uv} = -\varphi \cdot (f_v)_u = -\frac{v(G_0)_u}{2}\,. \tag{3.4}$$

Next (2.8) in mind, we have that

$$\varphi \cdot f_{uu} = \frac{(f_v \cdot f_u)_u - (f_{uv} \cdot f_u)}{v} = -\frac{(f_{uv} \cdot f_u)}{v} = -\frac{E_v}{2v} = -\frac{2E_0 + v(E_0)_v}{2}\,. \tag{3.5}$$

By (3.3), (3.4) and (3.5), f must satisfy the equation

$$(f_v, r_v, \varphi_v) = \tilde{\mathcal{F}}(u, v; \varphi_u, (\varphi, f_u, r_u))\,, \tag{3.6}$$

where $r := f_u$, and $\tilde{\mathcal{F}} := (\tilde{\mathcal{F}}^1, \tilde{\mathcal{F}}^2, \tilde{\mathcal{F}}^3)$ is given by

$$\begin{aligned}
&\tilde{\mathcal{F}}^1(u, v;\ x, (y_1, y_2, y_3)) := v y_1\,,\\
&\tilde{\mathcal{F}}^2(u, v;\ x, (y_1, y_2, y_3)) := v x\,,\\
&\tilde{\mathcal{F}}^3(u, v;\ x, (y_1, y_2, y_3))\\
&:= \frac{1}{2}\big((y_1, y_2, y_3)^T\big)^{-1} \begin{pmatrix} (G_0)_v \\ -v(G_0)_u \\ -3(E_0)_v - v(E_0)_{vv} - v(G_0)_{uu} + 2v(x \cdot x) \end{pmatrix}.
\end{aligned}$$

Applying the Cauchy-Kowalevski theorem, we can get a real analytic solution of the equation (3.6) under the same initial conditions as in the proof of Theorem A. Since the product curvature κ_Π can be reformulated as an invariant of the A_2-singular point of a given Kossowski metric as shown in [6], and the condition $K\,d\hat{A} \neq 0$ implies the condition $\kappa_\Pi \neq 0$. Thus we can prove that f has cuspidal edge singularity along the u-axis, and we get the assertion. $\square$

Appendix. The following assertion is applied to prove Proposition 5:

LEMMA. *Let S^2 be the unit sphere in $\boldsymbol{R}^3$ centered at the origin. Let $a, b \in S^2$ be two mutually orthogonal unit vectors, and μ a real number with $|\mu| < 1$. Then there exists a unit vector $w \in S^2$ satisfying*

$$w \cdot a = 0, \qquad w \cdot b = \mu .$$

Moreover, such a vector w is uniquely determined under the assumption that the determinant $\det(a, b, w)$ *is positive (resp. negative).*

PROOF. Let $M := (a, b, a \times b) \in \mathrm{SO}(3)$, where "$\times$" is the vector product of $\boldsymbol{R}^3$. Then the vector

$$w := M \begin{pmatrix} 0 \\ \mu \\ \pm\sqrt{1-\mu^2} \end{pmatrix}$$

has the desired property. The uniqueness can be shown immediately. □

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