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HAMILTONIAN STABILITY OF THE GAUSS IMAGES OF HOMOGENEOUS ISOPARAMETRIC HYPERSURFACES II

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Abstract. In this paper we determine the Hamiltonian stability of *Gauss images*, i.e., the images of the Gauss maps, of homogeneous isoparametric hypersurfaces of exceptional type with $g = 6$ or 4 distinct principal curvatures in spheres. Combining it with our previous results in [12] and Part I [14], we determine the Hamiltonian stability for the Gauss images of all homogeneous isoparametric hypersurfaces. In addition, we discuss the exceptional Riemannian symmetric space $(E_6, U(1) \cdot Spin(10))$ and the corresponding Gauss image, which have their own interest from the viewpoint of symmetric space theory.

1. Introduction. This paper is a continuation of Part I ([14]). The image of the Gauss map, i.e., the so-called *Gauss image*, of any compact oriented isoparametric hypersurface in the standard unit sphere $S^{n+1}(1)$ is a compact minimal Lagrangian submanifold embedded in the complex hyperquadric $Q_n(\mathbf{C})$. This provides a nice class of compact Lagrangian submanifolds embedded in complex hyperquadrics. For example, the Gauss image of a compact oriented isoparametric hypersurface with g distinct constant principal curvatures in $S^{n+1}(1)$ is orientable if and only if $2n/g$ is even ([19]). Moreover, it is a compact monotone and cyclic embedded Lagrangian submanifold with minimal Maslov number $2n/g = m_1 + m_2$ ([14]). Here m_i is the multiplicity of the i-th distinct principal curvature which satisfies $m_i = m_{i+2}$ for $i \pmod g$. Thus inspired by Y. G. Oh's work ([18]), it is an interesting problem to investigate the Hamiltonian stability of those compact minimal Lagrangian embedded submanifolds in $Q_n(\mathbf{C})$ obtained as the Gauss images of isoparametric hypersurfaces in $S^{n+1}(1)$.

It is well-known that all homogeneous isoparametric hypersurfaces in the standard unit sphere are obtained as principal orbits of the isotropy representation of rank 2 Riemannian symmetric pair (U, K) ([9] , [22]). We have determined the Hamiltonian stability of Gauss images of homogeneous isoparametric hypersurfaces of classical type with $g = 1, 2, 3$ ([12]) and $g = 4$ distinct principal curvatures ([14]). In this paper we study the remaining cases for Gauss images of homogeneous isoparametric hypersurfaces of exceptional type

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with $g = 6$ or 4 distinct principal curvatures. Our method is based on harmonic analysis over compact homogeneous spaces and fibrations on homogeneous isoparametric hypersurfaces. In particular, we prove the following theorem in the case when (U, K) is an exceptional symmetric space of rank 2, which was announced and proved for classical symmetric spaces (U, K) of rank 2 in Part I ([14]).

THEOREM 1.1. *Suppose that* (U, K) *is not of type* EIII, *that is*, $(U, K) \neq (E_6, U(1) \cdot Spin(10))$. *Then the Gauss image* $L = \mathcal{G}(N)$ *is not Hamiltonian stable if and only if* $m_2 - m_1 \geq 3$. *Moreover if* (U, K) *is of type* EIII, *then* $(m_1, m_2) = (6, 9)$ *but* $L = \mathcal{G}(N)$ *is strictly Hamiltonian stable.*

General discussion on our method was described in [14]. Sections 1, 2 and 3 of [14] are especially prerequisite for what follows. We continue to use the notation and the general introduction of Part I. This paper is organized as follows: For the convenience of readers, we review the most necessary background and formulations briefly in Section 2. In Sections 3 and 4, we determine the strict Hamiltonian stability of the Gauss images of compact homogeneous isoparametric hypersurfaces with $g = 6$. In Section 5, we discuss the exceptional Riemannian symmetric space $(E_6, U(1) \cdot Spin(10))$ and determine the strict Hamiltonian stability of the corresponding Gauss image, which may have their own interest from the viewpoint of symmetric space theory.

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2. Preliminaries.

Let N^n be a compact oriented homogeneous isoparametric hypersurface immersed in the standard unit sphere $S^{n+1}(1) \subset \mathbf{R}^{n+1}$ with g distinct principal curvatures, where g must be 1, 2, 3, 4 or 6 ([17]). Denote by $\mathbf{x}$ its position vector of a point of N and $\mathbf{n}$ the unit normal vector field of N in $S^{n+1}(1)$. Then the image $\mathcal{G}(N^n)$ of the *Gauss map* defined by

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \cong [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$$

is a compact minimal Lagrangian submanifold embedded in the complex hyperquadric $Q_n(\mathbf{C})$ with the Deck transformation group $\mathbf{Z}_g$. Let $g^{std}_{Q_n(\mathbf{C})}$ be the standard Kähler metric of $Q_n(\mathbf{C})$ induced from the standard inner product of $\mathbf{R}^{n+2}$. Note that the Einstein constant of $g^{std}_{Q_n(\mathbf{C})}$ is equal to n. The Gauss image $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is Hamiltonian stable if and only if the first (positive) eigenvalue of $\mathcal{G}(N^n)$ is n (cf. [2]). By Hsiang-Lawson ([9]) and Takagi-Takahashi ([22]), any homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of the isotropy representation of a compact Riemannian symmetric pair (U, K) of rank 2. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{u}$ as a symmetric Lie algebra of a symmetric pair (U, K) of rank 2 and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Define an AdU-invariant inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ of $\mathfrak{u}$ from the Killing-Cartan form of $\mathfrak{u}$. Then the vector space $\mathfrak{p}$ equipped with the inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ can be identified with the Euclidean space $\mathbf{R}^{n+2}$ and $S^{n+1}(1)$ denotes the $(n+1)$-dimensional unit standard sphere in $\mathfrak{p}$. The linear isotropy action $\mathrm{Ad}_{\mathfrak{p}}$ of K on $\mathfrak{p}$ and thus on $S^{n+1}(1)$ induces the group action of K on $\widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_n(\mathbf{C})$. For each *regular* element H of $\mathfrak{a} \cap S^{n+1}(1)$, we get a homogeneous isoparametric hypersurface in the unit sphere

$$N^n = (\mathrm{Ad}_{\mathfrak{p}} K)H \subset S^{n+1}(1) \subset \mathfrak{p} \cong \mathbf{R}^{n+2}.$$

Its Gauss image is

$$L^n = \mathcal{G}(N^n) = K \cdot [\mathfrak{a}] = [(\mathrm{Ad}_{\mathfrak{p}} K)\mathfrak{a}] \subset \widetilde{Gr}_2(\mathfrak{p}) \cong Q_n(\mathbf{C}).$$

Here N and $\mathcal{G}(N^n)$ have homogeneous space expressions $N \cong K/K_0$ and $\mathcal{G}(N^n) \cong K/K_{[\mathfrak{a}]}$, where we define

$$\begin{aligned} K_0 &:= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H\} \\ &= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H \text{ for each } H \in \mathfrak{a}\}, \\ K_{\mathfrak{a}} &:= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(\mathfrak{a}) = \mathfrak{a}\}, \\ K_{[\mathfrak{a}]} &:= \{k \in K_{\mathfrak{a}} \mid \mathrm{Ad}_{\mathfrak{p}}(k) : \mathfrak{a} \to \mathfrak{a} \text{ preserves the orientation of } \mathfrak{a}\}. \end{aligned}$$

Let $\Sigma(U,K)$ be the set of (restricted) roots of $(\mathfrak{u},\mathfrak{k})$ and $\Sigma^+(U,K)$ be its subset of positive roots. We have the following root space decompositions of $\mathfrak{k}$ and $\mathfrak{p}$ as follows:

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{k}_\gamma, \qquad \mathfrak{p} = \mathfrak{a} + \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{p}_\gamma,$$

where

$$\begin{aligned} \mathfrak{k}_0 :=& \{X \in \mathfrak{k} \mid [X,\mathfrak{a}] \subset \mathfrak{a}\} \\ =& \{X \in \mathfrak{k} \mid [X,H] = 0 \quad \text{for each } H \in \mathfrak{a}\}, \\ \mathfrak{k}_\gamma :=& \{X \in \mathfrak{k} \mid (\mathrm{ad}H)^2 X = (\gamma(H))^2 X \text{ for each } H \in \mathfrak{a}\}, \\ \mathfrak{p}_\gamma :=& \{Y \in \mathfrak{p} \mid (\mathrm{ad}H)^2 Y = (\gamma(H))^2 Y \text{ for each } H \in \mathfrak{a}\}. \end{aligned}$$

For each $\gamma \in \Sigma^+(U,K)$, set $m(\gamma) := \dim \mathfrak{k}_\gamma = \dim \mathfrak{p}_\gamma$. Define

$$\mathfrak{m} := \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{k}_\gamma \quad \text{and} \quad \mathfrak{a}^\perp := \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{p}_\gamma. \tag{1}$$

Then the tangent vector spaces $T_{eK_0}(K/K_0)$ and $T_{eK_{[\mathfrak{a}]}}(K/K_{[\mathfrak{a}]})$ can be identified with the vector subspace $\mathfrak{m}$ of $\mathfrak{k}$. We can choose an orthonormal basis of $\mathfrak{m}$ and $\mathfrak{a}^\perp$ with respect to $\langle\ ,\ \rangle_{\mathfrak{u}}$

$$\{X_{\gamma,i} \in \mathfrak{k}_\gamma \mid \gamma \in \Sigma^+(U,K), i = 1,2,\ldots,m(\gamma)\}$$

and

$$\{Y_{\gamma,i} \in \mathfrak{p}_\gamma \mid \gamma \in \Sigma^+(U,K), i = 1,2,\ldots,m(\gamma)\} \tag{2}$$

such that

$$[H, X_{\gamma,i}] = \sqrt{-1}\gamma(H) Y_{\gamma,i}, \quad [H, Y_{\gamma,i}] = -\sqrt{-1}\gamma(H) X_{\gamma,i} \tag{3}$$

for each $H \in \mathfrak{a}$. Let $\langle\ ,\ \rangle$ denote the $\mathrm{Ad}_{\mathfrak{m}}(K_0)$-invariant inner product of $\mathfrak{m}$ corresponding to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ on K/K_0. Thus we know ([12]) that

$$\left\{ \frac{1}{\|\gamma\|_{\mathfrak{u}}} X_{\gamma,i} \mid \gamma \in \Sigma^+(U,K), i = 1, 2, \dots, m(\gamma) \right\}$$

is an orthonormal basis of $\mathfrak{m}$ relative to $\langle\ ,\ \rangle$.

The Laplace operator $\triangle^0_{L^n} = \delta d$ acting on $C^\infty(K/K_0, \mathbf{C})$ with respect to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ corresponds to the linear differential operator $-\mathcal{C}_{L^n}$ on $C^\infty(K, \mathbf{C})_{K_0}$, where $\mathcal{C}_{L^n}$ is the Casimir operator relative to the $\mathrm{Ad}_{\mathfrak{m}}(K_0)$-invariant inner product $\langle\ ,\ \rangle$ of $\mathfrak{m}$ defined by

$$\mathcal{C}_{L^n} := \sum_{\gamma \in \Sigma^+(U,K)} \sum_{i=1}^{m(\gamma)} \frac{1}{\|\gamma\|^2_{\mathfrak{u}}} (X_{\gamma,i})^2 . \tag{4}$$

Suppose that $\Sigma(U,K)$ is irreducible. Let γ_0 denote the highest root of $\Sigma(U,K)$. For $g = 3, 4$, or 6, the restricted root system $\Sigma(U,K)$ is of type A_2, B_2, BC_2 or G_2. Then for each $\gamma \in \Sigma^+(U,K)$,

$$\frac{\|\gamma\|^2_{\mathfrak{u}}}{\|\gamma_0\|^2_{\mathfrak{u}}} = \begin{cases} 1 & \text{if } \Sigma(U,K) \text{ is of type } A_2 , \\ 1 \text{ or } 1/3 & \text{if } \Sigma(U,K) \text{ is of type } G_2 , \\ 1 \text{ or } 1/2 & \text{if } \Sigma(U,K) \text{ is of type } B_2 , \\ 1, 1/2 \text{ or } 1/4 & \text{if } \Sigma(U,K) \text{ is of type } BC_2 . \end{cases}$$

Set

$$\Sigma^+_1(U,K) := \{\gamma \in \Sigma^+(U,K) \mid \|\gamma\|^2_{\mathfrak{u}} = \|\gamma_0\|^2_{\mathfrak{u}}\} . \tag{5}$$

Define a symmetric Lie subalgebra $(\mathfrak{u}_1, \mathfrak{k}_1)$ by

$$\mathfrak{k}_1 := \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+_1(U,K)} \mathfrak{k}_\gamma , \quad \mathfrak{p}_1 := \mathfrak{a} + \sum_{\gamma \in \Sigma^+_1(U,K)} \mathfrak{p}_\gamma ,$$
$$\mathfrak{u}_1 := \mathfrak{k}_1 + \mathfrak{p}_1 .$$

Let K_1 and U_1 denote connected compact Lie subgroups of K and U generated by $\mathfrak{k}_1$ and $\mathfrak{u}_1$.

Suppose that $\Sigma^+(U,K)$ is of type BC_2. Define

$$\Sigma^+_2(U,K) := \{\gamma \in \Sigma^+(U,K) \mid \|\gamma\|^2_{\mathfrak{u}} = \|\gamma_0\|^2_{\mathfrak{u}} \text{ or } \|\gamma_0\|^2_{\mathfrak{u}}/2\} . \tag{6}$$

Define a symmetric Lie subalgebra $(\mathfrak{u}_2, \mathfrak{k}_2)$ by

$$\mathfrak{k}_2 := \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+_2(U,K)} \mathfrak{k}_\gamma , \quad \mathfrak{p}_2 := \mathfrak{a} + \sum_{\gamma \in \Sigma^+_2(U,K)} \mathfrak{p}_\gamma ,$$
$$\mathfrak{u}_2 := \mathfrak{k}_2 + \mathfrak{p}_2 .$$

Let K_2 and U_2 denote connected compact Lie subgroups of K and U generated by $\mathfrak{k}_2$ and $\mathfrak{u}_2$. We have the following subgroups of K in each case:

$$\begin{aligned} &K_0 \subset K\,, && \text{if} \quad \Sigma(U,K) \text{ is of type } A_2;\\ &K_0 \subset K_1 \subset K\,, && \text{if} \quad \Sigma(U,K) \text{ is of type } B_2 \text{ or } G_2;\\ &K_0 \subset K_1 \subset K_2 \subset K\,, && \text{if} \quad \Sigma(U,K) \text{ is of type } BC_2\,. \end{aligned}$$

Set

$$\begin{aligned} \mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} &:= \sum_{\gamma\in\Sigma^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2\,,\\ \mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} &:= \sum_{\gamma\in\Sigma_1^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2\,,\\ \mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} &:= \sum_{\gamma\in\Sigma_2^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2\,. \end{aligned} \tag{7}$$

Then the Casimir operator $\mathcal{C}_{L^n}$ can be decomposed as follows:

LEMMA 2.1.

$$\mathcal{C}_{L^n} = \begin{cases} \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} \quad \textit{if} \quad \Sigma(U,K) \textit{ is of type } A_2\,;\\[2ex] \dfrac{3}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} \quad \textit{if } \Sigma(U,K) \textit{ is of type } G_2\,;\\[2ex] \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} \quad \textit{if } \Sigma(U,K) \textit{ is of type } B_2\,;\\[2ex] \dfrac{4}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}}\\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } \Sigma(U,K) \textit{ is of type } BC_2\,. \end{cases}$$

Moreover, the Casimir operators $\mathcal{C}_{K/K_0}, \mathcal{C}_{K_1/K_0}$ *(and* $\mathcal{C}_{K_2/K_0}$*) commute with each other.*

For homogeneous isoparametric hypersurfaces of exceptional type with $g = 6$ or 4 distinct principal curvatures, we have the following interesting fibrations on homogeneous isoparametric hypersurfaces by lower dimensional homogeneous isoparametric hypersurfaces, which give rise to the decompositions of the Casimir operators.

(i) In the case $g = 6$ and $(U,K) = (G_2, SO(4))$, $(m_1, m_2) = (1,1)$.

$$
\begin{array}{l}
N^6 = K/K_0 = SO(4)/(\mathbf{Z}_2 + \mathbf{Z}_2) \\
\quad \Big\downarrow \; K_1/K_0 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2) \\
K/K_1 = SO(4)/SO(3) \cong S^3
\end{array}
$$

(ii) In the case $g = 6$ and $(U, K) = (G_2 \times G_2, G_2)$, $(m_1, m_2) = (2, 2)$.

$$
\begin{array}{l}
N^{12} = K/K_0 = G_2/T^2 \\
\quad \Big\downarrow \; K_1/K_0 = SU(3)/T^2 \\
K/K_1 = G_2/SU(3) \cong S^6
\end{array}
$$

(iii) In the case $g = 4$ and $(U, K) = (E_6, U(1) \cdot Spin(10))$, $(m_1, m_2) = (6, 9)$.

$$
\begin{array}{ccc}
N^{30} = \frac{K}{K_0} = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} & \xrightarrow{=} & \frac{K}{K_0} = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} \\
\Big\downarrow \; \frac{K_1}{K_0} = \frac{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}{S^1\cdot Spin(6)} & & \Big\downarrow \; \frac{K_2}{K_0} = \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot Spin(6)} \\
\frac{K}{K_1} = \frac{U(1)\cdot Spin(10)}{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))} & \xrightarrow{\frac{K_2}{K_1} = \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}} & \frac{K}{K_2} = \frac{U(1)\cdot Spin(10)}{U(1)\cdot(Spin(2)\cdot Spin(8))}
\end{array}
$$

3. The case $(U, K) = (G_2 \times G_2, G_2)$. Let $U = G_2 \times G_2$, $K = \{(x, x) \in U \mid x \in G_2\}$ and (U, K) is of type G_2. Then $K_0 = \{k \in K \mid \mathrm{Ad}(k)H = H \text{ for each } H \in \mathfrak{a}\} \cong T^2$ is a maximal torus of G_2 and $N^{12} = K/K_0 \cong G_2/T^2$ is a maximal flag manifold of dimension $n = 12$. Thus its Gauss image is $L^{12} = \mathcal{G}(N^{12}) (\cong N^{12}/\mathbf{Z}_6) = K \cdot [\mathfrak{a}] \cong (K/K_{[\mathfrak{a}]}) \subset Q_{12}(\mathbf{C})$.

Set $\langle\ ,\ \rangle_{\mathfrak{u}} = -B_{\mathfrak{u}}(\ ,\ )$, where $B_{\mathfrak{u}}(\ ,\ )$ denotes the Killing-Cartan form of $\mathfrak{u}$. Let $\langle\ ,\ \rangle$ be the inner product of $\mathfrak{m}$ corresponding to the invariant induced metric on L^n from $(Q_n(\mathbf{C}), g^{\mathrm{std}}_{Q_n(\mathbf{C})})$.

The restricted root system $\Sigma(U, K)$ of type G_2, can be given as follows ([5]):

$$
\begin{aligned}
\Sigma(U, K) = \{ & \pm(\varepsilon_1 - \varepsilon_2) = \pm\alpha_1\,, \pm(\varepsilon_3 - \varepsilon_1) = \pm(\alpha_1 + \alpha_2)\,, \\
& \pm(\varepsilon_3 - \varepsilon_2) = \pm(2\alpha_1 + \alpha_2)\,, \pm(-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \pm\alpha_2\,, \\
& \pm(\varepsilon_1 - 2\varepsilon_2 + \varepsilon_3) = \pm(3\alpha_1 + \alpha_2)\,, \\
& \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2) = \pm(3\alpha_1 + 2\alpha_2) = \tilde{\alpha}\}\,,
\end{aligned}
$$

where $\Pi(U, K) = \{\alpha_1 = \varepsilon_1 - \varepsilon_2\,, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$ is its fundamental root system. Here

$$\|\gamma\|_{\mathfrak{u}}^2 = \begin{cases} \dfrac{1}{24} & \text{if } \gamma \text{ is short}\,, \\[2ex] \dfrac{1}{8} & \text{if } \gamma \text{ is long}\,. \end{cases}$$

Now $K_1 = SU(3)$ and $K_0 = T^2 \subset K_1 = SU(3) \subset K = G_2$.

In Lemma 2.1 the Casimir operator

$$\mathcal{C}_L = \frac{3}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} - \frac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\,,\,\rangle_{\mathfrak{u}}}\,,$$

of L^n with respect to $\langle\,,\,\rangle$ corresponding to $-\Delta_{L^{12}}$ becomes

$$\begin{aligned} \mathcal{C}_L &= 24\,\mathcal{C}_{K/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} - 16\,\mathcal{C}_{K_1/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} \\ &= 12\,\mathcal{C}^{\mathfrak{k}}_{K/K_0} - 8\,\mathcal{C}^{\mathfrak{k}}_{K_1/K_0} \\ &= 12\,\mathcal{C}^{\mathfrak{k}}_{K/K_0} - 6\,\mathcal{C}^{\mathfrak{k}_1}_{K_1/K_0}\,, \end{aligned} \tag{8}$$

where $\mathcal{C}^{\mathfrak{k}}_{K/K_0}$ and $\mathcal{C}^{\mathfrak{k}}_{K_1/K_0}$ denote the Casimir operators of K/K_0 and K_1/K_0 relative to the K_0-invariant inner product induced from the Killing-Cartan form of $\mathfrak{k}$, respectively, and $\mathcal{C}^{\mathfrak{k}_1}_{K_1/K_0}$ denotes the Casimir operator of K_1/K_0 relative to the K_0-invariant inner product induced from the Killing-Cartan form of $\mathfrak{k}_1$.

Let $\{\alpha_1, \alpha_2\}$ be the fundamental root system of G_2 and $\{\Lambda_1, \Lambda_2\}$ be the fundamental weight system of G_2. In our work we frequently use Satoru Yamaguchi's results ([26]) on the spectra of maximal flag manifolds.

LEMMA 3.1 (Branching law of (G_2, T^2) [26]).

$$\begin{aligned} D(K, K_0) &= D(G_2, T^2) = D(G_2) \\ &= \{\Lambda = m_1\Lambda_1 + m_2\Lambda_2 \mid m_1, m_2 \in \mathbf{Z}, m_1 \geq 0, m_2 \geq 0\} \\ &= \{\Lambda = p_1\alpha_1 + p_1\alpha_2 \mid p_1, p_2 \in \mathbf{Z}, p_1 \geq 1, p_2 \geq 1\}\,. \end{aligned} \tag{9}$$

The eigenvalue formula of the Casimir operator $\mathcal{C}_{K/K_0}$ relative to the inner product induced from the Killing-Cartan form of G_2 is

$$-c(\Lambda, \langle\,,\,\rangle_{\mathfrak{g}_2}) = \frac{1}{24}(m_1p_1 + 3m_2p_2 + 2p_1 + 6p_2) \tag{10}$$

for each $\Lambda \in D(G_2, T^2) = D(G_2)$.

Since

$$-\mathcal{C}_L = -\Bigg(4\,\mathcal{C}^{\mathfrak{g}_2}_{K/K_0} + \sum_{\gamma:\text{short}} 16\,(X_{\gamma,i})^2\Bigg) \geq -4\,\mathcal{C}^{\mathfrak{g}_2}_{K/K_0}\,,$$

if the eigenvalue $-c_L$ of $-\mathcal{C}_L$ satisfies $-c_L \leq n = 12$, then $-c_\Lambda \leq 3$.

By using the formula (10), we obtain

$$\{\Lambda \in D(G_2, T^2) \mid -c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) \leq 3\}$$
$$= \{0, \Lambda_1((p_1, p_2) = (2,1)), 2\Lambda_1((p_1, p_2) = (4,2)), 3\Lambda_1((p_1, p_2) = (6,3)),$$
$$\Lambda_2((p_1, p_2) = (3,2)), 2\Lambda_2((p_1, p_2) = (6,4)), \Lambda_1+\Lambda_2((p_1, p_2)=(5,3)),$$
$$2\Lambda_1 + \Lambda_2((p_1, p_2) = (7,4))\}.$$

Let $\{\alpha'_1, \alpha'_2\}$ be the fundamental root system of $SU(3)$ and $\{\Lambda'_1, \Lambda'_2\}$ be the fundamental weight system of $SU(3)$. For each $\Lambda \in D(G_2, T^2)$ with $-c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) \leq 3$, by using the branching law of $(G_2, SU(3))$ in [15], we can determine all irreducible $SU(3)$-submodule $V_{\Lambda'}$ with the highest weight $\Lambda' = m'_1\Lambda'_1 + m_2\Lambda'_2$ contained in an irreducible G_2-module V_Λ as in Table 1.

Since

$$\mathfrak{g}_2^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha\in\Sigma(G_2)} \mathfrak{g}^\alpha = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha:\text{short}} \mathfrak{g}^\alpha + \sum_{\alpha:\text{long}} \mathfrak{g}^\alpha\,,$$
$$\mathfrak{su}(3)^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha:\text{long}} \mathfrak{g}^\alpha\,,$$

we know that

$$\begin{aligned} T^2\cdot\mathbf{Z}_6 =& \{a \in G_2 \mid \mathrm{Ad}(a)(\mathfrak{t}) = \mathfrak{t} \text{ preserving the orientation of } \mathfrak{t}\} \\ \supset& \{a \in SU(3) \mid \mathrm{Ad}(a)(\mathfrak{t}) = \mathfrak{t} \text{ preserving the orientation of } \mathfrak{t}\} \\ =& \,T^2\cdot\mathbf{Z}_3\,. \end{aligned}$$

Now we use results on $SU(3)/T^2$, which were already treated in the case of $g = 3$ and $m = 2$ ([12]).

LEMMA 3.2 (Branching law of $(SU(3), T^2)$ [26]).

$$\begin{aligned} D(K_1, K_0) =& D(SU(3), T^2) \\ =& \{\Lambda' = m'_1\Lambda'_1 + m'_2\Lambda'_2 \mid m'_i \in \mathbf{Z}, m'_i \geq 0\} \qquad (11) \\ =& \{\Lambda' = p'_1\alpha'_1 + p'_2\alpha'_2 \mid p'_i \in \mathbf{Z}, p'_i \geq 1\}\,, \end{aligned}$$

where

$$m'_1 = 2p'_1 - p'_2 \geq 0\,, \quad m'_2 = -p'_1 + 2p'_2 \geq 0\,.$$

The eigenvalue formula is

$$-c(\Lambda', \langle\ ,\ \rangle_{\mathfrak{su}(3)}) = \frac{1}{6}(m'_1p'_1 + m'_2p'_2 + 2p'_1 + 2p'_2) \qquad (12)$$

for each $\Lambda' \in D(SU(3), T^2)$.

Using Lemma 3.2, we get that $\Lambda' = m'_1\Lambda'_1 + m'_2\Lambda'_2 \in D(SU(3), T^2)$ such that $V_{\Lambda'} \subset V_\Lambda$ for some $\Lambda \in D(G_2, T^2)$ with $-c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) \leq 3$ satisfies

$$(m'_1, m'_2) \in \{(1,1), (3,0), (0,3), (2,2)\}\,.$$

By using the formula (12), we compute the corresponding eigenvalues of $\mathcal{C}_{K_1/K_0}$ given in

TABLE 1. All irreducible $SU(3)$-submodule $V_{\Lambda'}$ contained in an irreducible G_2-module V_Λ.

(m_1, m_2)	(p_1, p_2)	$-c$	$\dim_{\mathbf{C}} V_\Lambda$	irred. $SU(3)$-submodules (m_1', m_2')
(1, 0)	(2, 1)	$\frac{1}{2}$	7	(1, 0), (0, 1), (0, 0)
(2, 0)	(4, 2)	$\frac{7}{6}$	27	(2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)
(3, 0)	(6, 3)	2	77	(3, 0), (2, 1), (1, 2), (0, 3), (2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)
(0, 1)	(3, 2)	1	14	(1, 1), (1, 0), (0, 1)
(0, 2)	(6, 4)	$\frac{5}{2}$	77	(2, 2), (2, 1), (1, 2), (2, 0), (1, 1), (0, 2)
(1, 1)	(5, 3)	$\frac{7}{4}$	64	(2, 1), (1, 2), (2, 0), 2(1, 1), (0, 2), (1, 0), (0, 1)
(2, 1)	(7, 4)	$\frac{8}{3}$	189	(3, 1), (2, 2), (1, 3), (3, 0), 2(2, 1), 2(1, 2), (0, 3), (2, 0), 2(1, 1), (0, 2), (1, 0), (0, 1)

TABLE 2. Eigenvalue computation for $\mathcal{C}_{K_1/K_0}$.

(p_1', p_2')	(m_1', m_2')	$-c' = -c(\Lambda', \langle\ ,\ \rangle_{\mathfrak{su}(3)})$
(1, 1)	(1, 1)	1
(2, 1)	(3, 0)	2
(1, 2)	(0, 3)	2
(2, 2)	(2, 2)	$\frac{8}{3}$

Table 2.

Therefore, for all $\Lambda \in D(G_2, T^2)$ and all $\Lambda' \in D(SU(3), T^2)$ such that $V_{\Lambda'} \subset V_\Lambda$ and $-c(\Lambda, \langle\ ,\ \rangle_{\mathfrak{g}_2}) \leq 3$, the corresponding eigenvalues of $-\mathcal{C}_L = -12\,\mathcal{C}^{\mathfrak{k}}_{K/K_0} + 6\,\mathcal{C}^{\mathfrak{k}_1}_{K_1/K_0} = -12c + 6c'$ are given in Table 3:

Since $\Lambda_1' + \Lambda_2'$ ($(m_1', m_2') = (1, 1)$) corresponds to the complexified adjoint representation of $SU(3)$, we see that $(V'_{\Lambda_1'+\Lambda_2'})_{T^2} \cong \mathfrak{t}^2$ and $(V'_{\Lambda_1'+\Lambda_2'})_{T^2 \cdot \mathbf{Z}_3} = \{0\}$. Then

$$\Lambda_1' + \Lambda_2' \notin D(SU(3), T^2 \cdot \mathbf{Z}_3)\,,$$

and thus

$$2\Lambda_1, \Lambda_2 \notin D(G_2, T^2 \cdot \mathbf{Z}_6)\,.$$

Now we obtain that $\mathcal{G}(G_2/T^2) \subset Q_{12}(\mathbf{C})$ is Hamiltonian stable.

TABLE 3. Eigenvalues of $-\mathcal{C}_L$ for $L = G_2/(T_2 \cdot \mathbf{Z}_6)$.

(m_1, m_2)	(p_1, p_2)	$\dim_{\mathbf{C}} V_\Lambda$	$-c$	(m_1', m_2')	$-c'$	$-12c + 6c'$
(2, 0)	(4, 2)	27	$\frac{7}{6}$	(1, 1)	1	8
(3, 0)	(6, 3)	77	2	(1, 1)	1	18
(3, 0)	(6, 3)	77	2	(3, 0)	2	12
(3, 0)	(6, 3)	77	2	(0, 3)	2	12
(0, 1)	(3, 2)	14	1	(1, 1)	1	6
(0, 2)	(6, 4)	77	$\frac{5}{2}$	(1, 1)	1	24
(0, 2)	(6, 4)	77	$\frac{5}{2}$	(2, 2)	$\frac{8}{3}$	14
(1, 1)	(5, 3)	64	$\frac{7}{4}$	2(1, 1)	1	15
(2, 1)	(7, 4)	189	$\frac{8}{3}$	2(1, 1)	1	26
(2, 1)	(7, 4)	189	$\frac{8}{3}$	(3, 0)	2	20
(2, 1)	(7, 4)	189	$\frac{8}{3}$	(0, 3)	2	20
(2, 1)	(7, 4)	189	$\frac{8}{3}$	(2, 2)	$\frac{8}{3}$	16

We need to examine whether $3\Lambda_1 \in D(K, K_{[\mathfrak{a}]}) = D(G_2, T^2 \cdot \mathbf{Z}_6)$ or not. Consider

$$(V_{3\Lambda_1})_{T^2} = (V'_{3\Lambda_1'})_{T^2} \oplus (V'_{3\Lambda_2'})_{T^2} \oplus (V'_{\Lambda_1'+\Lambda_2'})_{T^2} .$$

Since

$$V'_{3\Lambda_1'} \cong \mathrm{Sym}^3(\mathbf{C}^3) = \mathrm{span}_{\mathbf{C}}\{e_{i_1} \cdot e_{i_2} \cdot e_{i_3} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq 3\} ,$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbf{C}^3$, we get

$$(V'_{3\Lambda_1'})_{T^2} = (V'_{3\Lambda_1'})_{T^2\cdot\mathbf{Z}_3} = \mathrm{span}_{\mathbf{C}}\{e_1 \cdot e_2 \cdot e_3\} .$$

Similarly, we get $V'_{3\Lambda_2'} \cong \mathrm{Sym}^3(\bar{\mathbf{C}}^3)$ and $(V'_{3\Lambda_2'})_{T^2} = (V'_{3\Lambda_2'})_{T^2\cdot\mathbf{Z}_3}$ with dimension 1. On the other hand we know that $(V'_{\Lambda_1'+\Lambda_2'})_{T^2} \cong \mathfrak{t}$ and $(V'_{\Lambda_1'+\Lambda_2'})_{T^2\cdot\mathbf{Z}_3} = \{0\}$. Hence we get $\dim_{\mathbf{C}}(V_{3\Lambda_1})_{T^2} = 4$ and $\dim_{\mathbf{C}}(V_{3\Lambda_1})_{T^2\cdot\mathbf{Z}_3} = 2$. However $\dim_{\mathbf{C}}(V_{3\Lambda_1})_{T^2\cdot\mathbf{Z}_6} = 1$. In fact, $T^2 \cdot \mathbf{Z}_6 \subset G_2$, $T^2 \cdot \mathbf{Z}_6 \not\subset SU(3)$, $T^2 \cdot \mathbf{Z}_3 \subset SU(3)$ and $(T^2 \cdot \mathbf{Z}_6)/(T^2 \cdot \mathbf{Z}_3) \cong \mathbf{Z}_2$. Thus there exists an element $u \in T^2 \cdot \mathbf{Z}_6 \subset G_2$ with $u \notin SU(3)$ which satisfies $\mathrm{Ad}(u)(SU(3)) \subset SU(3)$ and provides the generators of both $(T^2 \cdot \mathbf{Z}_6)/T^2 \cong \mathbf{Z}_6$ and $(T^2 \cdot \mathbf{Z}_6)/(T^2 \cdot \mathbf{Z}_3) \cong \mathbf{Z}_2$. Then we observe that $\rho_{3\Lambda_1'} \circ \mathrm{Ad}(u)|_{SU(3)} \cong \rho_{3\Lambda_2'}$ and $\rho_{3\Lambda_1}(u)(V'_{3\Lambda_1'}) = V'_{3\Lambda_2'}$. Thus $\rho_{3\Lambda_1}(u)(V'_{3\Lambda_1'})_{T^2\cdot\mathbf{Z}_3} = (V'_{3\Lambda_2'})_{T^2\cdot\mathbf{Z}_3}$ and $(\rho_{3\Lambda_1}(u))^2|_{(V'_{3\Lambda_1'})_{T^2\cdot\mathbf{Z}_3}} = (\rho_{3\Lambda_1}(u^2))|_{(V'_{3\Lambda_1'})_{T^2\cdot\mathbf{Z}_3}} = \mathrm{Id}$, because $u^2 \in T^2 \cdot \mathbf{Z}_3$. Hence we have $(V_{3\Lambda_1})_{T^2\cdot\mathbf{Z}_6} \subset (V'_{3\Lambda_1'})_{T^2\cdot\mathbf{Z}_3} \oplus (V'_{3\Lambda_2'})_{T^2\cdot\mathbf{Z}_3}$ and $\dim(V_{3\Lambda_1})_{T^2\cdot\mathbf{Z}_6} = 1$. Therefore $3\Lambda_1 \in D(G_2, T^2 \cdot \mathbf{Z}_6)$ and its multiplicity is equal to 1. It follows that

$$n(L^{12}) = \dim_{\mathbf{C}}(V_{3\Lambda_1}) = 77 = 91 - 14 = \dim(SO(14)) - \dim(G_2) = n_{hk}(L^{12}) ,$$

that is, $\mathcal{G}(G_2/T^2) \subset Q_{12}(\mathbf{C})$ is Hamiltonian rigid.

Let $\bigwedge^2 \mathbf{R}^{14} = \mathfrak{o}(n+2) = \mathrm{ad}_{\mathfrak{p}}(\mathfrak{g}_2) + \mathcal{V} \cong \mathfrak{g}_2 + \mathcal{V}$. Then

$$\bigwedge^2 \mathbf{C}^{14} = \left(\bigwedge^2 \mathbf{R}^{14}\right)^{\mathbf{C}} = \mathfrak{o}(n+2)^{\mathbf{C}} = \mathfrak{o}(n+2,\mathbf{C}) = \mathrm{ad}_{\mathfrak{p}}(\mathfrak{g}_2^{\mathbf{C}}) + \mathcal{V}^{\mathbf{C}} \cong \mathfrak{g}_2^{\mathbf{C}} + \mathcal{V}^{\mathbf{C}},$$

where $\dim \mathcal{V} = 77$ and $\dim_{\mathbf{C}} \mathcal{V}^{\mathbf{C}} = 77$. More precisely, we observe that $\mathcal{V}$ is a real 77-dimensional irreducible G_2-module with $(\mathcal{V})_{T^2\cdot\mathbf{Z}_6} \neq \{0\}$, and $\mathcal{V}^{\mathbf{C}}$ is a complex 77-dimensional G_2-module with $(\mathcal{V})^{\mathbf{C}}_{T^2\cdot\mathbf{Z}_6} \neq \{0\}$. Moreover, we have $\mathcal{V}^{\mathbf{C}} \cong V_{3\Lambda_1}$ with $\dim_{\mathbf{C}}(\mathcal{V})^{\mathbf{C}}_{T^2\cdot\mathbf{Z}_6} = 1$.

From these arguments we conclude that

THEOREM 3.3. *The Gauss image* $L^{12} = \mathcal{G}(G_2/T^2) = G_2/(T^2\cdot\mathbf{Z}_6) \subset Q_{12}(\mathbf{C})$ *is strictly Hamiltonian stable.*

4. The case $(U,K) = (G_2, SO(4))$**.** Let $U = G_2$, $K = SO(4)$ and (U,K) is of type G_2. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the orthogonal symmetric Lie algebra of $(G_2, SO(4))$ and $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$. Here $\mathfrak{u} = \mathfrak{g}_2$, $\mathfrak{k} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Let

$$p : \widetilde{K} = Spin(4) = SU(2)\times SU(2) \longrightarrow K = SO(4)$$

be the universal covering Lie group homomorphism with kernel $\mathbf{Z}_2$.

Recall that the complete set of all inequivalent irreducible unitary representations of $SU(2)$ is given by

$$\mathcal{D}(SU(2)) = \{(V_m, \rho_m) \mid m \in \mathbf{Z}, m \geq 0\},$$

where V_m denotes the complex vector space of complex homogeneous polynomials of degree m with two variables z_0, z_1 and the representation ρ_m of $SU(2)$ on V_m is defined by $(\rho_m(g)f)(z_0,z_1) = f((z_0,z_1)g)$ for each $g \in SU(2)$. Set

$$v_k^{(m)}(z_0,z_1) := \frac{1}{\sqrt{k!(m-k)!}} z_0^{m-k} z_1^k \in V_m \quad (k = 0,1,\ldots,m) \tag{13}$$

and define the standard Hermitian inner product of V_m invariant under $\rho_m(SU(2))$ such that $\{v_0^{(m)},\ldots,v_m^{(m)}\}$ is a unitary basis of V_m. Let $(V_l \otimes V_m, \rho_l \boxtimes \rho_m)$ denote an irreducible unitary representation of $SU(2)\times SU(2)$ of complex dimension $(l+1)(m+1)$ obtained by taking the exterior tensor product of V_l and V_m and then

$$\{(V_l \otimes V_m, \rho_l \boxtimes \rho_m) \mid l, m \in \mathbf{Z},\ l, m \geq 0\}$$

is the complete set of all inequivalent irreducible unitary representations of $SU(2)\times SU(2)$.

The isotropy representation of $(G_2, SO(4))$ is explicitly described as follows (cf. [8]): Suppose that $(l,m) = (3,1)$. The real 8-dimensional vector subspace W of $V_3 \otimes V_1$ spanned over $\mathbf{R}$ by

$$\begin{aligned}\Big\{ & v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)}\,,\ \sqrt{-1}\,(v_0^{(3)} \otimes v_0^{(1)} - v_3^{(3)} \otimes v_1^{(1)})\,,\\ & v_1^{(3)} \otimes v_0^{(1)} - v_2^{(3)} \otimes v_1^{(1)}\,,\ \sqrt{-1}\,(v_1^{(3)} \otimes v_0^{(1)} + v_2^{(3)} \otimes v_1^{(1)})\,,\\ & v_0^{(3)} \otimes v_1^{(1)} - v_3^{(3)} \otimes v_0^{(1)}\,,\ \sqrt{-1}\,(v_0^{(3)} \otimes v_1^{(1)} + v_3^{(3)} \otimes v_0^{(1)})\,,\\ & v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)}\,,\ \sqrt{-1}\,(v_2^{(3)} \otimes v_0^{(1)} - v_1^{(3)} \otimes v_1^{(1)}) \Big\}\end{aligned}$$

gives an irreducible orthogonal representation of $SU(2) \times SU(2)$ whose complexification is $V_3 \otimes V_1$, i.e., W is a *real form* of $V_3 \otimes V_1$. Then the isotropy representation $\mathrm{Ad}_{\mathfrak{p}}$ of $(G_2, SO(4))$ is given by $\mathrm{Ad}_{\mathfrak{p}}^{\mathbf{C}} \circ p \cong \rho_3 \boxtimes \rho_1$ and the vector space $\mathfrak{p}$ is linearly isomorphic to W. Moreover $\mathfrak{a}$ corresponds to a vector subspace

$$\mathbf{R}(v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)}) + \mathbf{R}(v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)})\ .$$

For each $X = \begin{pmatrix} \sqrt{-1}x & u \\ -\bar{u} & -\sqrt{-1}x \end{pmatrix}$, $Y = \begin{pmatrix} \sqrt{-1}y & w \\ -\bar{w} & -\sqrt{-1}y \end{pmatrix} \in \mathfrak{su}(2)$, the following useful formula holds:

LEMMA 4.1.

$$\begin{aligned} &[d(\rho_l \boxtimes \rho_m)(X,Y)]\left(v_i^{(l)} \otimes v_j^{(m)} \pm v_{l-i}^{(l)} \otimes v_{m-j}^{(m)}\right)\\ &\quad = \{(2i-l)x + (2j-m)y\}\,\sqrt{-1}(v_i^{(l)} \otimes v_j^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j}^{(m)})\\ &\qquad - \sqrt{i(l-i+1)}\,\mathrm{Re}(u)\,(v_{i-1}^{(l)} \otimes v_j^{(m)} \mp v_{l-i+1}^{(l)} \otimes v_{m-j}^{(m)})\\ &\qquad + \sqrt{i(l-i+1)}\,\mathrm{Im}(u)\,\sqrt{-1}(v_{i-1}^{(l)} \otimes v_j^{(m)} \pm v_{l-i+1}^{(l)} \otimes v_{m-j}^{(m)})\\ &\qquad - \sqrt{j(m-j+1)}\,\mathrm{Re}(w)\,(v_i^{(l)} \otimes v_{j-1}^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j+1}^{(m)})\\ &\qquad + \sqrt{j(m-j+1)}\,\mathrm{Im}(w)\,\sqrt{-1}(v_i^{(l)} \otimes v_{j-1}^{(m)} \pm v_{l-i}^{(l)} \otimes v_{m-j+1}^{(m)})\\ &\qquad + \sqrt{(l-i)(i+1)}\,\mathrm{Re}(u)\,(v_{i+1}^{(l)} \otimes v_j^{(m)} \mp v_{l-i-1}^{(l)} \otimes v_{m-j}^{(m)})\\ &\qquad + \sqrt{(l-i)(i+1)}\,\mathrm{Im}(u)\,\sqrt{-1}(v_{i+1}^{(l)} \otimes v_j^{(m)} \pm v_{l-i-1}^{(l)} \otimes v_{m-j}^{(m)})\\ &\qquad + \sqrt{(m-j)(j+1)}\,\mathrm{Re}(w)\,(v_i^{(l)} \otimes v_{j+1}^{(m)} \mp v_{l-i}^{(l)} \otimes v_{m-j-1}^{(m)})\\ &\qquad + \sqrt{(m-j)(j+1)}\,\mathrm{Im}(w)\,\sqrt{-1}(v_i^{(l)} \otimes v_{j+1}^{(m)} \pm v_{l-i}^{(l)} \otimes v_{m-j-1}^{(m)})\ . \end{aligned} \tag{14}$$

REMARK 4.2. By using the formula (14) we can check that the real vector subspace W is invariant under the action of $SU(2) \times SU(2)$ via $\rho_3 \boxtimes \rho_1$.

Define an orthonormal basis of the real vector space $W \cong \mathfrak{p}$ by

$$\begin{aligned} H_1 &:= \tfrac{1}{\sqrt{2}}(v_0^{(3)} \otimes v_0^{(1)} + v_3^{(3)} \otimes v_1^{(1)})\,,\\ H_2 &:= \tfrac{1}{\sqrt{2}}(v_2^{(3)} \otimes v_0^{(1)} + v_1^{(3)} \otimes v_1^{(1)})\,,\\ E_1 &:= \tfrac{1}{\sqrt{2}}\sqrt{-1}\,(v_0^{(3)} \otimes v_0^{(1)} - v_3^{(3)} \otimes v_1^{(1)})\,, \end{aligned}$$

$$E_2 := \tfrac{1}{\sqrt{2}}(v_1^{(3)} \otimes v_0^{(1)} - v_2^{(3)} \otimes v_1^{(1)}),$$
$$E_3 := \tfrac{1}{\sqrt{2}}\sqrt{-1}\,(v_1^{(3)} \otimes v_0^{(1)} + v_2^{(3)} \otimes v_1^{(1)}),$$
$$E_4 := \tfrac{1}{\sqrt{2}}(v_0^{(3)} \otimes v_1^{(1)} - v_3^{(3)} \otimes v_0^{(1)}),$$
$$E_5 := \tfrac{1}{\sqrt{2}}\sqrt{-1}\,(v_0^{(3)} \otimes v_1^{(1)} + v_3^{(3)} \otimes v_0^{(1)}),$$
$$E_6 := \tfrac{1}{\sqrt{2}}\sqrt{-1}\,(v_2^{(3)} \otimes v_0^{(1)} - v_1^{(3)} \otimes v_1^{(1)}) .$$

Then we have the matrix expression as follows:

$$[d(\rho_3 \boxtimes \rho_1)(X, Y)]\,(H_1, H_2) = (E_1, E_2, E_3, E_4, E_5, E_6)\begin{pmatrix} -(3x+y) & 0 \\ \sqrt{3}\,\mathrm{Re}(u) & -(2\,\mathrm{Re}(u)+\mathrm{Re}(w)) \\ \sqrt{3}\,\mathrm{Im}(u) & 2\,\mathrm{Im}(u)+\mathrm{Im}(w) \\ \mathrm{Re}(w) & -\sqrt{3}\,\mathrm{Re}(u) \\ \mathrm{Im}(w) & \sqrt{3}\,\mathrm{Im}(u) \\ 0 & x-y \end{pmatrix}.$$

The inner product $\langle\ ,\ \rangle$ corresponding to the metric induced from $g^{std}_{Q_6(\mathbf{C})}$ is given as follows: For $(X, X'), (Y, Y') \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$,

$$\begin{aligned}
&\langle (X, X'), (Y, Y') \rangle \\
&\quad := (3x+x')(3y+y') \\
&\qquad + 3\,\mathrm{Re}(u)\mathrm{Re}(w) + (2\,\mathrm{Re}(u)+\mathrm{Re}(u'))(2\,\mathrm{Re}(w)+\mathrm{Re}(w')) \\
&\qquad + 3\,\mathrm{Im}(u)\mathrm{Im}(w) + (2\,\mathrm{Im}(u)+\mathrm{Im}(u'))(2\,\mathrm{Im}(w)+\mathrm{Im}(w')) \\
&\qquad + \mathrm{Re}(u')\mathrm{Re}(w') + 3\mathrm{Re}(u)\mathrm{Re}(w) \\
&\qquad + \mathrm{Im}(u')\mathrm{Im}(w') + 3\mathrm{Im}(u)\mathrm{Im}(w) \\
&\qquad + (x-x')(y-y') \\
&\quad = 10xy + 2x'y + 2xy' + 2x'y' \\
&\qquad + 10\,\mathrm{Re}(u)\mathrm{Re}(w) + 2\,\mathrm{Re}(u')\mathrm{Re}(w) + 2\,\mathrm{Re}(u)\mathrm{Re}(w') + 2\mathrm{Re}(u')\mathrm{Re}(w') \\
&\qquad + 10\,\mathrm{Im}(u)\mathrm{Im}(w) + 2\,\mathrm{Im}(u')\mathrm{Im}(w) + 2\,\mathrm{Im}(u)\mathrm{Im}(w') + 2\mathrm{Im}(u')\mathrm{Im}(w') .
\end{aligned}$$

Thus the Casimir operator of $(\widetilde{K}, \widetilde{K}_{[\mathfrak{a}]})$ relative to the inner product $\langle\ ,\ \rangle$ is given as follows:

$$\begin{aligned}
\mathcal{C}_L = {} & \frac{1}{2}\,(X_1,0)\cdot(X_1,0) + \frac{1}{2}\,(X_2,0)\cdot(X_2,0) + \frac{1}{2}\,(X_3,0)\cdot(X_3,0) \\
& + \frac{5}{2}\,(0,X_1)\cdot(0,X_1) + \frac{5}{2}\,(0,X_2)\cdot(0,X_2) + \frac{5}{2}\,(0,X_3)\cdot(0,X_3) \\
& - (X_1,0)\cdot(0,X_1) - (X_2,0)\cdot(0,X_2) - (X_3,0)\cdot(0,X_3),
\end{aligned} \tag{15}$$

where

$$X_1 := \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad X_2 := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 := \frac{1}{2}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

is a basis of $\mathfrak{su}(2)$ and $\{(X_1,0), (X_2,0), (X_3,0), (0,X_1), (0,X_2), (0,X_3)\}$ is a basis of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Hence, we have the following formula for the Casimir operator:

LEMMA 4.3.

$$\begin{aligned}
&[d(\rho_l \boxtimes \rho_m)(\mathcal{C}_L)](v_i^{(l)} \otimes v_a^{(m)}) \\
&= -\left\{\frac{l(l+2)}{8} + \frac{5m(m+2)}{8} - \frac{(2i-l)(4a-m)}{4}\right\}(v_i^{(l)} \otimes v_a^{(m)}) \\
&+ \frac{1}{2}\sqrt{(i+1)(l-i)a(m-a+1)}(v_{i+1}^{(l)} \otimes v_{a-1}^{(m)}) \\
&+ \frac{1}{2}\sqrt{i(l-i+1)(a+1)(m-a)}(v_{i-1}^{(l)} \otimes v_{a+1}^{(m)})\,.
\end{aligned}$$

Set

$$\widetilde{K}_0 := \{(A,B) \in \widetilde{K} \mid \mathrm{Ad}(p(A,B))H = H \text{ for each } H \in \mathfrak{a}\}\,.$$

Then using this description of the isotropy representation we can compute directly

$$\begin{aligned}
\widetilde{K}_0 = \Bigg\{ & \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right), \left(\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}, \begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}\right), \\
& \left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right), \left(\begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}, \begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}\right), \\
& \left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right), \left(\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}, \begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}\right), \\
& \left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right), \left(\begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}, \begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}\right)\Bigg\}.
\end{aligned}$$

In particular, the order of $\widetilde{K}_0$ is 8. This result is consistent with thoes of [3, p.611], [4, p.651] and [25, p.573] in the topology of transformation group theory. Moreover we obtain

$$\begin{aligned}
K_0 = \Bigg\{ & p\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = p\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right), \\
& p\left(\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}, \begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}\right) = p\left(\begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}, \begin{pmatrix}-\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}\right), \\
& p\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) = p\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right), \\
& p\left(\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}, \begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}\right) = p\left(\begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}, \begin{pmatrix}0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}\right)\Bigg\} \\
\cong\ & \mathbf{Z}_2 + \mathbf{Z}_2\,.
\end{aligned}$$

Hence the order of group K_0 is equal to 4 and

$$\widetilde{K}/\widetilde{K}_0 \cong K/K_0 = SO(4)/\mathbf{Z}_2 + \mathbf{Z}_2\,.$$

For each $l, m \in \mathbf{Z}$ with $l, m \geq 0$, the vector subspace of $V_l \otimes V_m$

$$(V_l \otimes V_m)_{\widetilde{K}_0} :=\{\xi \in V_l \otimes V_m \mid [(\rho_l \boxtimes \rho_m)(A, B)](\xi) = \xi \text{ for any } (A, B) \in \widetilde{K}_0 \}$$

can be described explicitly as follows:

LEMMA 4.4. *When $(l+m)/2$ is even,*

$$(V_l \otimes V_m)_{\widetilde{K}_0} = \left\{ \xi = \sum_{i+a:even} \xi_{i,a}(v_i^{(l)} \otimes v_a^{(m)} + v_{l-i}^{(l)} \otimes v_{m-a}^{(m)}) \mid \xi_{i,a} \in \mathbf{C} \right\}$$

and when $(l+m)/2$ is odd,

$$(V_l \otimes V_m)_{\widetilde{K}_0} = \left\{ \xi = \sum_{i+a:odd} \xi_{i,a}(v_i^{(l)} \otimes v_a^{(m)} - v_{l-i}^{(l)} \otimes v_{m-a}^{(m)}) \mid \xi_{i,a} \in \mathbf{C} \right\}.$$

Next we describe the subgroups of $\widetilde{K}$ defined as

$$\widetilde{K}_{\mathfrak{a}} :=\{(A, B) \in \widetilde{K} \mid [(\rho_3 \boxtimes \rho_1)(A, B)](\mathfrak{a}) = \mathfrak{a}\},$$
$$\widetilde{K}_{[\mathfrak{a}]} :=\{(A, B) \in \widetilde{K} \mid [(\rho_3 \boxtimes \rho_1)(A, B)](\mathfrak{a}) = \mathfrak{a} \text{ preserving the orientation of } \mathfrak{a}\} \subset \widetilde{K}_{\mathfrak{a}}.$$

For $(A, B) \in \widetilde{K} = SU(2) \times SU(2)$, we compute that $(A, B) \in \widetilde{K}_{\mathfrak{a}}$ if and only if (A, B) is one of the following elements:

$$\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta_1'} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1'} \end{pmatrix} \right),$$

where $\theta_1 = \frac{\pi}{4}k_1$, $\theta_1' = \frac{\pi}{4}k_1'$, $k_1, k_1' \in \mathbf{Z}$, $k_1 - k_1' \in 4\mathbf{Z}$,

$$\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2'} \\ e^{\sqrt{-1}\theta_2'} & 0 \end{pmatrix} \right),$$

where $\theta_2 = \frac{\pi}{4}k_2$, $\theta_2' = \frac{\pi}{4}k_2'$, $k_2, k_2' \in \mathbf{Z}$, $k_2 - k_2' \in 4\mathbf{Z}$,

$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1'} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2'} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2'} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1'} \end{pmatrix} \right)$$

where $\theta_1 = \frac{\pi}{4}k_1$, $\theta_2 = \frac{\pi}{4}k_2$, $\theta_1' = \frac{\pi}{4}k_1'$, $\theta_2' = \frac{\pi}{4}k_2'$ and $k_1, k_2, k_1', k_2' \in \mathbf{Z}$, $k_1 + k_2, k_1 - k_2, k_1' + k_2', k_1' - k_2' \in 2\mathbf{Z}$, $k_1 - k_1', k_2 - k_2' \in 4\mathbf{Z}$, $k_1 + k_2 - k_1' - k_2', k_1 - k_2 - k_1' + k_2' \in 8\mathbf{Z}$.

In particular, the order of $\widetilde{K}_{\mathfrak{a}}$ is equal to $16 + 16 + 32 + 32 = 96$.

Moreover, for $(A, B) \in \widetilde{K} = SU(2) \times SU(2)$, we have that $(A, B) \in \widetilde{K}_{[\mathfrak{a}]}$ if and only if (A, B) is one of the following elements:

$$\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta_1'} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1'} \end{pmatrix} \right)$$

where $\theta_1 = \frac{\pi}{4}k_1, \theta_1' = \frac{\pi}{4}k_1', k_1, k_1' \in 2\mathbf{Z}, k_1 - k_1' \in 4\mathbf{Z}$,

$$\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2'} \\ e^{\sqrt{-1}\theta_2'} & 0 \end{pmatrix}\right)$$

where $\theta_2 = \frac{\pi}{4}k_2, \theta_2' = \frac{\pi}{4}k_2', k_2, k_2' \in 2\mathbf{Z}, k_2 - k_2' \in 4\mathbf{Z}$,

$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1'} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2'} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2'} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1'} \end{pmatrix}\right)$$

where $\theta_1 = \frac{\pi}{4}k_1, \theta_2 = \frac{\pi}{4}k_2, \theta_1' = \frac{\pi}{4}k_1', \theta_2' = \frac{\pi}{4}k_2'$ and $k_1, k_2, k_1', k_2' \in 2\mathbf{Z}+1, k_1+k_2, k_1-k_2, k_1'+k_2', k_1'-k_2' \in 2\mathbf{Z}, k_1-k_1', k_2-k_2' \in 4\mathbf{Z}, k_1+k_2-(k_1'+k_2'), k_1-k_2-(k_1'-k_2') \in 8\mathbf{Z}$. In other words, $(A, B) \in \widetilde{K}_{[\mathfrak{a}]}$ if and only if (A, B) is one of the following elements:

$$\left(\begin{pmatrix} e^{\sqrt{-1}\theta_1} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} e^{\sqrt{-1}\theta_1'} & 0 \\ 0 & e^{-\sqrt{-1}\theta_1'} \end{pmatrix}\right),$$

where $\theta_1 = \frac{\pi}{2}l_1, \theta_1' = \frac{\pi}{2}l_1', l_1, l_1' \in \mathbf{Z}, l_1 - l_1' \in 2\mathbf{Z}$,

$$\left(\begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2} \\ e^{\sqrt{-1}\theta_2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -e^{-\sqrt{-1}\theta_2'} \\ e^{\sqrt{-1}\theta_2'} & 0 \end{pmatrix}\right),$$

where $\theta_2 = \frac{\pi}{2}l_2, \theta_2' = \frac{\pi}{2}l_2', l_2, l_2' \in \mathbf{Z}, l_2 - l_2' \in 2\mathbf{Z}$,

$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_1'} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_2'} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}\theta_2'} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}\theta_1'} \end{pmatrix}\right)$$

where $\theta_1 = \frac{\pi}{2}l_1 + \frac{\pi}{4}, \theta_2 = \frac{\pi}{2}l_2 + \frac{\pi}{4}, \theta_1' = \frac{\pi}{2}l_1' + \frac{\pi}{4}, \theta_2' = \frac{\pi}{2}l_2' + \frac{\pi}{4}, l_1, l_2, l_1', l_2' \in \mathbf{Z}, l_1 - l_1', l_2 - l_2' \in 2\mathbf{Z}, l_1 + l_2 - (l_1' + l_2'), l_1 - l_2 - (l_1' - l_2') \in 4\mathbf{Z}$. In particular, the order of $\widetilde{K}_{[\mathfrak{a}]}$ is equal to $8 + 8 + 16 + 16 = 48 = 8 \times 6 = \sharp\widetilde{K}_0 \times \sharp\mathbf{Z}_6$. Then we obtain

LEMMA 4.5. $\widetilde{K}_{[\mathfrak{a}]}/\widetilde{K}_0 \cong \mathbf{Z}_6$.

PROOF. We compute

$$\begin{aligned} A &= \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\sqrt{-1}(\frac{\pi}{2}l_1+\frac{\pi}{4})} & -\frac{1}{\sqrt{2}}e^{-\sqrt{-1}(\frac{\pi}{2}l_2+\frac{\pi}{4})} \\ \frac{1}{\sqrt{2}}e^{\sqrt{-1}(\frac{\pi}{2}l_2+\frac{\pi}{4})} & \frac{1}{\sqrt{2}}e^{-\sqrt{-1}(\frac{\pi}{2}l_1+\frac{\pi}{4})} \end{pmatrix}, \\ A^3 &= \begin{pmatrix} -\sqrt{2}\cos(\frac{\pi}{2}l_1 + \frac{\pi}{4}) & 0 \\ 0 & -\sqrt{2}\cos(\frac{\pi}{2}l_1 + \frac{\pi}{4}) \end{pmatrix} \\ &= \begin{cases} -\mathrm{I}_2 & \text{if } \ l_1 \equiv 0 \text{ or } 3 \pmod 4 \\ \mathrm{I}_2 & \text{if } \ l_1 \equiv 1 \text{ or } 2 \pmod 4 \end{cases}, \\ A^6 &= \mathrm{I}_2\,. \end{aligned}$$

TABLE 4. Small eigenvalues of $\mathcal{C}_L$ on $\tilde{K}/\tilde{K}_0$.

(l, m)	$\dim(V_l \otimes V_m)_{\tilde{K}_0}$	eigenvalues of $\mathcal{C}_L$	$-\lambda \leq 6$
(1, 1)	1	-3	*
(2, 0)	0		
(0, 2)	0		
(3, 1)	2	$-3, -3$	*
(1, 3)	2	$-9, -9$	
(4, 0)	2	$-3, -3$	*
(0, 4)	2	$-15, -15$	*
(2, 2)	3	$-5, -5, -8$	*
(5, 1)	3	$-8, -5, -8$	*
(6, 0)	1	-6	*
(4, 2)	3	$-6, -9, -9$	*
(3, 3)	4	$-9, -12, -12, -15$	
(8, 0)	2	$-10, -10$	
(7, 1)	4	$-12, -12, -8, -8$	
(6, 2)	5	$-15, -12, -8, -8, -12$	

The generator of $\widetilde{K}_{[\mathfrak{a}]}/\widetilde{K}_0 \cong \mathbf{Z}_6$ is represented by the element

$$\left(\begin{pmatrix} \frac{1+\sqrt{-1}}{2} & -\frac{1-\sqrt{-1}}{2} \\ \frac{1+\sqrt{-1}}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1+\sqrt{-1}}{2} & \frac{1-\sqrt{-1}}{2} \\ -\frac{1+\sqrt{-1}}{2} & -\frac{1-\sqrt{-1}}{2} \end{pmatrix} \right). \tag{16}$$

□

Then using Lemmas 4.3 and 4.4 we can determine directly all eigenvalues of $\mathcal{C}_L$ on $\tilde{K}/\tilde{K}_0$ less than or equal to $\dim L = 6$ and corresponding representations of $\tilde{K}$ as in Table 4. Hence we get

$$\begin{aligned} &\{(l, m) \mid -c_L \leq 6 \text{ and } (V_l \otimes V_m)_{\tilde{K}_0} \neq \{0\}\} \\ =&\{(1, 1), (4, 0), (2, 2), (3, 1), (6, 0), (5, 1), (4, 2)\}. \end{aligned}$$

Using the generator (16) of $\widetilde{K}_{[\mathfrak{a}]}/\widetilde{K}_0 \cong \mathbf{Z}_6$, we compute that $(V_l \otimes V_m)_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ for $(l, m) = (1, 1), (4, 0), (3, 1), (5, 1)$ and $\dim_{\mathbf{C}}(V_l \otimes V_m)_{\tilde{K}_{[\mathfrak{a}]}} = 1$ for $(l, m) = (2, 2), (6, 0), (4, 2)$. But we observe that the fixed vector in $(V_2 \otimes V_2)_{\tilde{K}_{[\mathfrak{a}]}} \neq \{0\}$ corresponds to the larger eigenvalue $8 > 6$. Hence we obtain that the Gauss image $L^6 = \mathcal{G}\left(\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}\right) = \frac{SO(4)}{(\mathbf{Z}_2+\mathbf{Z}_2)\cdot\mathbf{Z}_6} \subset Q_6(\mathbf{C})$ is Hamiltonian stable.

Moreover from the above dimension computation we have

$$\begin{aligned} n(L^6) &= \dim_{\mathbf{C}} V_6 \boxtimes V_0 + \dim_{\mathbf{C}} V_4 \boxtimes V_2 = 7 \times 1 + 5 \times 3 = 7 + 15 = 22 \\ &= \dim SO(8) - \dim SO(4) = n_{hk}(L)\,. \end{aligned}$$

Thus the Gauss image $L^6 = \mathcal{G}\left(\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}\right) = \frac{SO(4)}{(\mathbf{Z}_2+\mathbf{Z}_2)\cdot\mathbf{Z}_6} \subset Q_6(\mathbf{C})$ is Hamiltonian rigid. From these results we conclude

THEOREM 4.6. *The Gauss image* $L^6 = \mathcal{G}\left(\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}\right) = \frac{SO(4)}{(\mathbf{Z}_2+\mathbf{Z}_2)\cdot\mathbf{Z}_6} \subset Q_6(\mathbf{C})$ *is strictly Hamiltonian stable.*

5. The case $(U, K) = (E_6, U(1)\cdot Spin(10))$**.** Let $U = E_6$ and $K = U(1)\cdot Spin(10)$. Then (U, K) is of type BC_2. First we settle our notations for a symmetric space of type $EIII$, following [21], [27], [10] and the references therein.

5.1. Cayley algebra. Let $\mathbf{K}$ be the real Cayley algebra. Let $\{c_0 = 1, c_1, \dots, c_7\}$ be the standard basis of $\mathbf{K}$ satisfying the following relations ([21]):

$$\begin{aligned} &c_i c_{i+1} = -c_{i+1} c_i = c_{i+3}\,, \quad c_{i+1} c_{i+3} = -c_{i+3} c_{i+1} = c_i\,, \\ &c_{i+3} c_i = -c_i c_{i+3} = c_{i+1}\,, \quad c_i^2 = -1 \text{ for } i = 1, \dots, 7\,. \end{aligned}$$

$\mathbf{K}$ is a noncommutative and nonassociative normed division algebra with the conjugation $x \mapsto \bar{x}$ and the canonical inner product (,) defined respectively by

$$\overline{x_0 + \sum_{i=1}^{7} x_i c_i} = x_0 - \sum_{i=1}^{7} x_i c_i\,, \qquad \left(\sum_{i=0}^{7} x_i c_i, \sum_{i=0}^{7} y_i c_i\right) = \sum_{i=0}^{7} x_i y_i\,.$$

We extend the conjugation and the inner product $\mathbf{C}$-linearly to the complexified algebra $\mathbf{K}^{\mathbf{C}}$ of $\mathbf{K}$ and denote them by the same notions $x \mapsto \bar{x}$ and (,) respectively. The automorphism group

$$G_2 := \{\alpha \in \mathrm{GL}_{\mathbf{R}}(\mathbf{K}) \mid \alpha(xy) = \alpha(x)\alpha(y),\ \forall x, y \in \mathbf{K}\}$$

of the Cayley algebra $\mathbf{K}$ is well-known to be a simply connected, connected compact Lie group of type G_2.

5.2. Exceptional Jordan algebra. The exceptional Jordan algebra $H_3(\mathbf{K})$ is defined as a real vector space

$$H_3(\mathbf{K}) = \{u \in M_3(\mathbf{K}) \mid \bar{u}^t = u\}\,,$$

equipped with the Jordan product

$$u \circ v = \frac{1}{2}(uv + vu) \ \text{ for } u, v \in H_3(\mathbf{K})\,.$$

$H_3(\mathbf{K})$ is of real dimension 27 and a typical element

$$u = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_i \in \mathbf{R}, x_i \in \mathbf{K} \tag{17}$$

of $H_3(\mathbf{K})$ is expressed as

$$u = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3 .$$

In $H_3(\mathbf{K})$, we define a trace $\mathrm{tr}(u)$ and an inner product (u, v) respectively by

$$\mathrm{tr}(u) = \xi_1 + \xi_2 + \xi_3, \qquad (u, v) := \mathrm{tr}(u \circ v)$$

for each $u, v \in H_3(\mathbf{K})$. Moreover, the Freudenthal product $u \times v$ is defined by

$$u \times v := \frac{1}{2}(2u \circ v - \mathrm{tr}(u)v - \mathrm{tr}(v)u + (\mathrm{tr}(u)\mathrm{tr}(v) - (u, v))\mathrm{I}_3) ,$$

where I_3 is the identity matrix of degree 3, and a trilinear form (u, v, w) and the determinant $\det u$ are defined respectively by

$$(u, v, w) = (u, v \times w) , \quad \det u = \frac{1}{3}(u, u, u) .$$

Set

$$SH_3(\mathbf{K}) = \{u \in M_3(\mathbf{K}) \mid \bar{u}^t = -u , \mathrm{tr}(u) = 0\} .$$

Each element $u \in SH_3(\mathbf{K})$ of the form

$$u = \begin{pmatrix} z_1 & x_3 & -\bar{x}_2 \\ -\bar{x}_3 & z_2 & x_1 \\ x_2 & -\bar{x}_1 & z_3 \end{pmatrix} , \quad z_i, x_i \in \mathbf{K}, \bar{z}_i = -z_i, \Sigma z_i = 0 \tag{18}$$

is expressed as

$$u = z_1 e_1 + z_2 e_2 + z_3 e_3 + x_1 \bar{u}_1 + x_2 \bar{u}_2 + x_3 \bar{u}_3 .$$

Now we define two injective linear maps $R : H_3(\mathbf{K}) \to \mathfrak{gl}(H_3(\mathbf{K}))$ and $D : SH_3(\mathbf{K}) \to \mathfrak{gl}(H_3(\mathbf{K}))$ by

$$\begin{aligned} R(u)v &= u \circ v = \frac{1}{2}(uv + vu) \quad \text{for} \quad u, v \in H_3(\mathbf{K}) , \\ D(u)v &= \frac{1}{2}[u, v] = \frac{1}{2}(uv - vu) \quad \text{for} \quad u \in SH_3(\mathbf{K}) , v \in H_3(\mathbf{K}) . \end{aligned} \tag{19}$$

Denote by $\mathfrak{D}$ and $\mathfrak{R}$ the images of D and R in $\mathfrak{gl}(H_3(\mathbf{K}))$, respectively. Introduce real vector subspaces of $\mathfrak{D}$ and $\mathfrak{R}$ as follows:

$$\begin{aligned} \mathfrak{D}_0 &= \{\delta \in \mathfrak{D} \mid \delta(e_i) = 0 \ \ (i = 1, 2, 3)\} , \\ \mathfrak{D}_i &= \{D(x\bar{u}_i) \mid x \in \mathbf{K}\} \ \ \text{for} \ \ i = 1, 2, 3 , \\ \mathfrak{R}_0 &= \left\{R\left(\sum \xi_i e_i\right) \mid \xi_i \in \mathbf{R}, \sum \xi_i = 0\right\} , \\ \mathfrak{R}_i &= \{R(xu_i) \mid x \in \mathbf{K}\} \ \ \text{for} \ \ i = 1, 2, 3 . \end{aligned}$$

Remark that $\dim \mathfrak{D}_0 = 28$, $\dim \mathfrak{D}_1 = \dim \mathfrak{D}_2 = \dim \mathfrak{D}_3 = 8$, $\dim \mathfrak{R}_0 = 2$ and $\dim \mathfrak{R}_1 = \dim \mathfrak{R}_2 = \dim \mathfrak{R}_3 = 8$. Moreover, we know that $\mathfrak{D}_0$ is a subalgebra of $\mathfrak{gl}(H_3(\mathbf{K}))$ generated by the set $\{D(\sum z_i e_i) \mid z_i \in \mathbf{K}, \bar{z}_i = -z_i, \sum z_i = 0\}$. In fact, $\mathfrak{D}_0$ is isomorphic to the Lie algebra $\mathfrak{o}(8)$ and its basis can be chosen as $\{D_{i,r} (1 \leq r \leq 7), D_{i,pq} (1 \leq p < q \leq 7)\}$ for

$i = 1, 2$ or 3 ([6], [10], [27]). We now explain in details by using Ise's notions ([10, p.82]). Set

$$D_{i,r} = D(c_r(-e_j + e_k)), \quad (1 \le i \le 3, 1 \le r \le 7),$$

and

$$D_{i,pq} = [D_{i,p}, D_{i,q}], \quad (1 \le i \le 3, 1 \le p, q \le 7), \tag{20}$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. Then from

$$D\left(\sum_{i=1}^{3} z_i e_i\right)(v) = \frac{1}{2} \sum_{\{i,j,k\}} (z_j x_i - x_i z_k) u_i ,$$

we obtain

$$\begin{cases} D_{i,r}(xu_i) = \begin{cases} -c_r u_i & \text{if } x = c_0 \\ c_0 u_i & \text{if } x = c_r \\ 0 & \text{if } x = c_q (q \neq r), \end{cases} \\ D_{i,r}(xu_j) = \frac{1}{2}(c_r x) u_j , \\ D_{i,r}(xu_k) = \frac{1}{2}(x c_r) u_k , \end{cases}$$

and

$$\begin{cases} D_{i,pq}(xu_i) = \begin{cases} c_q u_i & \text{if } x = c_p \\ -c_p u_i & \text{if } x = c_q \\ 0 & \text{if } x = c_r \ \ (r \le 0, \neq p, q), \end{cases} \\ D_{i,pq}(xu_j) = \frac{1}{2}\{c_p(c_q x)\} u_j , \\ D_{i,pq}(xu_k) = \frac{1}{2}\{(x c_q) c_p\} u_k . \end{cases}$$

These mean that every $D_{i,r}, D_{i,pq}$ leave $\mathfrak{T}_i = \{xu_i | x \in \mathbf{K}\}$ invariant $(1 \le i \le 3, 1 \le p, q, r \le 7)$ and identifying $\mathfrak{T}_i$ with $\mathbf{K}$, it is represented as a skew-symmetric matrix with respect to the basis $\{c_0, c_1, \dots, c_7\}$, that is, $D_{i,r} = E_{0r} - E_{r0}$ and $D_{i,pq} = E_{qp} - E_{pq}$, where E_{pq} denotes the 8×8 matrix with all 0-components except (p, q)-component, 1. Moreover,

$$[D_{i,r}, D_{i,pq}] = D_{i,p}\delta_{qr} - D_{i,q}\delta_{rp} , \tag{21}$$

$$[D_{i,pq}, D_{i,rs}] = D_{i,pr}\delta_{sq} + D_{i,qs}\delta_{pr} + D_{i,rq}\delta_{sp} + D_{i,sp}\delta_{rq} , \tag{22}$$

where $1 \le i \le 3$ and $1 \le p, q, r, s \le 7$. Particularly, we have

$$[D_{i,r}, D_{i,pq}] = 0 , \quad [D_{i,pq}, D_{i,rs}] = 0 ,$$

if p, q, r, s are all different from each other. Denote by $\mathfrak{D}_{i,0}$ a real linear space spanned by all $D_{i,r}, D_{i,pq}$ $(1 \le p, q, r \le 7)$. Then all $\mathfrak{D}_{i,0} (1 \le i \le 3)$ are isomorphic to each other, and they are isomorphic to the Lie algebra $\mathfrak{o}(8)$. We shall use $\mathfrak{D}_0 = \mathfrak{D}_{1,0}$ in the next argument.

Let

$$H_3(\mathbf{K})^{\mathbf{C}} := H_3(\mathbf{K}) + \sqrt{-1} H_3(\mathbf{K})$$

be the complexification of $H_3(\mathbf{K})$. Then there are two complex conjugations on $H_3(\mathbf{K})^{\mathbf{C}}$, namely,

$$\overline{u_1 + \sqrt{-1}u_2} = \bar{u}_1 + \sqrt{-1}\bar{u}_2\,, \quad \tau(u_1 + \sqrt{-1}u_2) = u_1 - \sqrt{-1}u_2\,,$$

where $u_1, u_2 \in H_3(\mathbf{K})$. Then $H_3(\mathbf{K})^{\mathbf{C}}$ is canonically identified with

$$H_3(\mathbf{K}^{\mathbf{C}}) = \{u \in M_3(\mathbf{K}^{\mathbf{C}}) \mid \bar{u}^t = u\}\,.$$

An element $u \in H_3(\mathbf{K}^{\mathbf{C}})$ of the form (17), with $\xi_i \in \mathbf{C}$, $x_i \in \mathbf{K}^{\mathbf{C}}$, is also expressed as $u = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3$. The standard Hermitian inner product $\langle\ ,\ \rangle$ of $H_3(\mathbf{K}^{\mathbf{C}})$ is defined by $\langle u, v\rangle := (\tau u, v)$. Meanwhile, the complexification $SH_3(\mathbf{K})^{\mathbf{C}}$ of $SH_3(\mathbf{K})$ is identified with

$$SH_3(\mathbf{K}^{\mathbf{C}}) = \{u \in M_3(\mathbf{K}^{\mathbf{C}}) \mid \bar{u}^t = -u, \operatorname{tr}(u) = 0\}\,,$$

whose element u of the form (18), with $z_i, x_i \in \mathbf{K}^{\mathbf{C}}$, is also expressed as $u = z_1 e_1 + z_2 e_2 + z_3 e_3 + x_1 \bar{u}_1 + x_2 \bar{u}_2 + x_3 \bar{u}_3$. Then $D(u) \in \mathfrak{gl}(H_3(\mathbf{K}^{\mathbf{C}}))$ for $u \in SH_3(\mathbf{K}^{\mathbf{C}})$ and $R(u) \in \mathfrak{gl}(H_3(\mathbf{K}^{\mathbf{C}}))$ for $u \in H_3(\mathbf{K}^{\mathbf{C}})$ can be defined in the same way as (19).

5.3. The groups F_4 and E_6. We shall use the setup in [27] and [10]. The automorphism group of the Jordan algebra $H_3(\mathbf{K})$

$$\begin{aligned} F_4 :=&\{\alpha \in GL(H_3(\mathbf{K})) \mid \alpha(u \circ v) = \alpha u \circ \alpha v\} \\ =&\{\alpha \in GL(H_3(\mathbf{K})) \mid \det(\alpha u) = \det u, (\alpha u, \alpha v) = (u, v)\} \end{aligned}$$

is known to be a connected, simply connected, compact Lie group of type F_4. Its Lie algebra $\mathfrak{f}_4$ is thus given by

$$\mathfrak{f}_4 := \{\delta \in \mathfrak{gl}(H_3(\mathbf{K})) \mid \delta(u \circ v) = \delta u \circ v + u \circ \delta v\}\,,$$

which is isomorphic to $\mathfrak{D} = \mathfrak{D}_0 + \mathfrak{D}_1 + \mathfrak{D}_2 + \mathfrak{D}_3$. We are interested in the following two subgroups of F_4:

$$(F_4)_{e_1} := \{\alpha \in F_4 \mid \alpha e_1 = e_1\} \cong Spin(9)\,,$$
$$(F_4)_{e_1,e_2,e_3} := \{\alpha \in F_4 \mid \alpha e_i = e_i\ (i = 1, 2, 3)\} \cong Spin(8)\,.$$

Note that the Lie algebra of $(F_4)_{e_1,e_2,e_3}$ is isomorphic to $\mathfrak{D}_0$.

The group G_2 can be realized as a subgroup of F_4. For each $\alpha \in G_2$, we define an $\mathbf{R}$-linear transformation $\tilde{\alpha}$ of $H_3(\mathbf{K})$ by

$$\tilde{\alpha}\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} := \begin{pmatrix} \xi_1 & \alpha x_3 & \overline{\alpha x_2} \\ \overline{\alpha x_3} & \xi_2 & \alpha x_1 \\ \alpha x_2 & \overline{\alpha x_1} & \xi_3 \end{pmatrix},$$

for $x_i \in \mathbf{K}$ ($i = 1, 2, 3$). Then $\tilde{\alpha} \in F_4$. Identifying α with $\tilde{\alpha}$, we consider G_2 as a subgroup of F_4.

The groups E_6 and $E_6^{\mathbf{C}}$ defined by

$$E_6 := \{\alpha \in GL_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid \det(\alpha u) = \det u, \langle\alpha u, \alpha v\rangle = \langle u, v\rangle\}\,,$$
$$E_6^{\mathbf{C}} := \{\alpha \in GL_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid \det(\alpha u) = \det(u)\}\,,$$

are known to be a connected, simply connected, compact Lie group of type E_6 and its complexification, respectively. Hence F_4 is a compact subgroup of E_6. The Lie algebras of E_6 and $E_6^{\mathbf{C}}$ are given respectively by

$$\mathfrak{e}_6 := \{\phi \in \mathfrak{gl}_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid (\phi u, u, u) = 0, \langle \phi u, v\rangle + \langle u, \phi v\rangle = 0\},$$
$$\mathfrak{e}_6^{\mathbf{C}} := \{\phi \in \mathfrak{gl}_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid (\phi u, u, u) = 0\}.$$

We know ([27, p.68]) that any element $\phi \in \mathfrak{e}_6^{\mathbf{C}}$ is uniquely expressed as

$$\phi = \delta + \varsigma, \quad \delta \in \mathfrak{D}^{\mathbf{C}}, \quad \varsigma \in \mathfrak{R}^{\mathbf{C}},$$

where $\mathfrak{D}^{\mathbf{C}}$ and $\mathfrak{R}^{\mathbf{C}}$ denote the complexifications of $\mathfrak{D}$ and $\mathfrak{R}$ respectively. So we get the so-called Chevalley-Schafer model ([6]) of $\mathfrak{e}_6^{\mathbf{C}}$: $\mathfrak{e}_6^{\mathbf{C}} = \mathfrak{D}^{\mathbf{C}} + \mathfrak{R}^{\mathbf{C}}$ as a subalgebra of $\mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$. The inclusion $\phi : \mathfrak{e}_6^{\mathbf{C}} \subset \mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$ is a 27-dimensional irreducible representation of $\mathfrak{e}_6^{\mathbf{C}}$. Moreover, any element $\phi \in \mathfrak{e}_6$ is uniquely expressed as

$$\phi = \delta + \sqrt{-1}\varsigma, \quad \delta \in \mathfrak{D}, \quad \varsigma \in \mathfrak{R}.$$

Thus we have the direct sum decomposition $\mathfrak{e}_6 = \mathfrak{D} + \sqrt{-1}\mathfrak{R}$.

5.4. Basic formulas in $\mathfrak{e}_6$.

LEMMA 5.1. *For $v = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 + x_2 u_2 + x_3 u_3 \in H_3(\mathbf{K}^{\mathbf{C}})$, we have*

$$R\big(\sum \eta_l e_l\big)v = \eta_1\xi_1 e_1 + \eta_2\xi_2 e_2 + \eta_3\xi_3 e_3 + \frac{1}{2}(\eta_2+\eta_3)x_1u_1 + \frac{1}{2}(\eta_3+\eta_1)x_2u_2 + \frac{1}{2}(\eta_1+\eta_2)x_3u_3,$$
$$D\big(\sum z_l e_l\big)v = \frac{1}{2}(z_2x_1 - x_1z_3)u_1 + \frac{1}{2}(z_3x_2 - x_2z_1)u_2 + \frac{1}{2}(z_1x_3 - x_3z_2)u_3,$$
$$D(x\bar{u}_i)v = (x, x_i)(e_j - e_k) + \frac{1}{2}(\xi_k - \xi_j)xu_i - \frac{1}{2}(\bar{x}\bar{x}_k)u_j + \frac{1}{2}(\bar{x}_j\bar{x})u_k,$$
$$R(xu_i)v = (x, x_i)(e_j + e_k) + \frac{1}{2}(\xi_j + \xi_k)xu_i + \frac{1}{2}\bar{x}\bar{x}_k u_j + \frac{1}{2}\bar{x}_j\bar{x}u_k,$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$.

The relations (20), (21), (22) and the following list give commutation rules for $\mathfrak{e}_6^{\mathbf{C}}$. Here, $x, y, z_i \in \mathbf{K}^{\mathbf{C}}$, $\bar{z}_i = -z_i$ for $i = 1, 2, 3$, $\sum_i z_i = 0$, and $\xi_1, \xi_2, \xi_3 \in \mathbf{C}$ with $\sum_l \xi_l = 0$. In formulae (23) (31), (i, j, k) is a cyclic permutation of $(1, 2, 3)$. In formulae (33) and (34), $i = 1, 2, 3$.

(23) $$[R(xu_i), R(yu_j)] = -(1/2)D(\overline{xy}\,\bar{u}_k),$$

(24) $$\big[R(xu_i), D(yu_j)\big] = [D(x\bar{u}_i), R(yu_j)] = (1/2)R(\overline{xy}\,\bar{u}_k),$$

(25) $$\big[D(x\bar{u}_i), D(y\bar{u}_j)\big] = -(1/2)D(\overline{xy}\bar{u}_k),$$

(26) $$\big[D(x\bar{u}_i), R(y\bar{u}_i)\big] = (x, y)R(e_j - e_k),$$

(27) $$\Big[R\big(\sum \xi_l e_l\big), R(x\bar{u}_i)\Big] = (1/2)(\xi_j - \xi_k)D(x\bar{u}_i),$$

$$\left[R\left(\sum \xi_l e_l\right), D(x\bar{u}_i)\right] = (1/2)(\xi_j - \xi_k)R(x\bar{u}_i)\,, \tag{28}$$

$$\left[D\left(\sum z_l e_l\right), D(x\bar{u}_i)\right] = (1/2)D((z_j x - x z_k)\bar{u}_i)\,, \tag{29}$$

$$\left[D\left(\sum z_l e_l\right), R(x\bar{u}_i)\right] = (1/2)R((z_j x - x z_k)u_i)\,, \tag{30}$$

$$\begin{aligned}\left[R(xu_i), R(yu_i)\right] &= -[D(x\bar{u}_i), D(y\bar{u}_i)] \\ &= D\left(\left(\frac{y+\bar{y}}{2}\frac{x-\bar{x}}{2} - \frac{x+\bar{x}}{2}\frac{y-\bar{y}}{2}\right)(e_j - e_k)\right) \\ &\quad - \left[D\left(\frac{x-\bar{x}}{2}(e_j - e_k)\right), D\left(\frac{y-\bar{y}}{2}(e_j - e_k)\right)\right],\end{aligned} \tag{31}$$

$$\left[\mathfrak{R}_0^{\mathbf{C}}, \mathfrak{R}_0^{\mathbf{C}} + \mathfrak{D}_0^{\mathbf{C}}\right] = \{0\}\,, \tag{32}$$

$$\left[R(xu_i), [R(xu_i), R(yu_i)]\right] = R(((x,x)y - (x,y)x)u_i)\,, \tag{33}$$

$$\left[D(x\bar{u}_i), [D(x\bar{u}_i), D(y\bar{u}_i)]\right] = D(((x,y)x - (x,x)y)\bar{u}_i)\,. \tag{34}$$

We remark that the Killing-Cartan form B of $\mathfrak{e}_6^{\mathbf{C}}$ is given by ([10, p.88] or [27, p.74])

$$B(u,v) = 4\,\mathrm{tr}(uv), \tag{35}$$

for each $u, v \in \mathfrak{e}_6^{\mathbf{C}} \subset \mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$.

5.5. Realization of $E_6/(U(1)\cdot Spin(10))$. Consider a **C**-linear transformation σ of $H_3(\mathbf{K})^{\mathbf{C}}$ defined by

$$\sigma\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}.$$

Then $\sigma \in E_6$ and $\sigma^2 = 1$. σ induces an involutive automorphism of E_6 by $\alpha \mapsto \sigma\alpha\sigma$, which is also denoted by σ. In order to investigate the subgroup $(E_6)^\sigma$ of all fixed elements by σ,

$$(E_6)^\sigma = \{\alpha \in E_6 \mid \sigma\alpha = \alpha\}\,, \tag{36}$$

consider two subgroups

$$(E_6)_{e_1} = \{\alpha \in E_6 \mid \alpha e_1 = e_1\} \cong Spin(10)$$

and

$$U(1) = \{\phi(\theta) \in GL_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}}) \mid \theta = e^{\sqrt{-1}t/2}, t \in \mathbf{R}\}\,, \tag{37}$$

where $\phi(\theta) := \exp(t\sqrt{-1}R(2e_1 - e_2 - e_3)) \in GL_{\mathbf{C}}(H_3(\mathbf{K})^{\mathbf{C}})$ and

$$\phi(\theta)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \theta^4\xi_1 & \theta x_3 & \theta\bar{x}_2 \\ \theta\bar{x}_3 & \theta^{-2}\xi_2 & \theta^{-2}x_1 \\ \theta x_2 & \theta^{-2}\bar{x}_1 & \theta^{-2}\xi_3 \end{pmatrix}. \tag{38}$$

Here the subgroups $U(1)$ and $Spin(10)$ of $(E_6)^\sigma$ are elementwise commutative. Define a mapping

$$p : \tilde{K} = U(1) \times Spin(10) \ni (\theta, \alpha) \to \phi(\theta)\alpha \in K = (E)^\sigma\,,$$

which is a surjective Lie group homomorphism. Since

$$U(1) \cap Spin(10) = \{1 = \phi(1), \phi(-1), \phi(\sqrt{-1}), \phi(-\sqrt{-1})\},$$

we have $\mathrm{Ker}(p) = \{(1, \phi(1)), (-1, \phi(-1)), (\sqrt{-1}, \phi(-\sqrt{-1})), (-\sqrt{-1}, \phi(\sqrt{-1}))\}$, which is isomorphic to $\mathbf{Z}_4$. Thus

$$K = (E_6)^\sigma = \tilde{K}/\mathbf{Z}_4 = (U(1) \times Spin(10))/\mathbf{Z}_4 =: U(1) \cdot Spin(10),$$

and $U/K = E_6/(U(1) \cdot Spin(10))$. Correspondingly, we have

$$\mathfrak{k} = (\mathfrak{e}_6)_\sigma = \{\phi \in \mathfrak{e}_6 \mid \sigma_* \phi = \phi\} = (\mathfrak{e}_6)_{e_1} + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3).$$

Since for any $\phi \in \mathfrak{e}_6$ there exist $u \in SH_3(\mathbf{K})$ and $v \in H_3(\mathbf{K})$ such that

$$\phi e_1 = D(u)(e_1) + \sqrt{-1}R(v)(e_1),$$

it holds that $\phi e_1 = 0$ if and only if

$$u = z_1 e_1 + z_2 e_2 + z_3 e_3 + a_1 \bar{u}_1 \in SH_3(\mathbf{K}), \quad v = \xi_2 e_2 + \xi_3 e_3 + x_1 u_1 \in H_3(\mathbf{K}),$$

where $a_1, x_1 \in \mathbf{K}$, $\xi_2, \xi_3 \in \mathbf{R}$ with $\xi_2 + \xi_3 = 0$. Hence

$$\begin{aligned}(\mathfrak{e}_6)_{e_1} &:= \{\phi \in \mathfrak{e}_6 \mid \phi e_1 = 0\} \\ &= \mathfrak{D}_0 + \mathfrak{D}_1 + \mathbf{R}\sqrt{-1}R(e_2 - e_3) + \sqrt{-1}\mathfrak{R}_1 \cong \mathfrak{o}(10).\end{aligned}$$

Therefore, the Cartan decomposition of a compact simple Lie algebra $\mathfrak{u} = \mathfrak{e}_6$ of type EIII is given as

$$\begin{aligned}\mathfrak{u} &= \mathfrak{e}_6 = \mathfrak{D} + \sqrt{-1}\mathfrak{R}, \\ \mathfrak{k} &= (\mathfrak{e}_6)_\sigma = \mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_0 + \sqrt{-1}\mathfrak{R}_1, \\ \mathfrak{p} &= (\mathfrak{e})_{-\sigma} = \mathfrak{D}_2 + \mathfrak{D}_3 + \sqrt{-1}\mathfrak{R}_2 + \sqrt{-1}\mathfrak{R}_3,\end{aligned}$$

where $\mathfrak{k}$ is isomorphic to $\mathfrak{u}(1) + \mathfrak{o}(10)$,

$$[\mathfrak{k}, \mathfrak{k}] = \mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathbf{R}R(e_2 - e_3) + \sqrt{-1}\mathfrak{R}_1 = (\mathfrak{e}_6)_{e_1}$$

is isomorphic to $\mathfrak{o}(10)$ and the center of $\mathfrak{k}$ is spanned by

$$Z = \sqrt{-1}R(2e_1 - e_2 - e_3).$$

On the other hand, a compact Hermitian symmetric space of type EIII can be defined by ([1, p. 74–75])

$$\mathrm{EIII} = \{u \in H_3(\mathbf{K})^{\mathbf{C}} \mid u \times u = 0, u \neq 0\}/\mathbf{C}^* \subset P(H_3(\mathbf{K})^{\mathbf{C}}),$$

which is considered as a compact complex submanifold embedded in a complex projective space $\mathbf{C}P^{26}$. Since E_6 acts transitively on EIII and the isotropy subgroup of E_6 at $o = [e_1]$ is $(E_6)^\sigma$, we know that $\mathrm{EIII} \cong E_6/(E_6)^\sigma = E_6/(U(1) \cdot Spin(10))$. Under the Hopf fibration $H_3(\mathbf{K})^{\mathbf{C}} \supset S^{53}(1) \ni e_1 \mapsto o = [e_1] \in P(H_3(\mathbf{K})^{\mathbf{C}})$, the tangent vector space $T_o(U/K)$ at o is linearly isomorphic to a vector subspace

$$\begin{aligned}T_o(\mathrm{EIII}) &\cong \{u \in H_3(\mathbf{K})^{\mathbf{C}} \mid u \times e_1 = 0, \langle u, e_1 \rangle = 0\} \\ &= \{x_2 u_2 + x_3 u_3 \mid x_2, x_3 \in \mathbf{K}^{\mathbf{C}}\}.\end{aligned}$$

The differential of the natural projection $p : U = E_6 \to U/K = \mathrm{EIII}$ induces a linear isomorphism $p_* : \mathfrak{p} \to T_o(\mathrm{EIII})$. Then $p_*(\phi) = \phi(e_1)$ and

$$p_*\big(2(D(x_2\bar{u}_2) - D(x_3\bar{u}_3)) + 2\sqrt{-1}(R(x_2'u_2) + R(x_3'u_3))\big) \\ = (x_2 + \sqrt{-1}x_2')u_2 + (x_3 + \sqrt{-1}x_3')u_3 \,. \tag{39}$$

5.6. Restricted root systems of EIII. Define $H_1, H_2 \in \mathfrak{p}$ by

$$H_1 = D\bar{u}_2 + \sqrt{-1}R(c_4u_2)\,,$$
$$H_2 = D\bar{u}_2 - \sqrt{-1}R(c_4u_2)\,.$$

Then by (26), $[H_1, H_2] = 0$. Hence

$$\mathfrak{a} = \{H(\xi_1, \xi_2) = \xi_1 H_1 + \xi_2 H_2 \mid \xi_1, \xi_2 \in \mathbf{R}\} \tag{40}$$

is a maximal abelian subalgebra in $\mathfrak{p}$. We remark that this maximal abelian subalgebra $\mathfrak{a}$ is different from ones given by M. Ise and used in [21]. Then by direct computations using (20)–(34), we get the following restricted root system decomposition of $\mathfrak{k}$ and $\mathfrak{p}$:

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_{2\xi_1} + \mathfrak{k}_{2\xi_2} + \mathfrak{k}_{\xi_1+\xi_2} + \mathfrak{k}_{\xi_1-\xi_2} + \mathfrak{k}_{\xi_1} + \mathfrak{k}_{\xi_2}\,,$$
$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}_{2\xi_1} + \mathfrak{p}_{2\xi_2} + \mathfrak{p}_{\xi_1+\xi_2} + \mathfrak{p}_{\xi_1-\xi_2} + \mathfrak{p}_{\xi_1} + \mathfrak{p}_{\xi_2}\,,$$

where

$$\begin{aligned} \mathfrak{k}_0 &= \{X \in \mathfrak{k} \mid [X, H] = 0 \quad \text{for each } H \in \mathfrak{a}\}\,, \\ &= \mathrm{span}_{\mathbf{R}}\{\sqrt{-1}R(e_1 - 2e_2 + e_3)\} + \mathrm{span}_{\mathbf{R}}\{-D_{1,4} + D_{1,12}, D_{1,12} + D_{1,36}\,, \\ &\quad D_{1,36} + D_{1,57}, -D_{1,1} + D_{1,24}, -D_{1,2} - D_{1,14}, -D_{1,3} + D_{1,46}, -D_{1,5} - D_{1,47}\,, \\ &\quad -D_{1,6} + D_{1,34}, -D_{1,7} + D_{1,45}, D_{1,13} - D_{1,26}, D_{1,15} + D_{1,27}, D_{1,16} + D_{1,23}\,, \\ &\quad D_{1,17} - D_{1,25}, D_{1,35} - D_{1,67}, D_{1,37} - D_{1,56}\}\,, \end{aligned}$$

$$\mathfrak{k}_{2\xi_1} = \mathrm{span}_{\mathbf{R}}\left\{\frac{1}{2}(-D_{1,4} - D_{1,12} + D_{1,36} - D_{1,57}) + \sqrt{-1}R(e_3 - e_1)\right\},$$
$$\mathfrak{k}_{2\xi_2} = \mathrm{span}_{\mathbf{R}}\left\{\frac{1}{2}(D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57}) + \sqrt{-1}R(e_3 - e_1)\right\},$$

$$\begin{aligned} \mathfrak{k}_{\xi_1+\xi_2} = \mathrm{span}_{\mathbf{R}}\Big\{ &- D_{1,1} - D_{1,24} - D_{1,37} - D_{1,56} = 2D_{2,1}\,, \\ &- D_{1,2} + D_{1,14} - D_{1,35} - D_{1,67} = 2D_{2,2}\,, \\ &- D_{1,3} - D_{1,46} + D_{1,17} + D_{1,25} = 2D_{2,3}\,, \\ &- D_{1,5} + D_{1,47} + D_{1,16} - D_{1,23} = 2D_{2,5}\,, \\ &- D_{1,6} - D_{1,34} - D_{1,15} + D_{1,27} = 2D_{2,6}\,, \\ &- D_{1,7} - D_{1,45} - D_{1,13} - D_{1,26} = 2D_{2,7}\Big\}\,, \end{aligned}$$

$$\begin{aligned}\mathfrak{k}_{\xi_1-\xi_2} = \mathrm{span}_{\mathbf{R}}\Big\{ & -D_{1,1} - D_{1,24} + D_{1,37} + D_{1,56} = 2D_{2,24}\,,\\ & -D_{1,2} + D_{1,14} + D_{1,35} + D_{1,67} = 2D_{2,14}\,,\\ & -D_{1,3} - D_{1,46} - D_{1,25} - D_{1,17} = -2D_{2,46}\,,\\ & -D_{1,5} + D_{1,47} - D_{1,16} + D_{1,23} = 2D_{2,47}\,,\\ & -D_{1,6} - D_{1,34} + D_{1,15} - D_{1,27} = -2D_{2,34}\,,\\ & -D_{1,7} - D_{1,45} + D_{1,13} + D_{1,26} = -2D_{2,45}\Big\}\,,\end{aligned}$$

$$\begin{aligned}\mathfrak{k}_{\xi_1} = \mathrm{span}_{\mathbf{R}}\Big\{ & D(x_1\bar{u}_1) + \sqrt{-1}R(y_1u_1) \mid (x_1, y_1) = (1, c_4), (c_1, -c_2), (c_2, c_1)\,,\\ & (c_3, c_6), (c_4, -1), (c_5, -c_7), (c_6, -c_3), (c_7, c_5)\Big\}\,,\\ \mathfrak{k}_{\xi_2} = \mathrm{span}_{\mathbf{R}}\Big\{ & D(x_1\bar{u}_1) + \sqrt{-1}R(y_1u_1) \mid (x_1, y_1) = (1, -c_4), (c_1, c_2), (c_2, -c_1)\,,\\ & (c_3, -c_6), (c_4, 1), (c_5, c_7), (c_6, c_3), (c_7, -c_5)\Big\}\,,\end{aligned}$$

$$\begin{aligned}\mathfrak{p}_{2\xi_1} &= \mathrm{span}_{\mathbf{R}}\{D(c_4\bar{u}_2) - \sqrt{-1}Ru_2\}\,,\\ \mathfrak{p}_{2\xi_2} &= \mathrm{span}_{\mathbf{R}}\{D(c_4\bar{u}_2) + \sqrt{-1}Ru_2\}\,,\\ \mathfrak{p}_{\xi_1+\xi_2} &= \mathrm{span}_{\mathbf{R}}\{D(c_i\bar{u}_2),\ i = 1, 2, 3, 5, 6, 7\}\,,\\ \mathfrak{p}_{\xi_1-\xi_2} &= \mathrm{span}_{\mathbf{R}}\{\sqrt{-1}R(c_iu_2),\ i = 1, 2, 3, 5, 6, 7\}\,,\\ \mathfrak{p}_{\xi_1} &= \mathrm{span}_{\mathbf{R}}\{D(x_3\bar{u}_3) + \sqrt{-1}R(y_3u_3) \mid (x_3, y_3) = (1, c_4), (c_1, c_2), (c_2, c_1)\,,\\ &\qquad (c_3, c_6), (c_4, -1), (c_5, -c_7), (c_6, -c_3), (c_7, c_5)\}\,,\\ \mathfrak{p}_{\xi_2} &= \mathrm{span}_{\mathbf{R}}\{D(x_3\bar{u}_3) + \sqrt{-1}R(y_3u_3) \mid (x_3, y_3) = (1, -c_4), (c_1, -c_2), (c_2, c_1)\,,\\ &\qquad (c_3, c_6), (c_4, 1), (c_5, -c_7), (c_6, -c_3), (c_7, c_5)\}\,.\end{aligned}$$

Thus we see that

$$\begin{aligned}\mathfrak{k}_0 &= \mathfrak{k}_0' + \mathfrak{c}(\mathfrak{k}_0) = \mathfrak{k}_0' + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3) \cong \mathfrak{so}(6) + \mathbf{R}\,,\\ \mathfrak{k}_1 &:= \mathfrak{k}_0 + \mathfrak{k}_{2\xi_1} + \mathfrak{k}_{2\xi_2}\\ &= \mathfrak{k}_0' + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3) + \mathbf{R}(D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57})\\ &\quad + \mathbf{R}\sqrt{-1}R(e_3 - e_1)\\ &\cong \mathfrak{so}(6) + \mathbf{R} + \mathbf{R} + \mathbf{R}\,,\end{aligned}$$

$$\begin{aligned}
\mathfrak{k}_2 &:= \mathfrak{k}_1 + \mathfrak{k}_{\xi_1+\xi_2} + \mathfrak{k}_{\xi_1-\xi_2} = \mathfrak{D}_0 + \sqrt{-1}\mathfrak{R}_0 \\
&= \mathfrak{D}_0 + \mathbf{R}\sqrt{-1}R(e_1 - 2e_2 + e_3) + \mathbf{R}\sqrt{-1}R(e_3 - e_1) \\
&= \mathfrak{D}_0 + \mathbf{R}\sqrt{-1}R(e_2 - e_3) + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3) \\
&\cong \mathfrak{so}(8) + \mathbf{R} + \mathbf{R}, \\
\mathfrak{k} &:= \mathfrak{k}_2 + \mathfrak{k}_{\xi_1} + \mathfrak{k}_{\xi_2} = \mathfrak{D}_0 + \sqrt{-1}\mathfrak{R}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_1 \\
&= (\mathfrak{D}_0 + \mathfrak{D}_1 + \sqrt{-1}\mathfrak{R}_1 + \mathbf{R}\sqrt{-1}R(e_2 - e_3)) + \mathbf{R}\sqrt{-1}R(2e_1 - e_2 - e_3) \\
&= \mathfrak{k}' + \mathfrak{c}(\mathfrak{k}) \cong \mathfrak{so}(10) + \mathbf{R}.
\end{aligned}$$

Consider the subgroup

$$\tilde{K}_2 = U(1) \times Spin(2) \times Spin(8) \subset \tilde{K} = U(1) \times Spin(10),$$

where $U(1)$ is given by (37), $Spin(2) \subset Spin(10) \cong (E_6)_{e_1}$ is generated by

$$\alpha_{23}(t) := \exp(t\sqrt{-1}R(e_2 - e_3)) :$$

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & e^{\frac{t\sqrt{-1}}{2}} x_3 & e^{-\frac{t\sqrt{-1}}{2}} \bar{x}_2 \\ e^{\frac{t\sqrt{-1}}{2}} \bar{x}_3 & e^{t\sqrt{-1}} \xi_2 & x_1 \\ e^{-\frac{t\sqrt{-1}}{2}} x_2 & \bar{x}_1 & e^{-t\sqrt{-1}} \xi_3 \end{pmatrix},$$

and $Spin(8) = (E_6)_{e_1,e_2,e_3}$ whose Lie algebra is just $\mathfrak{D}_0$. Therefore,

$$Spin(2) \cap Spin(8) = \{\alpha_{23}(t) \mid e^{t\sqrt{-1}} = 1\} = \{\alpha_{23}(0), \alpha_{23}(2\pi)\}.$$

Then the natural projection

$$p_2 : Spin(2) \times Spin(8) \ni (\alpha_{23}(t), \beta) \mapsto \alpha_{23}(t)\beta \in K_2'$$

has the kernel

$$\begin{aligned}
\mathrm{Ker} p_2 &= \{(\alpha_{23}(t), \alpha_{23}(t)^{-1}) \mid t = 2k\pi, k \in \mathbf{Z}\} \\
&= \{(\alpha_{23}(0), \alpha_{23}(0)), (\alpha_{23}(2\pi), \alpha_{23}(2\pi))\} \cong \mathbf{Z}_2.
\end{aligned}$$

Hence $K_2' \cong (Spin(2) \times Spin(8))/\mathbf{Z}_2$.

On the other hand, we also have

$$\tilde{K}_2 = S^1 \times Spin(2) \times Spin(8),$$

where this S^1 is generated by

$$\exp(t\sqrt{-1}R(e_1 - 2e_2 + e_3)) :$$

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} e^{t\sqrt{-1}} \xi_1 & e^{-\frac{t\sqrt{-1}}{2}} x_3 & e^{t\sqrt{-1}} \bar{x}_2 \\ e^{-\frac{t\sqrt{-1}}{2}} \bar{x}_3 & e^{-2t\sqrt{-1}} \xi_2 & e^{-\frac{t\sqrt{-1}}{2}} x_1 \\ e^{t\sqrt{-1}} x_2 & e^{-\frac{t\sqrt{-1}}{2}} \bar{x}_1 & e^{t\sqrt{-1}} \xi_3 \end{pmatrix},$$

$Spin(2) \subset E_6$ is generated by

$$\alpha_{31}(t) := \exp(t\sqrt{-1}R(e_3 - e_1)) :$$

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} e^{-t\sqrt{-1}}\xi_1 & e^{-\frac{t\sqrt{-1}}{2}}x_3 & \bar{x}_2 \\ e^{-\frac{t\sqrt{-1}}{2}}\bar{x}_3 & \xi_2 & e^{\frac{t\sqrt{-1}}{2}}x_1 \\ x_2 & e^{\frac{t\sqrt{-1}}{2}}\bar{x}_1 & e^{t\sqrt{-1}}\xi_3 \end{pmatrix},$$

and $Spin(8) = (E_6)_{e_1,e_2,e_3}$. Here $Spin(2) \times Spin(8) \subset (E_6)_{e_2} \cong Spin(10)$. Similarly, here

$$Spin(2) \cap Spin(8) = \{\alpha_{31}(t) \mid e^{t\sqrt{-1}} = 1\} = \{\alpha_{31}(0), \alpha_{31}(2\pi)\}.$$

Then the natural projection

$$p_2' : Spin(2) \times Spin(8) \ni (\alpha_{31}(t), \beta) \mapsto \alpha_{31}(t)\beta \in K_2'$$

has the kernel

$$\begin{aligned} \mathrm{Ker} p_2' &= \{(\alpha_{31}(t), \alpha_{31}(t)^{-1}) \mid t = 2k\pi, k \in \mathbf{Z}\} \\ &= \{(\alpha_{31}(0), \alpha_{31}(0)), (\alpha_{31}(2\pi), \alpha_{31}(2\pi))\} \cong \mathbf{Z}_2 . \end{aligned}$$

Thus,

$$\begin{aligned} K_2 &= (S^1 \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4 , \\ Spin(2) \cdot Spin(8) &= (Spin(2) \times Spin(8))/\mathbf{Z}_2 . \end{aligned}$$

Furthermore, we have

$$Spin(8) \supset Spin(2) \cdot Spin(6) \cong (Spin(2) \times Spin(6))/\mathbf{Z}_2 ,$$

where

$$\begin{aligned} Spin(8) = \{(\alpha_1, \alpha_2, \alpha_3) \in SO(\mathbf{K}) \times SO(\mathbf{K}) \times SO(\mathbf{K}) \mid \\ (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})} \text{ for each } x, y \in \mathbf{K}\} \end{aligned}$$

acts on $H_3(\mathbf{K})$ by

$$(\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} := \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix},$$

$$Spin(2) := \{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(c_i) = c_i, \text{ if } i \neq 0, 4\}$$

is generated by $D_{1,4} + D_{1,12} - D_{1,36} + D_{1,57}$ and

$$Spin(6) := \{(\alpha_1, \alpha_2, \alpha_3) \in Spin(8) \mid \alpha_2(1) = 1, \alpha_2(c_4) = c_4\}$$

is generated by $\mathfrak{k}_0'$. Note that

$$Spin(2) \cap Spin(6) = \{(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id})\},$$

we see that $\mathbf{Z}_2 = \{((\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (\mathrm{Id}, \mathrm{Id}, \mathrm{Id})), ((-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}))\}$. Thus, a connected compact Lie subgroup K_1 of K generated by $\mathfrak{k}_1$ is

$$K_1 = (S^1 \times (Spin(2) \cdot (Spin(2) \cdot Spin(6))))/\mathbf{Z}_4 .$$

Moreover,

$$S^1 \cap Spin(6) = \{(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}) , (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id})\},$$

hence a connected compact Lie group K_0 of K generated by $\mathfrak{k}_0$ is

$$K_0 = (S^1 \times Spin(6))/\mathbf{Z}_2 ,$$

where $\mathbf{Z}_2 = \{((\mathrm{Id}, \mathrm{Id}, \mathrm{Id}), (\mathrm{Id}, \mathrm{Id}, \mathrm{Id})), ((-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}), (-\mathrm{Id}, \mathrm{Id}, -\mathrm{Id}))\}$.

5.7. Isotropy representation of $(E_6, U(1) \cdot Spin(10))$**.** Via the linear isomorphism $p_* : \mathfrak{p} \to T_o(\mathrm{EIII})$ given by (39), we can describe the isotropy representation of $(E_6, U(1) \cdot Spin(10))$.

LEMMA 5.2. (1) *For each* $a \in K$ *and each* $\xi \in \mathfrak{p}$,

$$p_*(\mathrm{Ad}(a)\xi) = (\mathrm{Ad}(a)\xi)(e_1) = (a \circ \xi \circ a^{-1})(e_1) .$$

(2) *For each* $T \in \mathfrak{k}$ *and each* $\xi \in \mathfrak{p}$,

$$p_*(\mathrm{ad}(T)\xi) = p_*([T, \xi]) = ([T, \xi])(e_1) .$$

The restriction $(\rho_K, V = H_3(\mathbf{K}^{\mathbf{C}}))$ of Chevally-Schafer's representation $(\tilde{\rho}, H_3(\mathbf{K}^{\mathbf{C}}))$ of E_6 to K can be decomposed into three irreducible representations

$$(\rho_K, V) = (\rho_1, V_1) \oplus (\rho_2, V_2) \oplus (\rho_3, V_3) ,$$

where V_1, V_2 and V_3 are given as follows:

$$\begin{aligned} V_1 &= \{\xi e_1 \mid \xi \in \mathbf{C}\} ,\\ V_2 &= (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma} = \{x_2 u_2 + x_3 u_3 \mid x_2, x_3 \in \mathbf{K}^{\mathbf{C}}\} \cong T_o(\mathrm{EIII}) ,\\ V_3 &= H_2(\mathbf{K}^{\mathbf{C}}) = \{\xi_2 e_2 + \xi_3 e_3 + x_1 u_1 \mid x_1 \in \mathbf{K}^{\mathbf{C}}, \xi_2, \xi_3 \in \mathbf{C}\} , \end{aligned}$$

and $V_1 \oplus V_3 = (H_3(\mathbf{K}^{\mathbf{C}}))_{\sigma}$. Note that ρ_1 is a scalar representation, the restriction of ρ_2 to $Spin(10)$ is equivalent to one of the half-spin representations of $Spin(10, \mathbf{C})$, denoted by $\triangle_{10}^{+}$, and the restriction of ρ_3 to $Spin(10)$ is equivalent to the standard representation of $Spin(10, \mathbf{C})$.

Now we discuss the linear isotropy action of an element $\phi(\theta) = \exp(t\sqrt{-1}R(2e_1 - e_2 - e_3)) : H_3(\mathbf{K}^{\mathbf{C}}) \to H_3(\mathbf{K}^{\mathbf{C}})$ generating the center $U(1)$ of K on both $\mathfrak{p}$ and $V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$, which are linearly isomorphic to $T_o(\mathrm{EIII})$. Using the formula (38) and Lemma

5.2, we compute

$$p_*(\mathrm{Ad}(\phi(\theta))D(x_2\bar{u}_2)) = \theta^{-3}p_*(D(x_2\bar{u}_2))\,,$$
$$p_*(\mathrm{Ad}(\phi(\theta))R(x_2u_2)) = \theta^{-3}p_*(R(x_2u_2))\,,$$
$$p_*(Ad(\phi(\theta))D(x_3\bar{u}_3)) = \theta^{-3}p_*(D(x_3\bar{u}_3))\,,$$
$$p_*(Ad(\phi(\theta))R(x_3u_3)) = \theta^{-3}p_*(R(x_3u_3))\,.$$

On the other hand, the tangent vector space $T_o(\mathrm{EIII})$ at $o = [e_1] \in \mathrm{EIII} \subset P(H_3(\mathbf{K})^{\mathbf{C}})$ is linearly isomorphic to a vector subspace $V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$, which is a horizontal vector subspace at a point e_1 under the Hopf fibration $H_3(\mathbf{K})^{\mathbf{C}} \supset S^{53}(1) \to P(H_3(\mathbf{K})^{\mathbf{C}})$. By the formula (38) we see that a vector $x_2u_2 + x_3u_3 \in (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$ at a point e_1 in a vector space $H_3(\mathbf{K})^{\mathbf{C}}$ representing a tangent vector of EIII at $o = [e_1]$ is moved by the linear action of $\phi(\theta)$ to a vector $\theta x_2u_2 + \theta x_3u_3 \in (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$ at $\theta^4 e_1$. Thus its corresponding tangent vector of EIII at $o = [e_1]$ must be $\theta^{-4}(\theta x_2u_2 + \theta x_3u_3) = \theta^{-3}(x_2u_2 + x_3u_3) \in V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$ at e_1. Hence the linear isotropy action of $\phi(\theta)$ on $V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$ is given by the multiplication by θ^{-3} on $V_2 = (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma}$. Therefore the linear isotropy representation of $(E_6, U(1)\cdot Spin(10))$ is $(\mu_3 \otimes_{\mathbf{C}} \triangle_{10}^{+})_{\mathbf{R}}$.

5.8. The subgroup $K_{[\mathfrak{a}]}$. The maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is described as follows:

$$\mathfrak{a} = \mathbf{R}H_1 \oplus \mathbf{R}H_2 = \mathbf{R}(D\bar{u}_2 + \sqrt{-1}R(\mathbf{c}_4u_2)) \oplus \mathbf{R}(D\bar{u}_2 - \sqrt{-1}R(\mathbf{c}_4u_2))$$

and

$$p_*(\mathfrak{a}) = \mathbf{R}(1+\sqrt{-1}\mathbf{c}_4)u_2 \oplus \mathbf{R}(1-\sqrt{-1}\mathbf{c}_4)u_2\,. \tag{41}$$

We shall use the map $\varphi : Sp(4) \to E_6$ given by Yokota ([27]) and known results for the case $(\check{U}, \check{K}) = (Sp(4), Sp(2)\times Sp(2))$ in order to describe a generator of $K_{[\mathfrak{a}]}$.

The Cayley algebra $\mathbf{K}$ naturally contains the field $\mathbf{H}$ of quaternions as

$$\mathbf{H} = \{x_0 + x_2c_2 + x_3c_3 + x_5c_5 | x_i \in \mathbf{R}\}\,.$$

Any element $x \in \mathbf{K}$ can be expressed by

$$\begin{aligned} x &= x_0 + x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 + x_5c_5 + x_6c_6 + x_7c_7 \\ &= (x_0 + x_2c_2 + x_3c_3 + x_5c_5) + (x_4 + x_1c_2 + x_6c_3 - x_7c_5)c_4 \\ &=: m + a\mathbf{e} \in \mathbf{H}\oplus\mathbf{He} = \mathbf{K}\,, \end{aligned}$$

where we set $m := x_0 + x_2c_2 + x_3c_3 + x_5c_5 \in \mathbf{H}$, $a := x_4 + x_1c_2 + x_6c_3 - x_7c_5 \in \mathbf{H}$ and $\mathbf{e} := c_4$. In $\mathbf{H}\oplus\mathbf{He}$, we define a multiplication by

$$(m + a\mathbf{e})(n + b\mathbf{e}) = (mn - \bar{b}a) + (a\bar{n} + bm)\mathbf{e}\,.$$

More explicitly,

$$(a\mathbf{e})n = (a\bar{n})\mathbf{e}\,,\quad m(b\mathbf{e}) = (bm)\mathbf{e}\,,\quad (a\mathbf{e})(b\mathbf{e}) = -\bar{b}a\,.$$

We can also define a conjugation and an $\mathbf{R}$-linear transformation γ on $\mathbf{H}\oplus\mathbf{He}$ respectively by

$$\overline{m+a\mathbf{e}}=\bar{m}-a\mathbf{e}\,,\qquad \gamma(m+a\mathbf{e})=m-a\mathbf{e}\,.$$

Thus $\gamma\in G_2\subset F_4$. Any element

$$X=\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}=\begin{pmatrix}\xi_1 & m_3 & \bar{m}_2\\ \bar{m}_3 & \xi_2 & m_1\\ m_2 & \bar{m}_1 & \xi_3\end{pmatrix}+\begin{pmatrix}0 & a_3\mathbf{e} & -a_2\mathbf{e}\\ -a_3\mathbf{e} & 0 & a_1\mathbf{e}\\ a_2\mathbf{e} & -a_1\mathbf{e} & 0\end{pmatrix},$$

of $H_3(\mathbf{K})$, where $x_i=m_i+a_i\mathbf{e}\in\mathbf{H}\oplus\mathbf{He}=\mathbf{K}$ and $\xi_i\in\mathbf{R}$, can be identified with an element

$$\begin{pmatrix}\xi_1 & m_3 & \bar{m}_2\\ \bar{m}_3 & \xi_2 & m_1\\ m_2 & \bar{m}_1 & \xi_3\end{pmatrix}+(a_1,a_2,a_3)$$

in $H_3(\mathbf{H})\oplus\mathbf{H}^3$. Hereafter, we often use an identification $H_3(\mathbf{K})\cong H_3(\mathbf{H})\oplus\mathbf{H}^3$.

Let the $\mathbf{C}$-linear mapping $\gamma : H_3(\mathbf{K}^{\mathbf{C}})\to H_3(\mathbf{K}^{\mathbf{C}})$ be the complexification of $\gamma\in G_2\subset F_4$. Then $\gamma\in E_6$ and $\gamma^2=1$. Recall that τ is the complex conjugation of $H_3(\mathbf{K}^{\mathbf{C}})$ with respect to $H_3(\mathbf{K})$. Consider an involutive complex conjugate linear transformation $\tau\gamma$ of $H_3(\mathbf{K}^{\mathbf{C}})$ and the following subgroup $(E_6)^{\tau\gamma}$ of E_6:

$$(E_6)^{\tau\gamma}=\{\alpha\in E_6\mid \tau\gamma\alpha=\alpha\tau\gamma\}\,.$$

Correspondingly, $H_3(\mathbf{K}^{\mathbf{C}})$ can be decomposed into the following two real vector subspaces:

$$H_3(\mathbf{K}^{\mathbf{C}})=(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma}\oplus(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma}\,,$$

where

$$\begin{aligned}
&(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma}:=\{X\in H_3(\mathbf{K}^{\mathbf{C}})\mid \tau\gamma X=X\}\\
&=\left\{\begin{pmatrix}\xi_1 & m_3 & \bar{m}_2\\ \bar{m}_3 & \xi_2 & m_1\\ m_2 & \bar{m}_1 & \xi_3\end{pmatrix}+\sqrt{-1}\begin{pmatrix}0 & a_3e & -a_2e\\ -a_3e & 0 & a_1e\\ a_2e & -a_1e & 0\end{pmatrix}\mid \xi_i\in\mathbf{R}, m_i, a_i\in\mathbf{H}\right\}\\
&=H_3(\mathbf{H})\oplus\sqrt{-1}\mathbf{H}^3\,,\\
&(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma}:=\{X\in H_3(\mathbf{K}^{\mathbf{C}})\mid \tau\gamma X=-X\}\\
&=\left\{\sqrt{-1}\begin{pmatrix}\xi_1 & m_3 & \bar{m}_2\\ \bar{m}_3 & \xi_2 & m_1\\ m_2 & \bar{m}_1 & \xi_3\end{pmatrix}+\begin{pmatrix}0 & a_3e & -a_2e\\ -a_3e & 0 & a_1e\\ a_2e & -a_1e & 0\end{pmatrix}\mid \xi_i\in\mathbf{R}, m_i, a_i\in\mathbf{H}\right\}\\
&=\sqrt{-1}H_3(\mathbf{H})\oplus\mathbf{H}^3\,.
\end{aligned}$$

In particular, $H_3(\mathbf{K}^{\mathbf{C}})=((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma})^{\mathbf{C}}$.

Let $H_4(\mathbf{H})_0:=\{P\in H_4(\mathbf{H})\mid \operatorname{tr}P=0\}$. Define a $\mathbf{C}$-linear isomorphism $g : H_3(\mathbf{K}^{\mathbf{C}})=H_3(\mathbf{H}^{\mathbf{C}})\oplus(\mathbf{H}^3)^{\mathbf{C}}\to H_4(\mathbf{H})_0^{\mathbf{C}}$ by

$$g(M+\mathbf{a}):=\begin{pmatrix}\frac{1}{2}\operatorname{tr}(M) & \sqrt{-1}\mathbf{a}\\ \sqrt{-1}\mathbf{a}^* & M-\frac{1}{2}\operatorname{tr}(M)\mathrm{I}\end{pmatrix},$$

for $M+\mathbf{a}\in H_3(\mathbf{K}^{\mathbf{C}})$. Then we have

$$g((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma}) = H_4(\mathbf{H})_0\,, \quad g((H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma}) = \sqrt{-1}H_4(\mathbf{H})_0\,.$$

The map $\varphi : Sp(4) \longrightarrow (E_6)^{\tau\gamma} \subset E_6$, defined by $\varphi(A)X := g^{-1}(A(gX)A^*)$ for each $X \in H_3(\mathbf{K}^{\mathbf{C}})$, is a surjective Lie group homomorphism and $\mathrm{Ker}(\varphi) = \{\mathrm{I}, -\mathrm{I}\} \cong \mathbf{Z}_2$. Therefore we obtain

$$Sp(4)/\mathbf{Z}_2 \cong (E_6)^{\tau\gamma}\,.$$

Consider real vector subspaces $(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,\sigma}, (H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma}$ of $(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma}$ and $(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma,\sigma}$, $(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma,-\sigma}$ of $(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma}$, which are eigenspaces of σ, respectively given by

$$\begin{aligned}(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,\sigma} &= \{X \in H_3(\mathbf{K}^{\mathbf{C}}) \mid \tau\gamma X = X, \sigma X = X\}\\ &= \left\{ \begin{pmatrix} \xi_1 & 0 & 0\\ 0 & \xi_2 & m_1\\ 0 & \bar{m}_1 & \xi_3 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & a_1\mathbf{e}\\ 0 & -a_1\mathbf{e} & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_1, a_1 \in \mathbf{H} \right\},\end{aligned}$$

$$\begin{aligned}(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma} &= \{X \in H_3(\mathbf{K}^{\mathbf{C}}) \mid \tau\gamma X = X, \sigma X = -X\}\\ &= \left\{ \begin{pmatrix} 0 & m_3 & \bar{m}_2\\ \bar{m}_3 & 0 & 0\\ m_2 & 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} 0 & a_3\mathbf{e} & -a_2\mathbf{e}\\ -a_3\mathbf{e} & 0 & 0\\ a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, m_3, a_2, a_3 \in \mathbf{H} \right\},\end{aligned}$$

$$\begin{aligned}(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma,\sigma} &= \{X \in H_3(\mathbf{K}^{\mathbf{C}}) \mid \tau\gamma X = -X, \sigma X = X\}\\ &= \left\{ \sqrt{-1}\begin{pmatrix} \xi_1 & 0 & 0\\ 0 & \xi_2 & m_1\\ 0 & \bar{m}_1 & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & a_1\mathbf{e}\\ 0 & -a_1\mathbf{e} & 0 \end{pmatrix} \mid \xi_i \in \mathbf{R}, m_1, a_1 \in \mathbf{H} \right\},\end{aligned}$$

$$\begin{aligned}(H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma,-\sigma} &= \{X \in H_3(\mathbf{K}^{\mathbf{C}}) \mid \tau\gamma X = -X, \sigma X = -X\}\\ &= \left\{ \sqrt{-1}\begin{pmatrix} 0 & m_3 & \bar{m}_2\\ \bar{m}_3 & 0 & 0\\ m_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_3\mathbf{e} & -a_2\mathbf{e}\\ -a_3\mathbf{e} & 0 & 0\\ a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, m_3, a_2, a_3 \in \mathbf{H} \right\}.\end{aligned}$$

Thus we have the following decompositions

$$\begin{aligned}(H_3(\mathbf{K}^{\mathbf{C}}))_{\sigma} &= (H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,\sigma} \oplus (H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma,\sigma}\,,\\ (H_3(\mathbf{K}^{\mathbf{C}}))_{-\sigma} &= (H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma} \oplus (H_3(\mathbf{K}^{\mathbf{C}}))_{-\tau\gamma,-\sigma}\,.\end{aligned}$$

Note that the images of $(H_3(\mathbf{K^C}))_{\tau\gamma,\sigma}$ and $(H_3(\mathbf{K^C}))_{\tau\gamma,-\sigma}$ of the homomorphism g defined above is expressed explicitly as follows:

$$g((H_3(\mathbf{K^C}))_{\tau\gamma,\sigma}) = \left\{ \begin{pmatrix} \frac{1}{2}(\xi_1+\xi_2+\xi_3) & -a_1 & 0 & 0 \\ -\bar{a}_1 & \frac{1}{2}(\xi_1-\xi_2-\xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1+\xi_2-\xi_3) & m_1 \\ 0 & 0 & \bar{m}_1 & \frac{1}{2}(-\xi_1-\xi_2+\xi_3) \end{pmatrix} \mid \xi_1,\xi_2,\xi_3 \in \mathbf{R}, a_1, m_1 \in \mathbf{H} \right\},$$

$$g((H_3(\mathbf{K^C}))_{\tau\gamma,-\sigma}) = \left\{ \begin{pmatrix} 0 & 0 & -a_2 & -a_3 \\ 0 & 0 & m_3 & \bar{m}_2 \\ -\bar{a}_2 & \bar{m}_3 & 0 & 0 \\ -\bar{a}_3 & m_2 & 0 & 0 \end{pmatrix} \mid a_2, a_3, m_2, m_3 \in \mathbf{H} \right\}.$$

For each element $A \in Sp(2) \times Sp(2) \subset Sp(4)$, we can check that $\varphi(A)\sigma = \sigma\varphi(A)$, hence $\varphi(A) \in (E_6)^\sigma$ and we have

$$\varphi : Sp(2) \times Sp(2) \longrightarrow (E_6)^{\tau\gamma,\sigma} \subset (E_6)^\sigma \cong U(1) \cdot Spin(10).$$

Next, the restriction of φ to the subgroup $Sp(1) \times Sp(1) \times Sp(1) \times Sp(1)$ gives

$$\varphi : Sp(1) \times Sp(1) \times Sp(1) \times Sp(1) \longrightarrow \{\alpha \in E_6 \mid \alpha(e_i) = e_i \ (i = 1, 2, 3)\} \cong Spin(8).$$

And the group $Sp(1) \times Sp(1)$ can be considered as the diagonal subgroup of $Sp(1) \times Sp(1) \times Sp(1) \times Sp(1)$, namely, each $(a, b) \in Sp(1) \times Sp(1)$ corresponds to $(a, b, a, b) \in Sp(1) \times Sp(1) \times Sp(1) \times Sp(1)$. Thus the restriction of φ to $Sp(1) \times Sp(1)$ is mapped to a subgroup $K_0 = S^1 \cdot Spin(6)$ of $K = E^\sigma = U(1) \cdot Spin(10)$. In fact, for a 2-dimensional real vector subspace

$$\tilde{\mathfrak{a}} := \left\{ \begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix} \mid a_2, m_2 \in \mathbf{R} \right\} \subset g((H_3(\mathbf{K^C}))_{\tau\gamma,-\sigma}),$$

it follows from

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a^* & 0 & 0 & 0 \\ 0 & b^* & 0 & 0 \\ 0 & 0 & a^* & 0 \\ 0 & 0 & 0 & b^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & m_2 \\ a_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}$$

that $\tilde{\mathfrak{a}}$ corresponds to the subspace

$$\left\{ \begin{pmatrix} 0 & 0 & m_2 - \sqrt{-1}a_2\mathbf{e} \\ 0 & 0 & 0 \\ m_2 + \sqrt{-1}a_2\mathbf{e} & 0 & 0 \end{pmatrix} \mid m_2, a_2 \in \mathbf{R} \right\} \subset (H_3(\mathbf{K})^{\mathbf{C}})_{\tau\gamma,-\sigma},$$

which corresponds to the image $p_*(\mathfrak{a})$ of the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ under the linear isomorphism p_* given by (41). It implies that φ maps the subgroup $\check{K}_0 = Sp(1) \times Sp(1)$ for the exceptional symmetric space $(E_6, Sp(4)/\mathbf{Z}_2)$ of type EI to the subgroup $K_0 = S^1 \cdot Spin(6)$ for the exceptional symmetric space $(E_6, U(1) \cdot Spin(10))$ of type EIII.

Recall that

$$\check{k} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \check{K}_{[\check{\mathfrak{a}}]} = (Sp(1) \times Sp(1)) \cdot \mathbf{Z}_4$$

is a generator of $\mathbf{Z}_4$. Its adjoint actions on $g((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,\sigma})$ and $g((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma})$ are given as

$$\check{k} \begin{pmatrix} \frac{1}{2}(\xi_1+\xi_2+\xi_3) & -a_1 & 0 & 0 \\ -\bar{a}_1 & \frac{1}{2}(\xi_1-\xi_2-\xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1+\xi_2-\xi_3) & m_1 \\ 0 & 0 & \bar{m}_1 & \frac{1}{2}(-\xi_1-\xi_2+\xi_3) \end{pmatrix} \check{k}^{-1}$$
$$= \begin{pmatrix} \frac{1}{2}(\xi_1-\xi_2-\xi_3) & -\bar{a}_1 & 0 & 0 \\ -a_1 & \frac{1}{2}(\xi_1+\xi_2+\xi_3) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(-\xi_1-\xi_2+\xi_3) & -\bar{m}_1 \\ 0 & 0 & -m_1 & -\frac{1}{2}(-\xi_1+\xi_2-\xi_3) \end{pmatrix},$$

$$\check{k} \begin{pmatrix} 0 & 0 & -a_2 & -a_3 \\ 0 & 0 & m_3 & \bar{m}_2 \\ -\bar{a}_2 & \bar{m}_3 & 0 & 0 \\ -\bar{a}_3 & m_2 & 0 & 0 \end{pmatrix} \check{k}^{-1} = \begin{pmatrix} 0 & 0 & -\bar{m}_2 & m_3 \\ 0 & 0 & a_3 & -a_2 \\ -m_2 & \bar{a}_3 & 0 & 0 \\ \bar{m}_3 & -\bar{a}_2 & 0 & 0 \end{pmatrix}.$$

Taking $(H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma} = (H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,\sigma} \oplus (H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma,-\sigma}$ and $H_3(\mathbf{K}^{\mathbf{C}}) = ((H_3(\mathbf{K}^{\mathbf{C}}))_{\tau\gamma})^{\mathbf{C}}$ into account, together with the above computation, we know that any element

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & m_3 + \sqrt{-1}a_3\mathbf{e} & \bar{m}_2 - \sqrt{-1}a_2\mathbf{e} \\ \bar{m}_3 - \sqrt{-1}a_3\mathbf{e} & \xi_3 & m_1 + \sqrt{-1}a_1\mathbf{e} \\ m_2 + \sqrt{-1}a_2\mathbf{e} & \bar{m}_1 - \sqrt{-1}a_1\mathbf{e} & \xi_3 \end{pmatrix}$$

in $H_3(\mathbf{K}^{\mathbf{C}})$ is mapped by the adjoint action of $\check{k}$ to an element

$$\begin{pmatrix} \xi_1 & a_3-\sqrt{-1}m_3\mathbf{e} & -a_2-\sqrt{-1}\bar{m}_2\mathbf{e} \\ \bar{a}_3+\sqrt{-1}m_3\mathbf{e} & -\xi_2 & -\bar{m}_1+\sqrt{-1}\bar{a}_1\mathbf{e} \\ -\bar{a}_2+\sqrt{-1}\bar{m}_2\mathbf{e} & -m_1-\sqrt{-1}\bar{a}_1\mathbf{e} & -\xi_3 \end{pmatrix}$$

$$= \begin{pmatrix} \xi_1 & \sqrt{-1}(-\sqrt{-1}a_3-m_3\mathbf{e}) & -\sqrt{-1}(-\sqrt{-1}a_2+\bar{m}_2\mathbf{e}) \\ \sqrt{-1}(-\sqrt{-1}\bar{a}_3+m_3\mathbf{e}) & -\xi_2 & -(\bar{m}_1+\sqrt{-1}\bar{a}_1\mathbf{e}) \\ -\sqrt{-1}(-\sqrt{-1}\bar{a}_2-\bar{m}_2\mathbf{e}) & -(m_1+\sqrt{-1}\bar{a}_1\mathbf{e}) & -\xi_3 \end{pmatrix}$$

$$=\alpha_{23}(\pi)\circ(\alpha_1,\alpha_2,\alpha_3)\left(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}\right),$$

where $\alpha_1, \alpha_2, \alpha_3 \in SO(\mathbf{K}) \cong SO(8)$ are defined by

$$\begin{aligned} \alpha_1(m_1+a_1\mathbf{e}) &:= -(\bar{m}_1-\bar{a}_1\mathbf{e}), \\ \alpha_2(m_2+a_2\mathbf{e}) &:= -\bar{a}_2-\bar{m}_2\mathbf{e}, \\ \alpha_3(m_3+a_3\mathbf{e}) &:= -a_3-m_3\mathbf{e}. \end{aligned} \tag{42}$$

By a simple computation, we have

$$\alpha_1(m_1+a_1\mathbf{e})\,\alpha_2(m_2+a_2\mathbf{e}) = \overline{\alpha_3(\overline{(m_1+a_1\mathbf{e})(m_2+a_2\mathbf{e})})}.$$

Hence, $(\alpha_1, \alpha_2, \alpha_3) \in Spin(8)$. Notice that

$$\alpha_{23}(\pi)(\alpha_1,\alpha_2,\alpha_3)(u_2) = \alpha_{23}(\alpha_2(u_2)) = \alpha_{23}(\pi)(-\mathbf{e}u_2) = \sqrt{-1}\mathbf{e}u_2,$$

$$\alpha_{23}(\pi)(\alpha_1,\alpha_2,\alpha_3)(\sqrt{-1}\mathbf{e}u_2) = \alpha_{23}(\pi)(\alpha_2(-\sqrt{-1}\mathbf{e}u_2)) = \alpha_{23}(\pi)(\sqrt{-1}u_2) = -u_2.$$

It follows that

$$\alpha_{23}(\pi)(\alpha_1,\alpha_2,\alpha_3) \in Spin(2)\cdot Spin(8) \subset (U(1)\times(Spin(2)\cdot Spin(8)))/\mathbf{Z}_4 = K_2$$

induces a linear isometry of the maximal abelian subspace $\mathfrak{a}$ of order 4 which is a $\pi/2$-rotation of $\mathfrak{a}$, we obtain

$$\alpha_{23}(\pi)(\alpha_1,\alpha_2,\alpha_3) \in K_{[\mathfrak{a}]}$$

and it is a generator of $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

5.9. Description of the Casimir operator. Define $\langle u, v\rangle_{\mathfrak{u}} := -\mathrm{tr}(uv)$ for each $u, v \in \mathfrak{e}_6 \subset \mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$. Now the positive restricted root system is $\Sigma^+(U,K) = \{2\xi_1, 2\xi_2, \xi_1+\xi_2, \xi_1-\xi_2, \xi_1, \xi_2\}$ and

$$H_{\xi_1} = \frac{1}{12}(D(\bar{u}_2)+\sqrt{-1}R(c_4u_2)), \quad H_{\xi_2} = \frac{1}{12}(D(\bar{u}_2)-\sqrt{-1}R(c_4u_2)).$$

The square lengths of the restricted roots with respect to $\langle\ ,\ \rangle_{\mathfrak{u}}$ are given by

$$\|\gamma\|_{\mathfrak{u}}^2 = \frac{1}{3}, \frac{1}{6} \quad \text{or} \quad \frac{1}{12}.$$

Then the Casimir operator $\mathcal{C}_L$ with respect to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_{30}(\mathbf{C})}$ can be expressed as

$$\mathcal{C}_L = 12\,\mathcal{C}_{K/K_0} - 6\,\mathcal{C}_{K_2/K_0} - 3\,\mathcal{C}_{K_1/K_0}, \tag{43}$$

where $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ are the Casimir operators of compact homogeneous spaces K/K_0, K_2/K_0 and K_1/K_0 relative to the K -invariant Riemannian metric induced from an inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ of E_6.

5.10. Descriptions of $D(K)$, $D(K_2)$, $D(K_1)$ and $D(K_0)$. A maximal torus $\tilde{T}^5$ of $Spin(10)$ can be given by

$$\begin{aligned}\tilde{T}^5 =&\{\tilde{t} = (\cos\theta_1 - e_1e_2\sin\theta_1)\cdot(\cos\theta_2 - e_3e_4\sin\theta_2)\cdot(\cos\theta_3 - e_5e_6\sin\theta_3)\\ &\cdot(\cos\theta_4 - e_7e_8\sin\theta_4)\cdot(\cos\theta_5 - e_9e_{10}\sin\theta_5) \mid \theta_i \in \mathbf{R}\ (i = 1,2,3,4,5)\}\,.\end{aligned}$$

Under the standard universal $\mathbf{Z}_2$-covering map $p : Spin(10) \to SO(10)$ defined by

$$(p(\alpha))\mathbf{x} := \alpha\cdot\mathbf{x}\cdot{}^t\alpha \in \mathbf{R}^{10} \subset Cl(\mathbf{R}^{10})$$

for each $\alpha \in Spin(10)$ and each $\mathbf{x} \in \mathbf{R}^{10}$, an element of the maximal torus $\tilde{T}^5$ of $Spin(10)$ is mapped to an element in the maximal torus T^5 of $SO(10)$ as

$$\begin{aligned}\tilde{T}^5 \ni&(\cos\theta_1 - e_1e_2\sin\theta_1)\cdot(\cos\theta_2 - e_3e_4\sin\theta_2)\cdot(\cos\theta_3 - e_5e_6\sin\theta_3)\\ &\cdot(\cos\theta_4 - e_7e_8\sin\theta_4)\cdot(\cos\theta_5 - e_9e_{10}\sin\theta_5)\\ &\longmapsto \begin{pmatrix}\begin{pmatrix}\cos 2\theta_1 & -\sin 2\theta_1\\ \sin 2\theta_1 & \cos 2\theta_1\end{pmatrix} & 0 & 0\\ 0 & \cdots & 0\\ 0 & 0 & \begin{pmatrix}\cos 2\theta_5 & -\sin 2\theta_5\\ \sin 2\theta_5 & \cos 2\theta_5\end{pmatrix}\end{pmatrix} \in T^5\,.\end{aligned}$$

Hence we have the exponential map as follows:

$$\begin{aligned}\exp : \tilde{\mathfrak{t}} =&\mathfrak{t} = \{(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \mid \theta_i \in \mathbf{R}\ (i = 1,2,3,4,5)\}\\ \longrightarrow \tilde{T} =&\{\,(\cos(\theta_1/2) - e_1e_2\sin(\theta_1/2))\cdot(\cos(\theta_2/2) - e_3e_4\sin(\theta_2/2))\\ &\cdot(\cos(\theta_3/2) - e_5e_6\sin(\theta_3/2))\cdot(\cos(\theta_4/2) - e_7e_8\sin(\theta_4/2))\\ &\cdot(\cos(\theta_5/2) - e_9e_{10}\sin(\theta_5/2))\\ &\mid \theta_i \in \mathbf{R}\ (i = 1,2,3,4,5)\} \subset Spin(10)\,.\end{aligned}$$

Thus

$$\begin{aligned}\varGamma(Spin(10)) &= \{\xi = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \in \tilde{\mathfrak{t}} \mid \exp(\xi) = e\}\\ &=\{\xi = 2\pi\ (k_1,k_2,k_3,k_4,k_5) \mid k_i \in \mathbf{Z}\ (i = 1,2,3,4,5), \sum_{i=1}^{5} k_i \in 2\mathbf{Z}\} \subset \varGamma(SO(10))\,.\end{aligned}$$

Denote by y_i $(i = 1,\dots,5)$ a linear function $y_i : \tilde{\mathfrak{t}} \ni \tilde{t} \mapsto \theta_i \in \mathbf{R}$. Then

$$\begin{aligned}D(Spin(10)) =\Big\{&\varLambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^*\\ &\mid (p_1,\dots,p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \text{where } \varepsilon = 0 \text{ or } \frac{1}{2},\\ &p_1 \geq p_2 \geq p_3 \geq p_4 \geq |p_5|\Big\} \supset D(SO(10))\,.\end{aligned}$$

A maximal torus T_K of $K = (U(1) \times Spin(10))/\mathbf{Z}_4$ can be given as follows:

$$\begin{aligned} T_K = &\Big\{ \Big(e^{\sqrt{-1}\theta_0}, \left(\cos\frac{\theta_1}{2} - e_1e_2\sin\frac{\theta_1}{2}\right)\left(\cos\frac{\theta_2}{2} - e_3e_4\sin\frac{\theta_2}{2}\right) \\ &\left(\cos\frac{\theta_3}{2} - e_5e_6\sin\frac{\theta_3}{2}\right)\left(\cos\frac{\theta_4}{2} - e_7e_8\sin\frac{\theta_4}{2}\right)\left(\cos\frac{\theta_5}{2} - e_9e_{10}\sin\frac{\theta_5}{2}\right)\Big) \\ &\mid \theta_0, \ldots, \theta_5 \in \mathbf{R} \Big\}/\mathbf{Z}_4 , \end{aligned}$$

where $t_0 = 2\theta_0$, $t_1 = \theta_1$, $U(1) = \{\exp(t_0\sqrt{-1}R(2e_1 - e_2 - e_3)) \mid t_0 \in \mathbf{R}\}$, $Spin(2) = \{\exp(t_1\sqrt{-1}R(e_2 - e_3)) \mid t_1 \in \mathbf{R}\}$ and

$$\mathbf{Z}_4 := \{(1,1), (-1,-1), (\sqrt{-1}, -e_1e_2\cdots e_{10}), (-\sqrt{-1}, e_1e_2\cdots e_{10})\} .$$

The corresponding maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is

$$\mathfrak{t} = \{(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \mid \theta_i \in \mathbf{R}\ (i = 0, 1, 2, 3, 4, 5)\} .$$

Then

$$\begin{aligned} \Gamma(K) = &\Big\{ \xi = 2\pi\left(\frac{k_0}{2}, k_1, k_2, k_3, k_4, k_5\right) + \pi\varepsilon\left(\frac{1}{2}, 1, 1, 1, 1, 1\right) \\ &\qquad \mid k_0, k_1, k_2, k_3, k_4, k_5 \in \mathbf{Z},\ \varepsilon = 0 \text{ or } 1,\ \sum_{\alpha=0}^{5} k_\alpha \in 2\mathbf{Z} \Big\} , \\ D(K) = &D((U(1) \times Spin(10))/\mathbf{Z}_4) \\ = &\Big\{ \Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in \mathfrak{t}^* \\ &\mid \frac{1}{2}p_0 + p_1 + p_2 + p_3 + p_4 + p_5 \in 2\mathbf{Z},\ p_0 \in \mathbf{Z}, \\ &(p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}, \\ &p_1 \geq p_2 \geq p_3 \geq p_4 \geq |p_5| \Big\} . \end{aligned}$$

Since T_K is also a maximal torus of $K_2 = (U(1) \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4 \subset K$, $\Gamma(K_2) = \Gamma(K)$ and

$$\begin{aligned} D(K_2) =& D((U(1) \times Spin(2) \cdot Spin(8))/\mathbf{Z}_4) \\ =& \Big\{ \Lambda = p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + p_5 y_5 \in \mathfrak{t}^* \\ & \mid \frac{1}{2} p_0 + p_1 + p_2 + p_3 + p_4 + p_5 \in 2\mathbf{Z}\,,\ p_0 \in \mathbf{Z}\,, \\ & (p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1)\,,\ \varepsilon = 0 \text{ or } \frac{1}{2}\,, \\ & p_2 \geq p_3 \geq p_4 \geq |p_5| \Big\}. \end{aligned}$$

On the other hand, $K_2 = (S^1 \times (Spin(2) \cdot Spin(8)))/\mathbf{Z}_4$, where

$$\begin{aligned} S^1 &= \{\exp(\hat{t}_0 \sqrt{-1} R(-e_1 + 2e_2 - e_3)) \mid \hat{t}_0 \in \mathbf{R}\}\,, \\ Spin(2) &= \{\exp(\hat{t}_1 \sqrt{-1} R(e_3 - e_1)) \mid \hat{t}_1 \in \mathbf{R}\} \end{aligned}$$

and here $Spin(2) \cdot Spin(8) \subset (E_6)_{e_2} \cong Spin(10)$. Since

$$\begin{aligned} &\exp(t_0 \sqrt{-1} R(2e_1 - e_2 - e_3)) \cdot \exp(t_1 \sqrt{-1} R(e_2 - e_3)) \\ =& \exp\left(-\frac{t_0 - t_1}{2} \sqrt{-1} R(-e_1 + 2e_2 - e_3)\right) \cdot \exp\left(-\frac{3t_0 + t_1}{2} \sqrt{-1} R(e_3 - e_1)\right), \end{aligned}$$

one can take $\hat{t}_0 = -\frac{t_0 - t_1}{2}$, $\hat{t}_1 = -\frac{3t_0 + t_1}{2}$ such that the maximal torus $T_{K_2} = T_K$ of K_2 can also be described as

$$\begin{aligned} \hat{T}_{K_2} = T_{K_2} = T_K = \Big\{ \hat{t} = \Big(e^{\sqrt{-1}\hat{\theta}_0}, &\left(\cos\frac{\hat{\theta}_1}{2} - e_1 e_2 \sin\frac{\hat{\theta}_1}{2}\right)\left(\cos\frac{\hat{\theta}_2}{2} - e_3 e_4 \sin\frac{\hat{\theta}_2}{2}\right) \\ \left(\cos\frac{\hat{\theta}_3}{2} - e_5 e_6 \sin\frac{\hat{\theta}_3}{2}\right) &\left(\cos\frac{\hat{\theta}_4}{2} - e_7 e_8 \sin\frac{\hat{\theta}_4}{2}\right)\left(\cos\frac{\hat{\theta}_5}{2} - e_9 e_{10} \sin\frac{\hat{\theta}_5}{2}\right)\Big) \\ \mid \hat{\theta}_0, \ldots, \hat{\theta}_5 \in \mathbf{R} \Big\} / \mathbf{Z}_4\,, \end{aligned}$$

where $\hat{\theta}_0 = \hat{t}_0/2, \hat{\theta}_1 = \hat{t}_1$. Taking account of the triality of $Spin(8) = (E_6)_{e_1,e_2,e_3} \subset (E_6)_{e_1} \cong (E_6)_{e_2} \cong Spin(10)$, we choose a new basis $\hat{y}_i : \hat{t} \mapsto \hat{\theta}_i$ for $\mathfrak{t}^*$ satisfying

$$\begin{aligned} &\hat{y}_0 = -\frac{1}{2} y_0 + \frac{1}{4} y_1\,, \quad \hat{y}_1 = -3y_0 - \frac{1}{2} y_1\,, \quad \hat{y}_2 := \frac{1}{2}(y_2 + y_3 + y_4 + y_5)\,, \\ &\hat{y}_3 := \frac{1}{2}(y_2 + y_3 - y_4 - y_5)\,, \quad \hat{y}_4 := \frac{1}{2}(y_2 - y_3 + y_4 - y_5)\,, \\ &\hat{y}_5 := \frac{1}{2}(-y_2 + y_3 + y_4 - y_5)\,. \end{aligned}$$

Thus any $\Lambda = p_0y_0 + p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 + p_5y_5 \in D(K_2)$ can also be written as $\Lambda = \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5$, where

$$\hat{p}_0 = -\frac{1}{2}p_0 + 3p_1\,, \quad \hat{p}_1 = -\frac{1}{4}p_0 - \frac{1}{2}p_1\,, \quad \hat{p}_2 = \frac{1}{2}(p_2 + p_3 + p_4 + p_5)\,,$$
$$\hat{p}_3 = \frac{1}{2}(p_2 + p_3 - p_4 - p_5)\,, \quad \hat{p}_4 = \frac{1}{2}(p_2 - p_3 + p_4 - p_5)\,,$$
$$\hat{p}_5 = \frac{1}{2}(-p_2 + p_3 + p_4 - p_5)\,.$$

Thus $D(K_2)$ has the following another expression:

$$\begin{aligned} D(K_2) =& D((S^1 \times Spin(2) \cdot Spin(8))/\mathbf{Z}_4) \\ =& \Big\{ \Lambda = \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5 \in \mathfrak{t}^* \\ & \mid \frac{1}{2}\hat{p}_0 + \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{p}_5 \in 2\mathbf{Z},\ \hat{p}_0 \in \mathbf{Z}, \\ & (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}, \\ & \hat{p}_2 \geq \hat{p}_3 \geq \hat{p}_4 \geq |\hat{p}_5| \Big\}. \end{aligned}$$

Notice that the subgroup $K_1 = (S^1 \times (Spin(2) \cdot (Spin(2) \cdot Spin(6))))/\mathbf{Z}_4$ also has the same maximal torus $T_{K_1} = \hat{T}_{K_2} = T_{K_2} = T_K$ and the corresponding maximal abelian subalgebra $\mathfrak{t}_{\mathfrak{k}_1}$ of $\mathfrak{k}_1$ is

$$\mathfrak{t}_{\mathfrak{k}_1} = \hat{\mathfrak{t}}_{\mathfrak{k}_2} = \{(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5) \mid \hat{\theta}_i \in \mathbf{R}\ (i = 0,1,2,3,4,5)\} = \mathfrak{t}_{\mathfrak{k}_2} = \mathfrak{t}\,,$$

we get

$$\begin{aligned} D(K_1) =& \Big\{ \Lambda = \hat{p}_0\hat{y}_0 + \hat{p}_1\hat{y}_1 + \hat{p}_2\hat{y}_2 + \hat{p}_3\hat{y}_3 + \hat{p}_4\hat{y}_4 + \hat{p}_5\hat{y}_5 \in \mathfrak{t}^*_{\mathfrak{k}_1} = \mathfrak{t}^* \\ & \mid \frac{1}{2}\hat{p}_0 + \hat{p}_1 + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{p}_5 \in 2\mathbf{Z}\,,\ \hat{p}_0 \in \mathbf{Z}\,, \\ & (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1)\,,\ \varepsilon = 0 \text{ or } \frac{1}{2}\,, \\ & \hat{p}_3 \geq \hat{p}_4 \geq |\hat{p}_5| \Big\}. \end{aligned}$$

Finally, the maximal torus of $K_0 = (S^1 \times Spin(6))/\mathbf{Z}_2$ is given as follows:

$$\begin{aligned} T_{K_0} =& \Big\{ \Big(e^{\sqrt{-1}\hat{\theta}_0}, \Big(\cos\frac{\hat{\theta}_3}{2} - e_5e_6 \sin\frac{\hat{\theta}_3}{2}\Big)\Big(\cos\frac{\hat{\theta}_4}{2} - e_7e_8 \sin\frac{\hat{\theta}_4}{2}\Big) \\ & \Big(\cos\frac{\hat{\theta}_5}{2} - e_9e_{10} \sin\frac{\hat{\theta}_5}{2}\Big)\Big) \mid \hat{\theta}_i \in \mathbf{R}\ (i = 0,3,4,5) \Big\}/\mathbf{Z}_2 \ \subset \hat{T}_{K_2} = T_K \end{aligned}$$

and the corresponding maximal abelian subalgebra of $\mathfrak{k}_0$ is

$$\mathfrak{t}_{\mathfrak{k}_0} = \{(\hat\theta_0, 0, 0, \hat\theta_3, \hat\theta_4, \hat\theta_5) \mid \hat\theta_i \in \mathbf{R}\ (i = 0, 3, 4, 5)\} \subset \mathfrak{t}_{\mathfrak{k}_2} = \mathfrak{t}\,.$$

Then

$$\begin{aligned} D(K_0) = \Big\{ & \Lambda = \hat q_0 \hat y_0 + \hat q_3 \hat y_3 + \hat q_4 \hat y_4 + \hat q_5 \hat y_5 \in \mathfrak{t}^*_{\mathfrak{k}_0} \\ & \mid \frac{1}{2}\hat q_0 + \hat q_3 + \hat q_4 + \hat q_5 \in 2\mathbf{Z},\ \hat q_0 \in \mathbf{Z}\,, \\ & (\hat q_3, \hat q_4, \hat q_5) \in \mathbf{Z}^3 + \varepsilon(1,1,1)\,, \varepsilon = 0 \text{ or } \frac{1}{2}\,, \\ & \hat q_3 \geq \hat q_4 \geq |\hat q_5| \Big\}\,. \end{aligned}$$

5.11. Branching laws. Based on the branching laws of $(SO(2n+2), SO(2)\times SO(2n))$ obtained by Tsukamoto ([24]), we formulate the following branching laws.

LEMMA 5.3 (Branching Law of $(Spin(10), Spin(2)\cdot Spin(8))$). *For each*

$$\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \delta p_5 y_5 \in D(Spin(10))\,,$$

with $\delta = 1$ *or* -1 *and*

$$\begin{aligned} & (p_1, p_2, p_3, p_4, p_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}\,, \\ & p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0\,, \end{aligned}$$

V_Λ *contains an irreducible* $Spin(2)\cdot Spin(8)$*-module with the highest weight*

$$\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + \delta' q_5 y_5 \in D(Spin(2)\cdot Spin(8))$$

with $\delta' = 1$ *or* -1 *and*

$$\begin{aligned} & (q_1, q_2, q_3, q_4, q_5) \in \mathbf{Z}^5 + \varepsilon(1,1,1,1,1)\,,\ \varepsilon = 0 \text{ or } \frac{1}{2}\,, \\ & q_2 \geq q_3 \geq q_4 \geq q_5 \geq 0\,, \end{aligned}$$

if and only if Λ' *satisfies the following conditions:*

(1)

$$\begin{aligned} p_1 + 1 > q_2 > p_3 - 1\,, \\ p_2 + 1 > q_3 > p_4 - 1\,, \\ p_3 + 1 > q_4 > p_5 - 1\,, \\ p_4 + 1 > q_5 \geq 0\,. \end{aligned}$$

(2) *The coefficient of* X^{q_1} *in the following power series expansion in* X *of*

$$X^{\delta\delta'\ell_5}\left(\prod_{i=1}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}}\right)$$

does not vanish. Here

$$\ell_1 := p_1 - \max\{p_2, q_2\}\,,$$
$$\ell_2 := \min\{p_2, q_2\} - \max\{p_3, q_3\}\,,$$
$$\ell_3 := \min\{p_3, q_3\} - \max\{p_4, q_4\}\,,$$
$$\ell_4 := \min\{p_4, q_4\} - \max\{p_5, q_5\}\,,$$
$$\ell_5 := \min\{p_5, q_5\}\,.$$

Moreover its multiplicity is equal to the coefficient of X^{q_1}.

LEMMA 5.4 (Branching Law of $(Spin(8), Spin(2)\cdot Spin(6))$). *For each*

$$\Lambda = p_2 y_2 + p_3 y_3 + p_4 y_4 + \delta p_5 y_5 \in D(Spin(8))\,,$$

with $\delta = 1$ *or* -1 *and*

$$(p_2, p_3, p_4, p_5) \in \mathbf{Z}^4 + \varepsilon(1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}\,,$$
$$p_2 \geq p_3 \geq p_4 \geq p_5 \geq 0\,,$$

V_Λ *contains an irreducible* $Spin(2)\cdot Spin(6)$*-module with the highest weight*

$$\Lambda' = q_2 y_2 + q_3 y_3 + q_4 y_4 + \delta' p_5 y_5 \in D(Spin(2)\cdot Spin(6))$$

with $\delta' = 1$ *or* -1 *and*

$$(q_2, q_3, q_4, q_5) \in \mathbf{Z}^4 + \varepsilon(1,1,1,1),\ \varepsilon = 0 \text{ or } \frac{1}{2}\,,$$
$$q_3 \geq q_4 \geq q_5 \geq 0\,,$$

if and only if Λ' *satisfies the following conditions:*

(1)

$$p_2 + 1 > q_3 > p_4 - 1\,,$$
$$p_3 + 1 > q_4 > p_5 - 1\,,$$
$$p_4 + 1 > q_5 \geq 0\,.$$

(2) *The coefficient of* X^{q_2}

$$X^{\delta\delta'\ell_5}\left(\prod_{i=2}^{4}\frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}}\right)$$

does not vanish. Here

$$\ell_2 := p_2 - \max\{p_3, q_3\}\,,$$
$$\ell_3 := \min\{p_3, q_3\} - \max\{p_4, q_4\}\,,$$
$$\ell_4 := \min\{p_4, q_4\} - \max\{p_5, q_5\}\,,$$
$$\ell_5 := \min\{p_5, q_5\}\,.$$

Moreover its multiplicity is equal to the coefficient of X^{q_2}.

5.12. Description of $D(K, K_0)$. Let

$$\begin{aligned}
\Lambda &= p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \varepsilon p_5 y_5 \in D(K)\,,\\
\Lambda' &= p_0' y_0 + p_1' y_1 + p_2' y_2 + p_3' y_3 + p_4' y_4 + \varepsilon' p_5' y_5\\
&= \hat{p}_0' \hat{y}_0 + \hat{p}_1' \hat{y}_1 + \hat{p}_2' \hat{y}_2 + \hat{p}_3' \hat{y}_3 + \hat{p}_4' \hat{y}_4 + \hat{\varepsilon}' \hat{p}_5' \hat{y}_5 \in D(K_2)\,,\\
\Lambda'' &= \hat{p}_0'' \hat{y}_0 + \hat{p}_1'' \hat{y}_1 + \hat{p}_2'' \hat{y}_2 + \hat{p}_3'' \hat{y}_3 + \hat{p}_4'' \hat{y}_4 + \hat{\varepsilon}'' \hat{p}_5'' \hat{y}_5 \in D(K_1)\,,\\
\Lambda''' &= \hat{p}_0''' \hat{y}_0 + \hat{p}_3''' \hat{y}_3 + \hat{p}_4''' \hat{y}_4 + \hat{\varepsilon}''' \hat{p}_5''' \hat{y}_5 \in D(K_0)\,.
\end{aligned}$$

Assume that the corresponding representation spaces satisfy

$$V_\Lambda \supset W_{\Lambda'} \supset U_{\Lambda''} = U_{\Lambda'''} \neq \{0\}\,.$$

Suppose that $U_{\Lambda'''} \neq \{0\}$ is a trivial representation of K_0, that is, $\Lambda''' = 0$. Then we have

$$\hat{p}_0''' = \hat{p}_0'' = 0\,, \quad \hat{p}_3''' = \hat{p}_3'' = 0\,, \quad \hat{p}_4''' = \hat{p}_4'' = 0\,, \quad \hat{p}_5''' = \hat{p}_5'' = 0\,.$$

Thus $\Lambda'' = \hat{p}_1'' \hat{y}_1 + \hat{p}_2'' \hat{y}_2 \in D(K_1)$ with $\hat{p}_1''$, $\hat{p}_2'' \in \mathbf{Z}$, $\hat{p}_1'' + \hat{p}_2'' \in 2\mathbf{Z}$.

By the branching law of $(Spin(8), Spin(2)\cdot Spin(6))$, we get

$$\begin{aligned}
\hat{p}_2' &\geq \hat{p}_3'' = 0 \geq \hat{p}_4'\,,\\
\hat{p}_3' &\geq \hat{p}_4'' = 0 \geq \hat{p}_5'\,,\\
\hat{p}_4' &\geq \hat{p}_5'' = 0 \geq 0\,.
\end{aligned}$$

Thus $(\hat{p}_4', \hat{p}_5') = (0, 0)$ and $\hat{p}_2' \geq 0$, $\hat{p}_3' \geq 0$. It follows that

$$\begin{aligned}
\ell_2 &= \hat{p}_2' - \max\{\hat{p}_3', \hat{p}_3''\} = \hat{p}_2' - \max\{\hat{p}_3', 0\} = \hat{p}_2' - \hat{p}_3'\,,\\
\ell_3 &= \min\{\hat{p}_3', \hat{p}_3''\} - \max\{\hat{p}_4', \hat{p}_4''\} = \min\{\hat{p}_3', 0\} - \max\{0, 0\} = 0 - 0 = 0\,,\\
\ell_4 &= \min\{\hat{p}_4', \hat{p}_4''\} - \max\{\hat{p}_5', \hat{p}_5''\} = \min\{0, 0\} - \max\{0, 0\} = 0 - 0 = 0\,,\\
\ell_5 &= \min\{\hat{p}_5', \hat{p}_5''\} = \min\{0, 0\} = 0\,.
\end{aligned}$$

Then the coefficient of $X^{\hat{p}_2''}$ in the (finite) power series expansion in X

$$X^{\hat{\varepsilon}'\hat{\varepsilon}''\ell_5} \prod_{i=2}^{4} \frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}} = \frac{X^{\hat{p}_2'-\hat{p}_3'+1} - X^{-(\hat{p}_2'-\hat{p}_3')-1}}{X - X^{-1}}$$

is equal to its multiplicity. Hence we have

$$-(\hat{p}_2' - \hat{p}_3') \leq \hat{p}_2'' = \hat{p}_2' - \hat{p}_3' - 2i \leq \hat{p}_2' - \hat{p}_3'$$

for some $i \in \mathbf{Z}$ with $0 \leq i \leq \hat{p}_2' - \hat{p}_3'$. Moreover, $\hat{p}_0' = \hat{p}_0'' = 0$, $\hat{p}_1' = \hat{p}_1''$. Thus we get

$$\Lambda' = \hat{p}_1' \hat{y}_1 + \hat{p}_2' \hat{y}_2 + \hat{p}_3' \hat{y}_3 \in D(K_2)$$

with

$$\begin{aligned}
&\hat{p}_1' = \hat{p}_1''\,, \hat{p}_2', \hat{p}_3' \in \mathbf{Z}\,, \quad \hat{p}_1' + \hat{p}_2' + \hat{p}_3' \in 2\mathbf{Z}\,,\\
&-(\hat{p}_2' - \hat{p}_3') \leq \hat{p}_2'' = \hat{p}_2' - \hat{p}_3' - 2i \leq \hat{p}_2' - \hat{p}_3'
\end{aligned}$$

for some $i \in \mathbf{Z}$ with $0 \leq i \leq \hat{p}_2' - \hat{p}_3'$. Therefore,

$$\Lambda' = p_0' y_0 + p_1' y_1 + p_2' y_2 + p_3' y_3 + p_4' y_4 + \varepsilon' p_5' y_5 \in D(K_2)$$

with

$$\begin{aligned}
p'_0 &= -\frac{1}{2}\hat{p}'_0 - 3\hat{p}'_1 = -3\hat{p}'_1\,,\\
p'_1 &= \frac{1}{4}\hat{p}'_0 - \frac{1}{2}\hat{p}'_1 = -\frac{1}{2}\hat{p}'_1\,,\\
p'_2 &= \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3 + \hat{p}'_4 - \hat{\varepsilon}'\hat{p}'_5) = \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3)\,,\\
p'_3 &= \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3 - \hat{p}'_4 + \hat{\varepsilon}'\hat{p}'_5) = \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3)\,,\\
p'_4 &= \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3 + \hat{p}'_4 + \hat{\varepsilon}'\hat{p}'_5) = \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3)\,,\\
\varepsilon' p'_5 &= \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3 - \hat{p}'_4 - \hat{\varepsilon}'\hat{p}'_5) = \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3)\,.
\end{aligned}$$

In particular, $\varepsilon' = 1$, $p'_2 = p'_3 = \frac{1}{2}(\hat{p}'_2 + \hat{p}'_3)$, $p'_4 = p'_5 = \frac{1}{2}(\hat{p}'_2 - \hat{p}'_3)$. Then $p_0 = p'_0$ and by the branching laws of $(Spin(10), Spin(2)\cdot Spin(8))$, we get

$$\begin{aligned}
&p_1 \geq p'_2 \geq p_3\,, \quad p_2 \geq p'_3 = p'_2 \geq p_4\,,\\
&p_3 \geq p'_4 \geq p_5\,, \quad p_4 \geq p'_5 = p'_4 \geq 0\,.
\end{aligned}$$

Thus $p_1 \geq p_2 \geq p'_2 = p'_3 \geq p_3 \geq p_4 \geq p'_4 = p'_5 \geq p_5 \geq 0$. It follows that

$$\begin{aligned}
\ell_1 &= p_1 - \max\{p_2, p'_2\} = p_1 - p_2\,,\\
\ell_2 &= \min\{p_2, p'_2\} - \max\{p_3, p'_3\} = p'_2 - p'_3 = 0\,,\\
\ell_3 &= \min\{p_3, p'_3\} - \max\{p_4, p'_4\} = p_3 - p_4\,,\\
\ell_4 &= \min\{p_4, p'_4\} - \max\{p_5, p'_5\} = p'_4 - p'_5 = 0\,,\\
\ell_5 &= \min\{p_5, p'_5\} = p_5\,.
\end{aligned}$$

Then the coefficient of $X^{p'_1} = X^{-\frac{1}{2}\hat{p}'_1} = X^{-\frac{1}{2}\hat{p}''_1}$ in the (finite) power series expansion in X

$$\begin{aligned}
&X^{\varepsilon\varepsilon'\ell_5}\prod_{i=1}^{4}\frac{X^{\ell_i+1} - X^{-\ell_i-1}}{X - X^{-1}}\\
&=X^{\varepsilon\varepsilon' p_5}\frac{X^{p_1-p_2+1} - X^{-(p_1-p_2+1)}}{X - X^{-1}}\frac{X^{p_3-p_4+1} - X^{-(p_3-p_4+1)}}{X - X^{-1}}\\
&=X^{\varepsilon\varepsilon' p_5}\sum_{i=0}^{p_1-p_2}\sum_{j=0}^{p_3-p_4} X^{(p_1-p_2)+(p_3-p_4)-2(i+j)}
\end{aligned}$$

is equal to its multiplicity. Then we have $\Lambda = p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \varepsilon p_5 y_5 \in D(K, K_0)$ with $p_0 = p'_0 = -3\hat{p}'_1 = 6p'_1 \in 3\mathbf{Z}$.

5.13. Eigenvalue computation. Recall that the standard basis $\mathbf{e}_\alpha$ $(\alpha = 0, 1, \ldots, 5)$ of $\mathfrak{t} = \{(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \mid \theta_\alpha \in \mathbf{R}\}$ corresponds to $2\sqrt{-1}R(2e_1 - e_2 - e_3) \in \mathfrak{u}(1)$ and $\sqrt{-1}R(e_2 - e_3)$, $D_{1,4}$, $D_{1,12}$, $D_{1,36}$, $D_{1,57} \in spin(10)$, respectively. With respect to the inner product $\langle u, v\rangle_{\mathfrak{u}} = -\mathrm{tr} uv$ for $u, v \in \mathfrak{k} \subset \mathfrak{e}_6 \subset \mathfrak{gl}(H_3(\mathbf{K})^{\mathbf{C}})$,

$$\langle \mathbf{e}_0, \mathbf{e}_0 \rangle = 72\,, \quad \langle \mathbf{e}_i, \mathbf{e}_i \rangle = 6\,, \quad \langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle = 0$$

for $1 \le i \le 5$ and $0 \le \alpha \ne \beta \le 5$. It follows that the inner products of the dual bases $\{y_0, y_1, y_2, y_3, y_4, y_5\}$ of $\mathfrak{t}^*$ corresponding to $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ of $\mathfrak{t}$ are given by

$$\begin{aligned} &\langle y_\alpha, y_\beta \rangle = 0\,, \quad (0 \le \alpha \ne \beta \le 5)\,, \\ &\langle y_0, y_0 \rangle = \frac{1}{72}\,, \quad \langle y_i, y_j \rangle = \frac{1}{6}\,, \quad (1 \le i \ne j \le 5)\,. \end{aligned}$$

For

$$\begin{aligned} \Lambda &= p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \varepsilon p_5 y_5 \in D(K, K_0)\,, \\ \Lambda' &= p_0 y_0 + \frac{p_0}{6} y_1 + p_2' y_2 + p_2' y_3 + p_4' y_4 + p_4' y_5 \\ &= -\frac{p_0}{3}\hat{y}_1 + (p_2' + p_4')\hat{y}_2 + (p_2' - p_4')\hat{y}_3 \in D(K_2, K_0)\,, \\ \Lambda'' &= -\frac{p_0}{3}\hat{y}_1 + \hat{p}_2''\hat{y}_2 \in D(K_1, K_0)\,, \end{aligned}$$

the eigenvalue formulas of the Casimir operators $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$ and $\mathcal{C}_{K_1/K_0}$ with respect to the inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ are given respectively by

$$\begin{aligned} -c_\Lambda &= \frac{1}{72}p_0^2 + \frac{1}{6}\{(p_1+8)p_1 + (p_2+6)p_2 + (p_3+4)p_3 + (p_4+2)p_4 + (p_5)^2\}\,, \\ -c_{\Lambda'} &= \frac{1}{72}(p_0')^2 + \frac{1}{6}\{(p_1')^2 + (p_2'+6)p_2' + (p_3'+4)p_3' + (p_4'+2)p_4' + (p_5')^2\} \\ &= \frac{1}{72}(\hat{p}_0')^2 + \frac{1}{6}\{(\hat{p}_1')^2 + (\hat{p}_2'+6)\hat{p}_2' + (\hat{p}_3'+4)\hat{p}_3' + (\hat{p}_4'+2)\hat{p}_4' + (\hat{p}_5')^2\} \\ &= \frac{1}{72}(p_0)^2 + \frac{1}{6}\left\{\left(\frac{1}{6}p_0\right)^2 + (p_2'+6)p_2' + (p_2'+4)p_2' + (p_4'+2)p_4' + (p_4')^2\right\}\,, \\ -c_{\Lambda''} &= \frac{1}{72}(\hat{p}_0'')^2 + \frac{1}{6}\{(\hat{p}_1'')^2 + (\hat{p}_2'')^2 + (\hat{p}_3''+4)\hat{p}_3'' + (\hat{p}_4''+2)\hat{p}_4'' + (\hat{p}_5'')^2\} \\ &= \frac{1}{6}\left\{\left(\frac{1}{3}p_0\right)^2 + (\hat{p}_2'')^2\right\}\,. \end{aligned}$$

Then for each $\Lambda \in D(K, K_0)$, we have the following eigenvalue formulas

$$
\begin{aligned}
-c_L &= -12c_\Lambda + 6c_{\Lambda'} + 3c_{\Lambda''}\\
&= 2\{(p_1+8)p_1 + (p_2+6)p_2 + (p_3+4)p_3 + (p_4+2)p_4 + (p_5)^2\}\\
&\quad - \{(p_2'+6)p_2' + (p_2'+4)p_2' + (p_4'+2)p_4' + (p_4')^2\} - \frac{1}{2}(\hat{p}_2'')^2\\
&= 2(p_1+8)p_1 + 2((p_2)^2 - (p_2')^2) + 12p_2 - 10p_2' + 2(p_3)^2 + 8p_3\\
&\quad + 2((p_4)^2 - (p_4')^2) + 4p_4 - 2p_4' + 2(p_5)^2 - \frac{1}{2}(\hat{p}_2'')^2\\
&= 2(p_1+8)p_1 + 2((p_2)^2 - (p_2')^2) + 2p_2 + 10(p_2 - p_2') + 2(p_3)^2 + 8p_3\\
&\quad + 2((p_4)^2 - (p_4')^2) + 2p_4 + 2(p_4 - p_4') + 2(p_5)^2 - \frac{1}{2}(\hat{p}_2'')^2\\
&\geq 2(p_1+8)p_1 + 2p_2 + 2(p_3)^2 - \frac{1}{2}(\hat{p}_2'')^2 + 8p_3 + 2p_4 + 2(p_5)^2\\
&= 2(p_1+8)p_1 + 2p_2 + (2(p_5')^2 - \frac{1}{2}(\hat{p}_2'')^2) + 8p_3 + 2p_4 + 2(p_5)^2\\
&\geq 2(p_1+8)p_1 + 2p_2 + 8p_3 + 2p_4 + 2(p_5)^2\,,
\end{aligned}
\tag{44}
$$

where the equalities hold if and only if $p_2 = p_2'$, $p_4 = p_4'$, $2p_3 = 2p_4 = 2p_4' = 2p_5' = |\hat{p}_2''|$ since we have

$$
\begin{aligned}
&p_1 \geq p_2 \geq p_2' = p_3' \geq p_3 \geq p_4 \geq p_4' = p_5' \geq p_5 \geq 0\,,\\
&-2p_4' = -2p_5' = -(\hat{p}_2' - \hat{p}_3') \leq \hat{p}_2'' \leq \hat{p}_2' - \hat{p}_3' = 2p_5' = 2p_4'\,.
\end{aligned}
$$

Notice that if $p_1 = 0$, then $-c_L = 0$ and if $p_1 \geq 2$, then $-c_L \geq 40 > 30$. In case $p_1 = 3/2$, the possible $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) \in D(K, K_0)$ are

$$
\left(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right), \left(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right), \left(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}\right), \left(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right),
$$
$$
\left(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(p_0, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right), \left(p_0, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(p_0, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right),
$$
$$
\left(p_0, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(p_0, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right).
$$

In these cases, the eigenvalue of the Casimir operator $\mathcal{C}_L$ is given by

$$
\begin{aligned}
-c_L &\geq 2(p_1+8)p_1 + 2p_2 + 8p_3 + 2p_4 + 2(p_5)^2\\
&\geq 2\cdot\left(\frac{3}{2}+8\right)\cdot\frac{3}{2} + 2\cdot\frac{1}{2} + 8\cdot\frac{1}{2} + 2\cdot\frac{1}{2} + 2\cdot\left(\frac{1}{2}\right)^2\\
&= 35 > 30.
\end{aligned}
$$

Hence in order to determine the Hamiltonian stability, i.e., to compare the first eigenvalue $-c_L$ and 30, we have only to treat the cases when $p_1 = 1/2$ or 1.

TABLE 5. Small eigenvalues of $-\mathcal{C}_L$ for $L = U(1) \cdot Spin(10)/(S^1 \cdot Spin(6) \cdot \mathbf{Z}_4)$.

Λ	Λ'	Λ''	$-c_L$
$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, -1, 1, 0, 0, 0$	15
$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, -1, -1, 0, 0, 0$	15
$-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$-3, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 1, 1, 0, 0, 0$	15
$-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$-3, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$0, 1, -1, 0, 0, 0$	15
$6, 1, 0, 0, 0, 0$	$6, 1, 0, 0, 0, 0$	$0, -2, 0, 0, 0, 0$	18
$-6, 1, 0, 0, 0, 0$	$-6, -1, 0, 0, 0, 0$	$0, 2, 0, 0, 0, 0$	18
$0, 1, 1, 0, 0, 0$	$0, 0, 0, 0, 0, 0$	$0, 0, 0, 0, 0, 0$	32
$0, 1, 1, 0, 0, 0$	$0, 0, 1, 1, 0, 0$	$0, 0, 0, 0, 0, 0$	20
$6, 1, 1, 1, 0, 0$	$6, 1, 1, 1, 0, 0$	$0, -2, 0, 0, 0, 0$	30
$-6, 1, 1, 1, 0, 0$	$-6, -1, 1, 1, 0, 0$	$0, 2, 0, 0, 0, 0$	30
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 0, 0$	$0, 0, 0, 0, 0, 0$	36
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, 0, 0, 0, 0$	32
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, 2, 0, 0, 0$	30
$0, 1, 1, 1, 1, 0$	$0, 0, 1, 1, 1, 1$	$0, 0, -2, 0, 0, 0$	30
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, 2, 0, 0, 0$	32
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, -2, 0, 0, 0$	32
$6, 1, 1, 1, 1, 1$	$6, 1, 1, 1, 1, 1$	$0, -2, 0, 0, 0, 0$	34
$-6, 1, 1, 1, 1, -1$	$-6, -1, 1, 1, 1, 1$	$0, 2, 2, 0, 0, 0$	32
$-6, 1, 1, 1, 1, -1$	$-6, -1, 1, 1, 1, 1$	$0, 2, -2, 0, 0, 0$	32
$-6, 1, 1, 1, 1, -1$	$-6, -1, 1, 1, 1, 1$	$0, 2, 0, 0, 0, 0$	34

It follows from the description of $D(K, K_0)$ in Section 5.12 that the element in $D(K, K_0)$. when $p_1 = 1/2$ is given by

$$\left(p_0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \text{ or } \left(p_0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

and the element in $D(K, K_0)$ for $p_1 = 1$ is given by

$$(p_0, 1, 0, 0, 0, 0)\,, \quad (p_0, 1, 1, 0, 0, 0)\,, \quad (p_0, 1, 1, 1, 0, 0)\,,$$
$$(p_0, 1, 1, 1, 1, 0)\,, \quad (p_0, 1, 1, 1, 1, 1) \text{ or } (p_0, 1, 1, 1, 1, -1)\,.$$

Using the branching laws, the descriptions of $D(K_2, K_0)$, $D(K_1, K_0)$ in Section 5.12 and the eigenvalue formula (44), by direct computation we get the following small eigenvalues in Table 5. Here, $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) \in D(K, K_0)$, $\Lambda' = (p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) \in D(K_2, K_0)$ and $\Lambda'' = (\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) \in D(K_1, K_0)$. The next lemma follows from Table 5.

LEMMA 5.5. $\Lambda = p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + \varepsilon p_5 y_5 \in D(K, K_0)$ *has eigenvalue* $-c_L \leq 30$ *if and only if* $(p_0, p_1, p_2, p_3, p_4, p_5)$ *is one of*

$$\left\{0, \left(3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), (6,1,0,0,0,0), (-6,1,0,0,0,0)\right.$$
$$\left.(0,1,1,0,0,0), (6,1,1,1,0,0), (-6,1,1,1,0,0), (0,1,1,1,1,0)\right\}.$$

Since $\Lambda_1 = (3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ corresponds to the complexified isotropy representation of EIII and it is conjugate to $\Lambda_2 = (-3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, we see that $\Lambda_1, \Lambda_2 \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) = (6,1,0,0,0,0) \in D(K, K_0)$. Then by the branching laws we get $\Lambda' = 6y_0 + y_1 \in D(K_2, K_0)$, $\Lambda'' = -2\hat{y}_1 \in D(K_1, K_0)$ and $\Lambda''' = 0 \in D(K_0)$. Hence, the eigenvalue of the Casimir operator is $-c_L = 18 < 30$.

On the other hand,

$$V_\Lambda \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \mid \xi_2, \xi_3 \in \mathbf{C}, x_1 \in \mathbf{K}^{\mathbf{C}} \right\} \cong \mathbf{C}^{10}$$
$$\supset W_{\Lambda'} = U_{\Lambda''} = U_{\Lambda'''} = (V_\Lambda)_{K_0}$$

and $\rho_\Lambda = \mu_6 \boxtimes \sigma_{\mathbf{C}^{10}}$, where $\sigma_{\mathbf{C}^{10}}$ denotes the standard representation of $SO(10)$, and for each $\phi(\theta) \in U(1)$,

$$\mu_6(\phi(\theta)) \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} = \theta^{-6} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

where $\theta = e^{\sqrt{-1}t_0/2}$. Since for any $\exp(\hat{t}_0\sqrt{-1}R(e_1 - 2e_2 + e_3)) \in S^1 \subset K_0$,

$$\begin{aligned} &\exp(\hat{t}_0\sqrt{-1}R(e_1 - 2e_2 + e_3)) \\ &\quad = \exp\left(\hat{t}_0\frac{1}{2}\sqrt{-1}R(2e_1 - e_2 - e_3)\right)\exp\left(-\hat{t}_0\frac{3}{2}\sqrt{-1}R(e_2 - e_3)\right) \\ &\quad \in U(1)\cdot Spin(2) \subset K, \end{aligned}$$

we compute

$$
\begin{aligned}
&\rho_\Lambda(\exp(\hat{t}_0\sqrt{-1}R(e_1-2e_2+e_3)))\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}\\
=&(\mu_6\boxtimes\sigma_{\mathbf{C}^{10}})(\exp(\hat{t}_0\sqrt{-1}R(e_1-2e_2+e_3)))\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}\\
=&\mu_6\left(\exp\left(\hat{t}_0\frac{1}{2}\sqrt{-1}R(2e_1-e_2-e_3)\right)\right)\alpha_{23}\left(-\hat{t}_0\frac{3}{2}\right)\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}\\
=&(e^{\sqrt{-1}\frac{1}{2}\hat{t}_0\frac{1}{2}})^{-6}\begin{pmatrix}0&0&0\\0&e^{-\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_2&x_1\\0&\bar{x}_1&e^{\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_3\end{pmatrix}\\
=&e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}\begin{pmatrix}0&0&0\\0&e^{-\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_2&x_1\\0&\bar{x}_1&e^{\sqrt{-1}\hat{t}_0\frac{3}{2}}\xi_3\end{pmatrix}\\
=&\begin{pmatrix}0&0&0\\0&e^{-\sqrt{-1}3\hat{t}_0}\xi_2&e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}x_1\\0&e^{-\sqrt{-1}\frac{3}{2}\hat{t}_0}\bar{x}_1&\xi_3\end{pmatrix}.
\end{aligned}
$$

In particular,

$$
\rho_\Lambda(\exp(\hat{t}_0\sqrt{-1}R(e_1-2e_2+e_3)))\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_3\end{pmatrix}
$$

for each $\hat{t}_0\in\mathbf{R}$. Hence,

$$
(V_\Lambda)_{K_0}\cong\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_3\end{pmatrix}\mid\xi_3\in\mathbf{C}\right\}.
$$

But as a generator of $\mathbf{Z}_4$ of $K_{[\mathfrak{a}]}$, the action of $\alpha_{23}(\pi)(\alpha_1,\alpha_2,\alpha_3)\in K_{[\mathfrak{a}]}$ given by (42) is

$$
\begin{aligned}
&\rho_\Lambda(\alpha_{23}(\pi)(\alpha_1,\alpha_2,\alpha_3))\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_3\end{pmatrix}\\
=&(\alpha_{23}(\pi))\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-\xi_3\end{pmatrix}.
\end{aligned}
$$

Therefore $(V_\Lambda)_{K_{[\mathfrak{a}]}}=\{0\}$ and $\Lambda=6y_0+y_1\notin D(K,K_{[\mathfrak{a}]})$. Similarly, $\Lambda=-6y_0+y_1\notin D(K,K_{[\mathfrak{a}]})$.

Suppose $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) = (0, 1, 1, 0, 0, 0) \in D(K, K_0)$. Then by the branching laws we get

$$\Lambda' = (p'_0, p'_1, p'_2, p'_3, p'_4, p'_5) = (0, 0, 1, 1, 0, 0) \in D(K_2, K_0)\,,$$
$$\Lambda'' = (\hat{p}''_0, \hat{p}''_1, \hat{p}''_2, \hat{p}''_3, \hat{p}''_4, \hat{p}''_5) = (0, 0, 0, 0, 0, 0) \in D(K_1, K_0)\,.$$

Here $\rho'_{\Lambda'} = \mathrm{Id} \boxtimes \mathrm{Id} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{Spin(8)} = \mathrm{Id} \boxtimes \mathrm{Id} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{SO(8)} \in \mathcal{D}(K_2)$. Notice that $W_{\Lambda'} = \mathfrak{o}(8)^{\mathbf{C}} = \mathfrak{o}(2)^{\mathbf{C}} \oplus \mathfrak{o}(6)^{\mathbf{C}} \oplus M(2, 6; \mathbf{R})^{\mathbf{C}}$, and the subgroups $U(1)$ and $Spin(2)$ of $K_2 = (U(1) \times (Spin(2) \cdot Spin(8))/\mathbf{Z}_4$ act trivially on $\mathfrak{o}(8)^{\mathbf{C}}$. The subgroup $Spin(6)$ of $Spin(2) \cdot Spin(6)$ acts trivially on $\mathfrak{o}(2)^{\mathbf{C}}$, hence $(W_{\Lambda'})_{K_0} = \mathfrak{o}(2)^{\mathbf{C}}$. For $\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3) \in K_{[\mathfrak{a}]}$ a generator of $\mathbf{Z}_4$ given in (42), $\alpha_{23}(\pi)$ and $(\alpha_1, \alpha_2, \alpha_3)$ commute to each other. $\alpha_{23}(\pi) \in Spin(2)$ acts trivially on $\mathfrak{o}(2)^{\mathbf{C}}$. α_2 of $(\alpha_1, \alpha_2, \alpha_3)$ acts on $\mathbf{R}1 + \mathbf{Re}$ as $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ and preserves the vector subspace orthogonally complementary to $\mathbf{R}1 + \mathbf{Re}$ in $\mathbf{K} \cong \mathbf{R}^8$. Thus the $Spin(2)$-factor of $(\alpha_1, \alpha_2, \alpha_3)$ in $Spin(2) \cdot Spin(6)$ corresponds to $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right) \in O(2)$. Since its adjoint action on $\mathfrak{o}(2)^{\mathbf{C}}$ is $-\mathrm{Id}$, the adjoint action of $(\alpha_1, \alpha_2, \alpha_3) \in Spin(8)$ is not trivial on $\mathfrak{o}(2)^{\mathbf{C}}$. Hence $(W_{\Lambda'})_{K_{[\mathfrak{a}]}} = \{0\}$ and in particular we obtain $\Lambda = y_1 + y_2 \notin D(K, K_{[\mathfrak{a}]})$.

Suppose $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) = (6, 1, 1, 1, 0, 0) \in D(K, K_0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 120$. By the branching laws we get $\Lambda' = 6y_0 + y_1 + y_2 + y_3 = -2\hat{y}_1 + \hat{y}_2 + \hat{y}_3 \in D(K_2, K_0)$, $\Lambda'' = -2\hat{y}_1 \in D(K_1, K_0)$ and $\Lambda''' = 0 \in D(K_0)$. Hence, the eigenvalue of the Casimir operator is $-c_L = 30$.

On the other hand, $\rho'_{\Lambda'} = \mathrm{Id} \boxtimes \mu_{-2} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{Spin(8)} = \mathrm{Id} \boxtimes \mu_{-2} \boxtimes \mathrm{Ad}^{\mathbf{C}}_{SO(8)} \in \mathcal{D}(K_2)$. Here $W_{\Lambda'} = \mathfrak{o}(8)^{\mathbf{C}} = \mathfrak{o}(2)^{\mathbf{C}} \oplus \mathfrak{o}(6)^{\mathbf{C}} \oplus M(2, 6; \mathbf{R})^{\mathbf{C}}$. Same as the previous case, we get $(W_{\Lambda'})_{K_0} = \mathfrak{o}(2)^{\mathbf{C}}$. Notice that for the generator $\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)$ of $\mathbf{Z}_4$ in $K_{[\mathfrak{a}]}$ given by (42), the action of $\alpha_{23}(\pi) \in Spin(2)$ on $H_3(\mathbf{K}^{\mathbf{C}})$ is given by

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & -\sqrt{-1}\bar{x}_2 \\ \sqrt{-1}\bar{x}_3 & -\xi_2 & x_1 \\ -\sqrt{-1}x_2 & \bar{x}_1 & -\xi_3 \end{pmatrix}.$$

In particular, $\alpha_{23}(\pi)$ transforms u_2 to $-\sqrt{-1}u_2$ and $\mathbf{e}u_2$ to $-\sqrt{-1}\mathbf{e}u_2$, which says that $\alpha_{23}(\pi)$ acts on $\mathfrak{o}(2) \cong \mathbf{R}1 + \mathbf{Re}$ as the matrix multiplication by $\left(\begin{smallmatrix} -\sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{smallmatrix}\right)$. Thus $\mu_{-2}(\alpha_{23}(\pi))$ acts on $\mathfrak{o}(2) \cong \mathbf{R}1 + \mathbf{Re}$ is just the matrix multiplication by $-\mathrm{Id}$. On the other hand, α_2 of $(\alpha_1, \alpha_2, \alpha_3)$ acts on $\mathbf{R}1 + \mathbf{R}e$ as $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. Thus the $Spin(2)$-factor of $(\alpha_1, \alpha_2, \alpha_3)$ in $Spin(2) \cdot Spin(6)$ corresponds to $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right) \in O(2)$. Hence its adjoint action on $\mathfrak{o}(2)^{\mathbf{C}}$ is $-\mathrm{Id}$. Therefore, $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \mathfrak{o}(2)^{\mathbf{C}}$, i.e., $\Lambda = 6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]}) = \mathfrak{o}(2)^{\mathbf{C}}$. Thus $\Lambda = 6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1. Similarly, $\Lambda = -6y_0 + y_1 + y_2 + y_3 \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1.

Suppose $\Lambda = (p_0, p_1, p_2, p_3, p_4, p_5) = (0, 1, 1, 1, 1, 0) \in D(K, K_0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 210$. By Subsection 5.12, we describe explicitly $(V_\Lambda)_{K_0}$ as follows:

$$\begin{aligned}&(V_{\Lambda(0,1,1,1,1,0)})_{K_0}\\ &\quad = W_{\Lambda'_1(0,0,1,1,1,1)} \cap (V_\Lambda)_{K_0} \oplus W_{\Lambda'_2(0,0,1,1,0,0)} \cap (V_\Lambda)_{K_0}\\ &\quad = (U_{\Lambda''_1(0,0,0,0,0,0)} \oplus U_{\Lambda''_1(0,0,2,0,0,0)} \oplus U_{\Lambda''_1(0,0,-2,0,0,0)})_{K_0} \oplus U_{\Lambda''_2(0,0,0,0,0,0)} .\end{aligned}$$

Then the Casimir operator $-\mathcal{C}_L$ has eigenvalues $-c_L = 32, 30, 30$ or 36 along this decomposition.

On the other hand, $\Lambda'_1 = 2\hat{y}_2 \in D(K_2, K_0)$, $W_{\Lambda'_1} \cong S^2_0(\mathbf{C}^8) \cong S^2_0(\mathbf{K}^8)$ and

$$W_{\Lambda'_1} \cap (V_\Lambda)_{K_0} = U_{\Lambda''_1(0,0,0,0,0,0)} \oplus (U_{\Lambda''_1(0,0,2,0,0,0)})_{K_0} \oplus (U_{\Lambda''_1(0,0,-2,0,0,0)})_{K_0} .$$

Recall that $\{1, c_1, \ldots, c_7\}$ denote the standard basis of the Cayley algebra $\mathbf{K}$ and $\mathbf{e} := c_4$. Then

$$3(1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}) - (c_1 \cdot c_1 + c_2 \cdot c_2 + c_3 \cdot c_3 + c_5 \cdot c_5 + c_6 \cdot c_6 + c_7 \cdot c_7) \in S^2_0(\mathbf{K}^{\mathbf{C}}) .$$

For any $A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2)$, $A(1, \mathbf{e}) = (1, \mathbf{e})\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. Hence

$$\begin{aligned} A(1 \cdot 1) &= (\cos t 1 + \sin t \mathbf{e}) \cdot (\cos t 1 + \sin t \mathbf{e})\\ &= \cos^2 t (1 \cdot 1) + \sin^2 t (\mathbf{e} \cdot \mathbf{e}) + 2 \sin t \cos t (1 \cdot \mathbf{e}) ,\\ A(\mathbf{e} \cdot \mathbf{e}) &= (-\sin t 1 + \cos t \mathbf{e}) \cdot (-\sin t 1 + \cos t \mathbf{e})\\ &= \sin^2 t (1 \cdot 1) + \cos^2 t (\mathbf{e} \cdot \mathbf{e}) - 2 \sin t \cos t (1 \cdot \mathbf{e}) ,\\ A(1 \cdot \mathbf{e}) &= (\cos t 1 + \sin t \mathbf{e}) \cdot (-\sin t 1 + \cos t \mathbf{e})\\ &= -\frac{1}{2} \sin 2t (1 \cdot 1 - \mathbf{e} \cdot \mathbf{e}) + \cos 2t (1 \cdot \mathbf{e}) .\end{aligned}$$

In particular, $A(1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}) = 1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}$ and

$$\begin{aligned}&A(3(1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}) - (c_1 \cdot c_1 + c_2 \cdot c_2 + c_3 \cdot c_3 + c_5 \cdot c_5 + c_6 \cdot c_6 + c_7 \cdot c_7))\\ &\quad = 3(1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}) - (c_1 \cdot c_1 + c_2 \cdot c_2 + c_3 \cdot c_3 + c_5 \cdot c_5 + c_6 \cdot c_6 + c_7 \cdot c_7) .\end{aligned}$$

Thus, $3(1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}) - (c_1 \cdot c_1 + c_2 \cdot c_2 + c_3 \cdot c_3 + c_5 \cdot c_5 + c_6 \cdot c_6 + c_7 \cdot c_7) \in U_{\Lambda''(0,0,0,0,0,0)}$. On the other hand, $1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} - 2\sqrt{-1}(1 \cdot \mathbf{e})$, $1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} + 2\sqrt{-1}(1 \cdot \mathbf{e}) \in S^2_0(\mathbf{K}^{\mathbf{C}})$, and we see that

$$\begin{aligned} A(1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} - 2\sqrt{-1}1 \cdot \mathbf{e}) &= \mathbf{e}^{\sqrt{-1}2t}(1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} - 2\sqrt{-1}1 \cdot \mathbf{e}) ,\\ A(1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} + 2\sqrt{-1}1 \cdot \mathbf{e}) &= \mathbf{e}^{-\sqrt{-1}2t}(1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} + 2\sqrt{-1}1 \cdot \mathbf{e}) .\end{aligned}$$

Hence, $1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} - 2\sqrt{-1}1 \cdot \mathbf{e} \in U_{\Lambda''(0,0,2,0,0,0)}$, $1 \cdot 1 - e \cdot \mathbf{e} + 2\sqrt{-1}1 \cdot \mathbf{e} \in U_{\Lambda''(0,0,-2,0,0,0)}$. Therefore,

$$\begin{aligned}&(V_\Lambda)_{K_0} \cap W_{\Lambda'_1}\\ &\quad = \mathbf{C}(3(1 \cdot 1 + \mathbf{e} \cdot \mathbf{e}) - (c_1 \cdot c_1 + c_2 \cdot c_2 + c_3 \cdot c_3 + c_5 \cdot c_5 + c_6 \cdot c_6 + c_7 \cdot c_7))\\ &\qquad \oplus \mathbf{C}(1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} - 2\sqrt{-1}1 \cdot \mathbf{e}) \oplus \mathbf{C}(1 \cdot 1 - \mathbf{e} \cdot \mathbf{e} + 2\sqrt{-1}1 \cdot \mathbf{e}) .\end{aligned}$$

Since the action of the generator $\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(2\sqrt{-1}(1\cdot \mathbf{e})) = 2(\sqrt{-1}\mathbf{e}\cdot(-1)) = -2\sqrt{-1}(1\cdot\mathbf{e}),$$
$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(1\cdot 1 - \mathbf{e}\cdot\mathbf{e}) = 1\cdot 1 - \mathbf{e}\cdot\mathbf{e},$$
$$(\alpha_{23}(\pi)(\alpha_1, \alpha_2, \alpha_3))(1\cdot 1 + \mathbf{e}\cdot\mathbf{e}) = -(1\cdot 1 + \mathbf{e}\cdot\mathbf{e}),$$

we obtain

$$(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap W_{\Lambda'_1} = \mathbf{C}(1\cdot 1 - \mathbf{e}\cdot\mathbf{e}),$$

and thus $\Lambda = y_1 + y_2 + y_3 + y_4 \in D(K, K_{[\mathfrak{a}]})$, which has eigenvalue 30 of $-\mathcal{C}_L$ with the multiplicity 1. Therefore,

$$\begin{aligned} n(L^{30}) &= \dim_{\mathbf{C}} V_{(6,1,1,1,0,0)} + \dim_{\mathbf{C}} V_{(-6,1,1,1,0,0)} + \dim_{\mathbf{C}} V_{(0,1,1,1,0,0)} \\ &= 120 + 120 + 210 = 450 \\ &= \dim SO(32) - \dim U(1)\cdot Spin(10) = n_{kl}(L^{30}). \end{aligned}$$

Then we conclude that

THEOREM 5.6. *The Gauss image*

$$L^{30} = (U(1)\cdot Spin(10))/(S^1\cdot Spin(6)\cdot \mathbf{Z}_4) \subset Q_{30}(\mathbf{C})$$

is strictly Hamiltonian stable.

Combining with the results on Hamiltonian stabilities of Gauss images of all homogeneous isoparametric hypersurfaces, we obtain our main theorem.

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