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MÖBIUS INVARIANT ENERGIES AND AVERAGE LINKING WITH CIRCLES

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Abstract. We introduce a Möbius invariant energy associated to planar domains, as well as a generalization to space curves. This generalization is a Möbius version of Banchoff-Pohl's notion of area enclosed by a space curve. A relation with Gauss-Bonnet theorems for complete surfaces in hyperbolic space is also described.

1. Introduction.

For smooth knots $K \subset \mathbb{R}^3$ a functional called *Möbius energy* was introduced in [12] as the integral along K of a certain function $V(\cdot, K)$. This function, called r^{-2}*-renormalized potential*, was defined by

$$V(p,K) = \lim_{\varepsilon \to 0} \left(\int_{|q-p|>\varepsilon} \frac{dq}{|q-p|^2} - \frac{2}{\varepsilon} \right), \qquad p, q \in K. \tag{1}$$

This is an instance of a general procedure that can be used to *renormalize* diverging integrals (i.e., to associate them with finite values). First, one removes a certain ε-neighborhood of the set where the integrand blows up. Then one restricts the integration to the complement of this ε-neighborhood, and expands the result as a Laurent series in ε. The renormalized integral is then defined as the constant term of this series.

Similar energies have been considered, for instance taking different exponents in the denominator of (1). The Möbius energy however has the nice property of being invariant under conformal transformations of space (cf. [3]). Among recent investigations, let us mention the solution of a conjecture about the minimal energy of a link in [4].

A similar construction of Möbius invariant energy for closed surfaces is found in [1]. For surfaces with boundary, a Möbius invariant function in the style above has not been considered yet. Also, a similar Möbius invariant energy for domains $\Omega \subset \mathbb{R}^n$ is still not known. In this direction, a renormalized potential in the style of (1) is studied in [14]; for a regular domain $\Omega \subset \mathbb{R}^n$ and $\alpha \in (0, n)$ define

$$V^{(\alpha)}(p, \Omega) = \int_{\Omega} |q-p|^{\alpha-n} dq .$$

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This coincides up to a constant factor with the Riesz potential of the characteristic function of Ω. For $\alpha < 0$, the integral $\int_\Omega |q-p|^{\alpha-n}dq$ would diverge whenever $p \in \Omega$. Hence, the procedure described above was used in [14] to define a renormalized potential $V^{(\alpha)}(p,\Omega)$ for all α. These potentials appear in convexity theory under the name of *dual mixed volumes* or *dual Quermassintegrals* (cf. [11]). Unfortunately, for negative α the integral $\int_\Omega V^{(\alpha)}(p,\Omega)dp$ diverges, as $V^{(\alpha)}(p,\Omega)$ blows up when p approaches the boundary $\partial\Omega$.

In the present paper, we take $\alpha = -2$ and give a second renormalization to assign a finite value to the integral of $V^{(-2)}(\cdot,\Omega)$ in the case of planar domains $\Omega \subset \mathbb{R}^2$. The choice $\alpha = -2$ makes the resulting energy scale invariant. Our first main goal then is to show that it is actually Möbius invariant. This is done by means of a Gauss-Bonnet formula for complete surfaces in *hyperbolic space* obtained in [17].

Another basic tool in our work is the infinitesimal cross-ratio. This is a Möbius invariant complex-valued differential 2-form defined on the space of point pairs of $\mathbb{R}^2$. The real part of this 2-form also exists in higher dimensions. This real valued 2-form is uniquely characterized by invariance under Möbius transformations (cf. [9]). Following this line, we find natural extensions of our results to higher dimensions, not for domains but for space curves. Concretely, we find a new Möbius invariant functional for space curves, which can be described as a renormalization of the measure of the set of circles linked with the curve. Again, this functional appears in a Gauss-Bonnet formula for surfaces in hyperbolic space.

Besides connections to knot energies and hyperbolic geometry, our results may be interesting from the viewpoint of *integral geometry*. Indeed, owing to divergence problems, almost nothing is known about integral geometry under the Möbius group (an exception is [10]). Here, the use of renormalizations allows us to extend some results of euclidean integral geometry to Möbius geometry. For instance, our functional for space curves can be seen as a Möbius invariant version of Banchoff-Pohl's notion of the area enclosed by a space curve (cf. [5]).

Next we sketch our results briefly. For a planar region $\Omega \subset \mathbb{R}^2$ we consider the renormalized potential (cf. Definition 3.1)

$$V(w,\Omega) = \lim_{\varepsilon\to 0}\left(\int_{\Omega\setminus B_\varepsilon(w)} \frac{d^2z}{|z-w|^4} - \frac{\pi}{\varepsilon^2}\right),$$

where d^2z is the area element of $\mathbb{R}^2$, and $B_\varepsilon(w) = \{z\colon |z-w| \le \varepsilon\}$. After studying the blow-up of $V(\cdot,\Omega)$ near $\partial\Omega$ (cf. Proposition 3.4) we define the energy $E(\Omega)$ of a region Ω bounded by a smooth curve K of length $L(K)$ as

$$E(\Omega) = \lim_{\delta\to 0}\left(\int_{\Omega_\delta} V(w,\Omega)d^2w + \frac{\pi}{4\delta}L(K)\right), \tag{2}$$

where $\Omega_\delta \subset \Omega$ is the set of points whose distance from $\Omega^c = \mathbb{R}^2 \setminus \Omega$ is larger than δ. This is a renormalization of the integral of $V(\cdot,\Omega)$. Alternatively, given a smoothly embedded curve

$K \subset \mathbb{R}^2$ we define (cf. Definition 3.11)

$$E(K) = \lim_{\varepsilon \to 0} \left(\frac{2L(K)}{\varepsilon} - \int_{\Omega \times \Omega^c \setminus \Delta_\varepsilon} \frac{d^2w d^2z}{|z-w|^4} \right), \tag{3}$$

where $\Delta_\varepsilon \subset \mathbb{R}^2 \times \mathbb{R}^2$ consists of pairs (w, z) with $|z - w| < \varepsilon$. For $\Omega \subset \mathbb{R}^2$ with boundary K, both energies are related by $E(\Omega) = E(K) + \pi^2 \chi(\Omega)/4$ (cf. Proposition 3.13). Among several expressions for these energies we point out the following one which involves no limit:

$$E(K) = -\frac{1}{2} \int_{K \times K} \sin\theta_p \sin\theta_q \frac{dpdq}{|q-p|^2},$$

where dp, dq denote the arc-length elements, and θ_p (resp. θ_q) is the angle between $q - p$ and K at p (resp. at q).

Considering $\mathbb{R}^2$ as the ideal boundary of Poincaré half-space model of hyperbolic space $\mathbb{H}^3$, we can assume K to be the ideal boundary of a smooth surface $S \subset \mathbb{H}^3$ meeting $\mathbb{R}^2$ orthogonally. Then we have the following Gauss-Bonnet formula (cf. Proposition 3.17)

$$\int_S \kappa \, dS = 2\pi \chi(S) + \frac{2}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\#(\ell_{wz} \cap S) - \lambda^2(w, z; K)) \frac{d^2w d^2z}{|z-w|^4} - \frac{4}{\pi} E(K),$$

where κ denotes the extrinsic curvature of S, and ℓ_{wz} denotes the geodesic with ideal endpoints w, z, while $\lambda(w, z; K)$ is the algebraic intersection number of K with the segment $[zw] \subset \mathbb{R}^2$. As a consequence, we get the Möbius invariance of $E(K)$ and $E(\Omega)$ (cf. Corollary 3.18). Moreover, we get lower bounds for these energies in Corollary 3.20.

For a closed curve $K \subset \mathbb{R}^3$ we define $E(K)$ as the renormalized measure of the set of circles linked with K. Indeed, there is a natural (Möbius invariant) measure $d\gamma$ on $\mathcal{S}(1, 3)$, the space of oriented circles $\gamma \subset \mathbb{R}^3$. To be precise, we define

$$E(K) = \lim_{\varepsilon \to 0} \left(\frac{3\pi L(K)}{8\varepsilon} - \frac{3}{16\pi} \int_{\mathcal{S}_\varepsilon(1,3)} \lambda^2(\gamma, K) d\gamma \right)$$

where $\mathcal{S}_\varepsilon(1, 3)$ is the set of oriented circles with radii larger than ε, and $\lambda(\gamma, K)$ denotes the linking number between γ and K. This is motivated by (3), and indeed both definitions coincide when K is planar. Again, we find an expression of $E(K)$ that involves no renormalization:

$$E(K) = -\frac{1}{2} \int_{K \times K} \cos\tau \sin\theta_p \sin\theta_q \frac{dpdq}{|q-p|^2},$$

where τ is the angle between the two oriented planes through p, q tangent to K at p and q respectively. It is interesting to remark that replacing $\cos\tau$ by $\sin\tau$ gives the so-called *writhe* of K, a Möbius invariant functional for space curves discovered by Banchoff and White in [6]. Besides, if the power in the denominator is replaced by 1 or 0, one gets respectively the length of K and Banchoff-Pohl's notion of the *area* enclosed by K.

Again, $E(K)$ appears in a Gauss-Bonnet formula: if a surface $S \subset \mathbb{H}^4$ in Poincaré half-space model of 4-dimensional hyperbolic space meets the ideal boundary orthogonally along a closed curve K, then (cf. Corollary 4.16)

$$\frac{1}{\pi}\int_{N^1S}\kappa\, de\, dS = 2\pi\chi(S) + \frac{3}{4\pi^2}\int_{\mathcal{L}_2}(\#(\ell\cap S) - \lambda^2(\ell, K))d\ell - \frac{4}{\pi}E(K)\,,$$

where κ denotes the Lipschitz-Killing curvature defined on the unit normal bundle N^1S, and de denotes the length element of N^1_xS. The second integral is taken over the space $\mathcal{L}_2$ of oriented totally geodesic planes $\ell \subset \mathbb{H}^4$, which is naturally identified with $\mathcal{S}(1,3)$, and the measure $d\ell$ corresponds to $d\gamma$. By construction, these two integrals are invariant under isometries of $\mathbb{H}^4$. This shows the Möbius invariance of $E(K)$.

In Proposition 4.18 we provide an alternative construction of $E(K)$ inspired by (2). There, we consider an ε-parallel curve K_ε, integrate the product of linking numbers $\lambda(\gamma, K)\lambda(\gamma, K_\varepsilon)$ over all circles γ, and apply renormalization as ε goes down to 0. Finally, in Section 5 we go back to the planar case and give some Möbius invariant expressions of the energy of a domain.

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2. Möbius energy for pairs of planar domains.

2.1. Infinitesimal cross ratio. We start fixing some notations, and introducing some tools. We will be considering pairs of complex numbers $(w, z) = (u + iv, x + iy) \in \mathbb{C}\times\mathbb{C}$. We denote the diagonal in $\mathbb{C}\times\mathbb{C}$ by $\Delta = \{(w, w)\}$. The *infinitesimal cross ratio* (cf. [10]) is a complex valued 2-form on $\mathbb{C}\times\mathbb{C}\setminus\Delta$ given by

$$\omega_{cr} = \frac{dw\wedge dz}{(w-z)^2} = \frac{(du + idv)\wedge(dx + idy)}{(w-z)^2}\,.$$

Like the classical cross-ratio of 4-tuples of complex numbers, the 2-form ω_{cr} is invariant under the diagonal action of orientation preserving Möbius transformations: $h(z) = (az+b)/(cz+d)$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Recall that such an h defines a transformation $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ of the Riemann sphere $\mathbb{CP}^1 = \mathbb{C}\cup\{\infty\}$. For simplicity we will work with $\mathbb{C}$ instead of $\mathbb{CP}^1$. This causes no trouble, except that h is not defined in one point.

Both the real part and the imaginary part of the infinitesimal cross ratio are exact forms:

(4)
$$d\left(\mathfrak{Re}\,\frac{dw}{w-z}\right)=d\left(\mathfrak{Re}\,\frac{dz}{z-w}\right)=-\mathfrak{Re}\,\omega_{cr}, \quad d\left(\mathfrak{Im}\,\frac{dw}{w-z}\right)=d\left(\mathfrak{Im}\,\frac{dz}{z-w}\right)=-\mathfrak{Im}\,\omega_{cr}\,.$$

Direct computation shows

$$\mathfrak{Re}\,\omega_{cr}\wedge\mathfrak{Re}\,\omega_{cr} = \mathfrak{Im}\,\omega_{cr}\wedge\mathfrak{Im}\,\omega_{cr} = 2\frac{d^2w\wedge d^2z}{|z-w|^4}\,, \tag{5}$$

where $d^2w = du\wedge dv, d^2z = dx\wedge dy$ are the area elements in $\mathbb{C}$. At some places we will omit the wedges in the exterior product of forms, specially when these are understood as measures. Note that the form in (5) is invariant under all Möbius transformations, preserving orientation or not.

2.2. The Möbius energy of pairs of disjoint planar domains. Let Ω_1, Ω_2 be a pair of disjoint domains in $\mathbb{R}^2$ with smooth regular boundaries. Suppose each pair of particles in Ω_1 and Ω_2 has a mutual repelling force between them. Assume this force has magnitude r^{-5} where r denotes the distance between the particles. The reason for this exponent will be clear below. Under these assumptions the corresponding energy for the interaction of Ω_1 and Ω_2 would be the following

DEFINITION 2.1. The Möbius mutual energy between Ω_1 and Ω_2 is defined as

$$E(\Omega_1, \Omega_2) = \int_{\Omega_1 \times \Omega_2} \frac{d^2 w d^2 z}{|z - w|^4},$$

where $d^2 w$ (resp. $d^2 z$) denotes the area element in $\Omega_1 \subset \mathbb{R}^2$ (resp. $\Omega_2 \subset \mathbb{R}^2$).

This energy is invariant under Möbius transformations. Indeed, (5) implies

$$E(\Omega_1, \Omega_2) = \frac{1}{2} \int_{\Omega_1 \times \Omega_2} \mathfrak{Re}\, \omega_{cr} \wedge \mathfrak{Re}\, \omega_{cr} = \frac{1}{2} \int_{\Omega_1 \times \Omega_2} \mathfrak{Im}\, \omega_{cr} \wedge \mathfrak{Im}\, \omega_{cr}.$$

PROPOSITION 2.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be a pair of disjoint planar domains with smooth regular boundaries $K_1 = \partial\Omega_1$, $K_2 = \partial\Omega_2$. Then $E(\Omega_1, \Omega_2)$ can be expressed by the following double contour integrals*

$$E(\Omega_1, \Omega_2) = -\frac{1}{2} \int_{K_1 \times K_2} \cos\theta_1 \cos\theta_2 \frac{dp_1 dp_2}{|p_2 - p_1|^2}, \tag{6}$$

$$E(\Omega_1, \Omega_2) = -\frac{1}{2} \int_{K_1 \times K_2} \sin\theta_1 \sin\theta_2 \frac{dp_1 dp_2}{|p_2 - p_1|^2}, \tag{7}$$

where dp_i is the length element on K_i, and θ_i is the oriented angle from the positive tangent of K_i at p_i to the vector $p_2 - p_1$.

PROOF. Put

$$\lambda = -\mathfrak{Re} \frac{dw}{w - z}, \qquad \rho = -\mathfrak{Re} \frac{dz}{z - w}, \qquad \omega = \mathfrak{Re}\, \omega_{cr}, \tag{8}$$

so that $d\lambda = d\rho = \omega$. By Stokes' theorem

$$\int_{\Omega_1 \times \Omega_2} \omega \wedge \omega = \int_{(K_1 \times \Omega_2) \cup (\Omega_1 \times K_2)} \lambda \wedge \omega = \int_{K_1 \times \Omega_2} \lambda \wedge \omega.$$

Since $\lambda \wedge \omega = \omega \wedge \rho - d(\lambda \wedge \rho)$,

$$\int_{\Omega_1 \times \Omega_2} \omega \wedge \omega = \int_{K_1 \times \Omega_2} \omega \wedge \rho - \int_{K_1 \times K_2} \lambda \wedge \rho = -\int_{K_1 \times K_2} \lambda \wedge \rho.$$

Then (6) follows from elementary computations. The same arguments with $\mathfrak{Re}$ replaced by $\mathfrak{Im}$ in (8) proves (7). □

COROLLARY 2.3. *Under the above hypothesis*

$$E(\Omega_1, \Omega_2) = -\frac{1}{4} \int_{K_1 \times K_2} \frac{\overrightarrow{dp_1} \cdot \overrightarrow{dp_2}}{|p_2 - p_1|^2}, \tag{9}$$

where $\overrightarrow{dp_1}\cdot\overrightarrow{dp_2} = du_1\wedge du_2 + dv_1\wedge dv_2$ *is a* 2*-form on* $K_1\times K_2$ *where* $p_i = (u_i, v_i)\in K_i$.

PROOF. Let ϕ_i be the angle of the tangent vector to K_i from the x-axis ($i = 1, 2$). Then

$$\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1-\theta_2) = \cos(\phi_1-\phi_2) = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2.$$

Let dp_i denote the length element of K_i. As $du_i = \cos\phi_i dp_i, dv_i = \sin\phi_i dp_i$ the above equation implies

$$(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)dp_1dp_2 = \overrightarrow{dp_1}\cdot\overrightarrow{dp_2}\,. \tag{10}$$

Therefore, by averaging (6) and (7) we have

$$E(\Omega_1,\Omega_2) = -\frac{1}{4}\int_{K_1\times K_2}\frac{\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2}{|p_2-p_1|^2}\,dp_1dp_2 = -\frac{1}{4}\int_{K_1\times K_2}\frac{\overrightarrow{dp_1}\cdot\overrightarrow{dp_2}}{|p_2-p_1|^2}\,.$$

□

3. Renormalized Möbius energy of planar domains.

Let $\Omega\subset\mathbb{R}^2$ be a planar domain with compact smooth boundary $K = \partial\Omega$. We will define a Möbius invariant energy associated to Ω. One cannot take $E(\Omega,\Omega)$ because of the blow up of ω_{cr} near the diagonal $\Delta\subset\Omega\times\Omega$. We introduce two kinds of renormalizations and show that they produce essentially the same energy. The first one is described next, and the second one appears later in Subsection 3.3.

3.1. Renormalized potential.

The first approach consists of two steps. Firstly we define a renormalized potential at every point of the domain. The integral of this potential is divergent when the boundary is not empty. Hence we need a second step where this integral is renormalized.

DEFINITION 3.1 ([1, 14]). Let w be a point in $\Omega\setminus\partial\Omega$. We define the renormalized r^{-4}-potential of Ω at w by

$$V(w,\Omega) = \lim_{\varepsilon\to 0}\left(\int_{\Omega\setminus B_\varepsilon(w)}\frac{d^2z}{|z-w|^4} - \frac{\pi}{\varepsilon^2}\right). \tag{11}$$

PROPOSITION 3.2 ([14]). *The renormalized potential of* Ω *at an interior point* w *is given by*

$$V(w,\Omega) = -\int_{\Omega^c}\frac{d^2z}{|z-w|^4}\,, \tag{12}$$

where $\Omega^c = \mathbb{R}^2\setminus\Omega$ *denotes the complement of* Ω. *Hence* $-\infty < V(w,\Omega) < 0$.

PROOF. Since

$$\int_{\mathbb{R}^2\setminus B_\varepsilon(w)}\frac{d^2z}{|z-w|^4} = \frac{\pi}{\varepsilon^2}\,,$$

if $\varepsilon > 0$ is such that $B_\varepsilon(w)\subset\Omega$ then

$$\int_{\Omega\setminus B_\varepsilon(w)}\frac{d^2z}{|z-w|^4} - \frac{\pi}{\varepsilon^2} = \int_{\Omega\setminus B_\varepsilon(w)}\frac{d^2z}{|z-w|^4} - \int_{\mathbb{R}^2\setminus B_\varepsilon(w)}\frac{d^2z}{|z-w|^4} = -\int_{\mathbb{R}^2\setminus\Omega}\frac{d^2z}{|z-w|^4}\,.$$

□

In view of (12) one can interpret $-V(w, \Omega)$ as the area of the image of Ω^c after an inversion with respect to a circle of center w and radius 1. Indeed, the Jacobian of such an inversion is precisely $-|w-z|^{-4}$.

LEMMA 3.3. *Given $\kappa \in \mathbb{R}$, let Ω be a domain bounded by a curve of constant signed curvature κ (i.e., Ω is a disk for $\kappa > 0$, a half-plane for $\kappa = 0$ and the complement of a disk if $\kappa < 0$). Let $w \in \Omega$ be a point at distance $\delta > 0$ from $\partial\Omega$. Then,*

$$V(w, \Omega) = -\frac{\pi}{\delta^2}\frac{1}{(2-\kappa\delta)^2} = -\frac{\pi}{4\delta^2}(1 + \kappa\delta + O(\delta^2))\,.$$

PROOF. An easy computation shows in the three cases ($\kappa > 0, \kappa = 0$ and $\kappa < 0$) that the inversion with respect to a circle of center w and radius 1 maps $\partial\Omega$ to a circle of radius

$$R = \frac{1}{\delta(2-\kappa\delta)}\,.$$

Hence,

$$-V(w, \Omega) = \pi R^2 = \frac{\pi}{\delta^2}\frac{1}{(2-\kappa\delta)^2}\,.$$

□

PROPOSITION 3.4. *Let $\delta = d(\cdot, K)$ denote the distance function to K defined on Ω. Let $\varepsilon > 0$ be such that, whenever $\delta(w) < \varepsilon$, there is a unique $p \in K$ with $d(w, K) = |w-p|$. For every such w,*

$$V(w, \Omega) = -\left(\frac{\pi}{4\delta(w)^2} + \frac{\kappa(p)\pi}{4\delta(w)}\right) + O(1)\,, \tag{13}$$

where $\kappa(p)$ is the curvature of K at p, and $O(1)$ stands for a bounded function on Ω.

PROOF. After a motion of $\mathbb{R}^2$ we can assume that $w = (0, \delta)$, $p = (0, 0)$, and $\partial\Omega$ coincides near p with the graph of a function $g(x) = O(x^2)$. Let Ω_p be the domain bounded by a curve of constant curvature such that $\partial\Omega$ and $\partial\Omega_p$ have second order contact at p, and $w \in \Omega_p$. Let $\partial\Omega_p$ coincide locally with the graph of a function $f(x)$.

By the previous lemma, we only need to show that

$$\int_{-\varepsilon}^{\varepsilon}\int_{f(x)}^{g(x)} \frac{1}{(x^2+(y-\delta)^2)^2}dydx < C\,, \qquad \forall\delta > 0$$

for some $C > 0$ which can be chosen independently of w.

Let $F(x, y)$ be such that

$$\frac{\partial F}{\partial y} = \frac{1}{(x^2+y^2)^2}\,.$$

Then

$$\int_{f(x)}^{g(x)} \frac{1}{(x^2+(y-\delta)^2)^2}dy = F(x, g(x)-\delta) - F(x, f(x)-\delta)\,.$$

Let the previous integral be denoted by $I(\delta)$. Taylor's theorem for $F(x, y)$ on the variable y around $y = f(x) - \delta$ yields

$$I(\delta) = \frac{\partial F}{\partial y}(x, f(x) - \delta)(g(x) - f(x)) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(x, \xi)(g(x) - f(x))^2$$

for some $\xi = \xi(x)$ in $(f(x) - \delta, g(x) - \delta)$ or $(g(x) - \delta, f(x) - \delta)$. Hence

$$I(\delta) = \frac{g(x) - f(x)}{(x^2 + (f(x) - \delta)^2)^2} - \frac{2\xi(g(x) - f(x))^2}{(x^2 + \xi^2)^3} = \frac{g(x) - f(x)}{(x^2 + (f(x) - \delta)^2)^2} + O(\varepsilon),$$

since

$$\frac{|\xi|(g(x) - f(x))^2}{(x^2 + \xi^2)^3} \le \frac{(g(x) - f(x))^2}{x^5} = \frac{O(x^6)}{x^5} = O(\varepsilon).$$

Finally, we use that $g(x) - f(x) = ax^3 + O(x^4)$ for some $a \in \mathbb{R}$, so

$$\int_{-\varepsilon}^{\varepsilon} I(\delta)dx = \int_{-\varepsilon}^{\varepsilon} \frac{ax^3}{(x^2 + (f(x) - \delta)^2)^2}dx + \int_{-\varepsilon}^{\varepsilon} \frac{O(x^4)}{(x^2 + (f(x) - \delta)^2)^2}dx + O(\varepsilon).$$

This is uniformly bounded since

$$\int_{-\varepsilon}^{\varepsilon} \frac{x^3}{(x^2 + (f(x) - \delta)^2)^2}dx = 0$$

because $f(x)$ is even, and

$$\int_{-\varepsilon}^{\varepsilon} \frac{O(x^4)}{(x^2 + (f(x) - \delta)^2)^2}dx \le \int_{-\varepsilon}^{\varepsilon} \frac{O(x^4)}{x^4}dx.$$

□

3.2. Renormalized energy of planar domains. The potential $V(w, \Omega)$ is not integrable over Ω. Hence we take the following renormalization.

DEFINITION 3.5. We define the *renormalized Möbius energy* of the domain Ω by

$$E(\Omega) = \lim_{\delta \to 0}\left(\int_{\Omega_\delta} V(w, \Omega)d^2w + \frac{\pi}{4\delta}L(K)\right), \tag{14}$$

where $\Omega_\delta = \{w \in \Omega : d(w, K) \ge \delta\}$, and $L(K)$ denotes the length of the boundary.

In some sense, $E(\Omega)$ is also a renormalization of $E(\Omega, \Omega^c)$. Indeed, (12) shows that

$$E(\Omega) = \lim_{\delta \to 0}\left(\frac{\pi}{4\delta}L(K) - \int_{\Omega_\delta \times \Omega^c} \frac{d^2wd^2z}{|z - w|^4}\right). \tag{15}$$

Although $d^2wd^2z/|z - w|^4$ is invariant under Möbius transformations, it is not clear at this moment that $E(\Omega)$ is Möbius invariant. This will be shown later. First we must prove the convergence of (14).

PROPOSITION 3.6. *Given $\Omega \subset \mathbb{R}^2$ with compact smooth boundary, the limit in* (14) *exists and is finite.*

PROOF. Take ε as in Proposition 3.4. If $0 < \delta < \varepsilon$, then

$$\int_{\Omega_\delta} V(w,\Omega)d^2w = \int_{\Omega_\varepsilon} V(w,\Omega)d^2w + \int_{N_\varepsilon(K)\setminus N_\delta(K)} V(w,\Omega)d^2w\,,$$

where $N_\delta(K) = \{w \in \Omega \,:\, d(w,K) \le \delta\}$. If we express the arc-length parameter of K by s, Proposition 3.4 implies

$$\begin{aligned}\int_{N_\varepsilon(K)\setminus N_\delta(K)} V(w,\Omega)d^2w &= -\frac{\pi}{4}\int_0^{L(K)}\int_\delta^\varepsilon (1-\kappa(s)t)\left(\frac{1}{t^2}+\frac{\kappa(s)}{t}\right)dtds + O(\varepsilon)\\ &= -\frac{\pi}{4}\left(\frac{1}{\delta}-\frac{1}{\varepsilon}\right)L(K) - \frac{\pi}{4}(\varepsilon-\delta)\int_0^{L(K)}\kappa^2(s)ds + O(\varepsilon)\,,\end{aligned}$$

which completes the proof. □

COROLLARY 3.7. *If $\Omega \subset \mathbb{R}^2$ is a planar domain with smooth regular boundary K of length $L(K)$, then*

$$\begin{aligned}E(\Omega) &= \lim_{\delta\to 0}\left(\frac{\pi}{4\delta}L(K) - E(\Omega_\delta,\Omega^c)\right)\\ &= \lim_{\delta\to 0}\left(\frac{\pi}{4\delta}L(K) - \frac{1}{2}\int_{K_\delta\times K}\cos\theta_p\cos\theta_q\frac{dpdq}{|q-p|^2}\right)\\ &= \lim_{\delta\to 0}\left(\frac{\pi}{4\delta}L(K) - \frac{1}{2}\int_{K_\delta\times K}\sin\theta_p\sin\theta_q\frac{dpdq}{|q-p|^2}\right)\\ &= \lim_{\delta\to 0}\left(\frac{\pi}{4\delta}L(K) - \frac{1}{4}\int_{K_\delta\times K}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}\right),\end{aligned}$$

where $K_\delta = \partial\Omega_\delta$, and θ_p (resp. θ_q) denotes the oriented angle from the positive tangent vector of K (resp. K_δ) at p (resp. at q) to the vector $q-p$.

PROOF. The first equality is immediate from (15). The rest follows respectively from (6), (7) and (9). Remark that the orientations of K as $K = \partial\Omega$ and $K = \partial\Omega^c$ are opposite. The signs in the last three lines follow from this fact. □

THEOREM 3.8. *If $\Omega \subset \mathbb{R}^2$ is a planar domain with smooth regular boundary K of length $L(K)$, then*

$$E(\Omega) = \lim_{\varepsilon\to 0}\left(\frac{L(K)}{2\varepsilon} - \frac{1}{4}\int_{K\times K\setminus\Delta_\varepsilon}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}\right) + \frac{\pi}{8}\int_K \kappa(s)ds\,, \tag{16}$$

where $\Delta_\varepsilon = \{(p,q)\colon |p-q| \le \varepsilon\}$, and κ is the curvature of K with the orientation induced by Ω.

We postpone the proof to Proposition 4.21 where the latter equality is generalized to space curves.

PROPOSITION 3.9. *We have*

(17)
$$\lim_{\varepsilon\to 0}\left(\frac{L(K)}{\varepsilon}-\frac{1}{2}\int_{K\times K\setminus\Delta_\varepsilon}\cos\theta_p\cos\theta_q\frac{dpdq}{|q-p|^2}\right)=-\frac{1}{2}\int_{K\times K}\sin\theta_p\sin\theta_q\frac{dpdq}{|q-p|^2}\,.$$

Again, the proof is postponed to next section (cf. Propositions 4.8 and 4.11).

Finally we arrive at an expression of the energy that involves no renormalization.

THEOREM 3.10. *If $\Omega\subset\mathbb{R}^2$ is a planar domain with smooth regular boundary K of length $L(K)$, then*

$$E(\Omega)=\lim_{\varepsilon\to 0}\left(\frac{L(K)}{\varepsilon}-\frac{1}{2}\int_{K\times K\setminus\Delta_\varepsilon}\cos\theta_p\cos\theta_q\,\frac{dpdq}{|q-p|^2}\right)+\frac{\pi}{8}\int_K\kappa(s)ds \tag{18}$$

$$=-\frac{1}{2}\int_{K\times K}\sin\theta_p\sin\theta_q\,\frac{dpdq}{|q-p|^2}+\frac{\pi}{8}\int_K\kappa(s)ds\,, \tag{19}$$

where κ is the curvature of K, with the orientation induced by Ω.

Note the absence of limit in (19). We remark that the first term in (19) is equal (up to a factor) to the symmetric energy of [7] when K is a single convex curve. In case Ω is compact, the last term in (18) and (19) is $\pi^2\chi(\Omega)/4$.

PROOF. Both $\sin\theta_p$ and $\sin\theta_q$ are $O(|q-p|)$ as we will see in (36). Hence,

$$\lim_{\varepsilon\to 0}\int_{K\times K\setminus\Delta_\varepsilon}\sin\theta_p\sin\theta_q\,\frac{dpdq}{|q-p|^2}=\int_{K\times K}\sin\theta_p\sin\theta_q\,\frac{dpdq}{|q-p|^2}\,.$$

By applying (10), (17), and the above equation to (16), we obtain the conclusion. □

3.3. Renormalized energy of plane curves. Let us introduce an alternative renormalization of $E(\Omega,\Omega^c)$ (cf. (15)).

DEFINITION 3.11. Let $K\subset\mathbb{R}^2$ be a smooth compact curve bounding a region $\Omega\subset\mathbb{R}^2$. We define the *renormalized energy* of the curve K as

$$E(K)=\lim_{\varepsilon\to 0}\left(\frac{2L(K)}{\varepsilon}-\int_{\Omega\times\Omega^c\setminus\Delta_\varepsilon}\frac{d^2wd^2z}{|z-w|^4}\right),$$

where $\Delta_\varepsilon=\{(w,z)\colon\ |z-w|\le\varepsilon\}$, a neighborhood of the diagonal in $\mathbb{R}^2\times\mathbb{R}^2$.

It must be noticed that this energy $E(K)$ does not coincide with the classical Möbius energy of curves introduced in [12].

In the following, $\Omega\subset\mathbb{R}^2$ will always denote the *compact* domain bounded by K.

The following notation will be convenient. Let Ω induce an orientation on $K=\partial\Omega$. Given $w,z\in\mathbb{R}^2$ let us consider the linking number

$$\lambda(w,z,K)=\sum_{x\in[wz]\cap K}\epsilon(x) \tag{20}$$

where $[wz]$ denotes the (oriented) line segment from z to w, and ϵ is the sign of the intersection. Of course, $\lambda(w, z, K) = 0$ if w, z are both in Ω or both in Ω^c. Otherwise $\lambda(w, z, K) = \pm 1$.

PROPOSITION 3.12. *With the notation introduced above,*

$$E(K) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta} (\#([wz]\cap K) - \lambda^2(w, z, K))\frac{d^2w d^2z}{|z-w|^4}\,.$$

PROOF. Let $m = \frac{1}{2}(z+w)$, $r = |z-w|$, and θ be the angle between $[wz]$ and any fixed direction. Then

$$d^2w d^2z = -r d^2m d\theta dr\,.$$

Thus

$$\int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta_\varepsilon} \#([wz]\cap K)\frac{d^2w d^2z}{|z-w|^4} = \int_\varepsilon^\infty \left(\int_0^{2\pi}\int_{\mathbb{R}^2} \#([wz]\cap K) d^2m d\theta\right)\frac{dr}{r^3}\,.$$

Fixed $r > 0$, the integral between brackets runs over all the positions of an oriented segment of length r, and $d^2m d\theta$ is the Haar measure of the group of rigid plane motions. Hence Poincaré's formula (cf. [16, (7.11)]) gives

$$\int_\varepsilon^\infty \int_0^{2\pi}\int_{\mathbb{R}^2} \#([wz]\cap K) d^2m d\theta \frac{dr}{r^3} = \int_\varepsilon^\infty 4rL(K)\frac{dr}{r^3} = \frac{4}{\varepsilon}L(K)\,.$$

Finally

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\times\mathbb{R}^2\setminus\Delta_\varepsilon} (\#([wz]\cap K) - \lambda^2(w, z, K))\frac{d^2w d^2z}{|z-w|^4} = \lim_{\varepsilon\to 0}\left(\frac{4L(K)}{\varepsilon} - 2\int_{\Omega\times\Omega^c\setminus\Delta_\varepsilon}\frac{d^2w d^2z}{|z-w|^4}\right)\,.$$

□

The two energies $E(\Omega)$ and $E(K)$ do not coincide, but they are related as follows.

PROPOSITION 3.13. *Let $\Omega \subset \mathbb{R}^2$ be a compact domain with smooth boundary K. Then*

$$E(K) = -\frac{1}{2}\int_{K\times K} \sin\theta_p \sin\theta_q \frac{dp dq}{|q-p|^2} = E(\Omega) - \frac{\pi^2}{4}\chi(\Omega)\,,$$

where $\chi(\Omega)$ denotes the Euler characteristic of Ω.

PROOF. The second equality is a consequence of formula (19) and the Gauss-Bonnet theorem in $\mathbb{R}^2$. In order to prove the first equality, let us consider the space $A(1, 2)$ of lines in $\mathbb{R}^2$. This is a 2-dimensional manifold admitting an invariant measure given by $d\ell = dr \wedge d\theta$ where (r, θ) are the polar coordinates of the point in ℓ that is closest to the origin. We can describe each pair $(w, z) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta$ by the line ℓ through them, and two arc-length parameters t, s along ℓ. With this notation we have (cf. [16, equation (4.2)])

$$d^2w d^2z = |t-s| dt ds d\ell\,.$$

On the other hand,

$$\#([wz]\cap K) - \lambda^2(w, z, K) = \sum \epsilon(p)\epsilon(q)\,, \tag{21}$$

where the sum runs over all ordered pairs of distinct points p, q in $[wz] \cap K$. Indeed, if a and b are respectively the numbers of positive and negative intersections of $[wz]$ with K, then (21) boils down to

$$a + b - (a - b)^2 = a(a - 1) + b(b - 1) - 2ab\,.$$

Hence, by the previous proposition,

$$E(K) = -\frac{1}{2} \int_{A(1,2)} \sum_{p,q \in \ell \cap K} \frac{\epsilon(p)\epsilon(q)}{|q - p|} d\ell\,. \tag{22}$$

It was shown in [15, Section 2] that for any measurable function f on $K \times K$

$$\int_{A(1,2)} \sum_{p,q \in \ell \cap K} f(p, q)\epsilon(p)\epsilon(q) d\ell = \int_{K \times K} f(p, q) \sin\theta_p \sin\theta_q \frac{dpdq}{|q - p|}\,.$$

Taking $f(p, q) = |q - p|^{-1}$, the result follows. □

With Theorem 3.8 we get

COROLLARY 3.14. *Let $K \subset \mathbb{R}^2$ be a simple closed curve (not necessarily connected). Then*

$$E(K) = \lim_{\varepsilon \to 0} \left(\frac{L(K)}{2\varepsilon} - \frac{1}{4} \int_{K \times K \setminus \Delta_\varepsilon} \frac{\overrightarrow{dp} \cdot \overrightarrow{dq}}{|q - p|^2} \right)\,.$$

REMARK 3.15. In particular, if Ω is convex, we have from (22) that

$$E(K) = \int_{A(1,2)} \frac{1}{L(\ell \cap \Omega)} d\ell\,,$$

where $L(\ell \cap \Omega)$ is the length of the chord. This extends in some sense the Crofton formulas discussed in [16, Chapter 4].

It turns out that $E(K)$ appeared in a Gauss-Bonnet formula for complete surfaces in hyperbolic space, with a tame behaviour at infinity. Before recalling the formula, let us describe this condition on the asymptotic behaviour of the surfaces.

DEFINITION 3.16. Let $f : S \looparrowright \mathbb{H}^n$ be an immersion of a C^2-differentiable surface S in hyperbolic space. We say that S has *cone-like ends* if

i) S is the interior of a compact surface with boundary $\overline{S}$, and taking the Poincaré half-space model of hyperbolic space, f extends to a C^2-differentiable immersion $f : \overline{S} \looparrowright \mathbb{R}^n$,

ii) $C = f(\partial \overline{S})$ is a collection of connected simple closed curves contained in $\partial_\infty \mathbb{H}^n$, the boundary of the model, and

iii) $f(\overline{S})$ is orthogonal to $\partial_\infty \mathbb{H}^n$ along C.

The reason for the name *surfaces with cone-like ends* is that they are asymptotically close the 'cone' formed by the family of geodesics starting at a point and ending in C. However, it must be noted that there are other notions in the literature named similarly, and with different meanings.

PROPOSITION 3.17 ([17]). *Let $S \subset \mathbb{H}^3 \subset \mathbb{R}^3$ be a surface in Poincaré half-space model with cone-like ends on the curve $K = \partial_\infty S \subset \partial_\infty \mathbb{H}^3 \equiv \mathbb{R}^2$. Then the following holds with $\Phi(K)$ depending only on K;*

$$\int_S \kappa \, dS = 2\pi \chi(S) + \frac{2}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\#(\ell_{wz} \cap S) - \lambda^2(w, z, K)) \frac{d^2 w d^2 z}{|z-w|^4} - \Phi(K) , \tag{23}$$

where κ denotes the extrinsic curvature of S, and ℓ_{wz} denotes the geodesic with ideal endpoints w, z.

Given $K \subset \mathbb{R}^2$, and $R > 0$ we take the surface $S = K \times (0, R] \cup \Omega \times \{R\} \subset \mathbb{H}^3$. By taking limits as $R \to \infty$, the equation above becomes

$$\Phi(K) = \frac{2}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\#([wz] \cap K) - \lambda^2(w, z, K))) \frac{d^2 w d^2 z}{|z-w|^4} = \frac{4}{\pi} E(K) . \tag{24}$$

COROLLARY 3.18. *The energies considered are Möbius invariant in the following sense:*

1. *If $K \subset \mathbb{R}^2$ is a smooth closed curve, then $E(K) = E(f(K))$ for every Möbius transformation f leaving K closed.*
2. *If $\Omega \subset \mathbb{R}^2$ is bounded by a closed curve, then $E(\Omega) = E(f(\Omega))$ for every Möbius transformation f such that Ω, $f(\Omega)$ are both compact or both unbounded.*

PROOF. Clearly, $\Phi(K)$ is invariant since all other terms in (23) are invariant. Together with (24), this proves the first statement. The second part follows then by Proposition 3.13. □

PROPOSITION 3.19. *Let $\Omega \subset \mathbb{R}^2$ be a compact domain with smooth boundary $K = \partial\Omega$. Then*

$$E(K) \quad = \quad \frac{\pi^2}{2} \chi(\Omega) + \int_{NT(\Omega)} \frac{d^2 w d^2 z}{|z-w|^4} ,$$

where $NT(\Omega)$ is the set of pairs $(w, z) \in \Omega \times \Omega$ such that any circle γ containing w and z intersects K.

PROOF. Let $Q \subset \mathbb{H}^3$ be the intersection of all geodesic half-spaces (closed sets in $\mathbb{H}^3$ bounded by totally geodesic planes) containing Ω^c in its ideal set. This is a kind of convex hull of Ω^c, and is bounded by a surface S of class C^1. With the arguments of [17, Proposition 3], one can approximate S by a sequence of surfaces S_n with cone-like ends and with total curvatures converging to 0. Then Proposition 3.17 and (24) give

$$0 = 2\pi \chi(\Omega) + \frac{2}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (\#(\ell_{wz} \cap S) - \lambda^2(w, z, K))) \frac{d^2 w d^2 z}{|z-w|^4} - \frac{4}{\pi} E(K) .$$

Note that the integrand above is 2 if $w, z \in \Omega$ and $\ell_{wz} \cap Q \neq \emptyset$; otherwise it is 0. But ℓ meets the convex hull Q if and only if every geodesic plane $\wp$ containing ℓ meets Ω^c. □

COROLLARY 3.20. *Let Ω be compact with n connected components and let the boundary $K = \partial\Omega$ have k components. Then $E(\Omega) \geq (2n + k)\pi^2/4$ with equality only if $n = k = 1$ and Ω is a disk.*

PROOF. Given a compact domain $\Omega \subset \mathbb{R}^2$, we have from Theorem 3.10 that

$$E(\Omega) = E(\mathbb{R}^2 \setminus \Omega) + \frac{\pi^2}{2}\chi(\Omega)\,.$$

Let Ω be a compact connected domain with non-connected boundary. Then $\mathbb{R}^2 \setminus \Omega = \cup_{i=1}^k \Omega_i$ for a collection of domains Ω_i (one of them, say Ω_1, non-compact) with connected boundaries $K_i = \partial\Omega_i$. Hence

$$E(\Omega) = \sum_{i=1}^k E(\Omega_i) + \sum_{i\neq j} E(\Omega_i, \Omega_j) + \frac{\pi^2}{2}\chi(\Omega)\,. \tag{25}$$

Clearly $E(\Omega_i, \Omega_j) > 0$, and by the previous proposition

$$E(\Omega_i) = E(K_i) + \frac{\pi^2}{4} \geq \frac{3\pi^2}{4}\,, \text{ for } i > 1, \qquad E(\Omega_1) = E(K_1) - \frac{\pi^2}{4} \geq \frac{\pi^2}{4}\,.$$

Plugging these inequalities and $\chi(\Omega) = k - 2$ into (25) yields

$$E(\Omega) \geq \frac{3(k-1)\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi^2}{2}\chi(\Omega) = \frac{(k+2)\pi^2}{4}\,.$$

If Ω has n connected components, we just need to use again that mutual energies are positive to get the stated inequality.

Suppose now that we have the equality in the inequalities above. Then clearly $k = 1$, and $NT(\Omega)$ has empty interior. Let now D be a maximal closed disc contained in Ω. If $\Omega \neq D$, then Ω has a larger diameter than D. But then $(w, z) \in NT(\Omega)$ whenever $|z - w|$ is close to the diameter of Ω. We conclude that $\Omega = D$. □

4. A new Möbius invariant functional for space curves.

As a first step towards generalization of the previous results to higher dimensions, we look again at the infinitesimal cross-ratio.

4.1. The infinitesimal cross-ratio in higher dimensions.

4.1.1. The real part. The real part of the infinitesimal cross-ratio was extended to higher dimensions in [10]. To be precise, the following 2-form $\boldsymbol{\omega}$ in $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ was considered:

$$\begin{aligned} \boldsymbol{\omega} &= d\left(\frac{\sum_{i=1}^n (z_i - w_i)dw_i}{|z-w|^2}\right) \\ &= -\frac{\sum_{i=1}^n dw_i \wedge dz_i}{|z-w|^2} + 2\frac{(\sum_{i=1}^n (z_i - w_i)dw_i) \wedge (\sum_{j=1}^n (z_j - w_j)dz_j)}{|z-w|^4}\,. \end{aligned} \tag{26}$$

This form is invariant under the diagonal action of Möbius transformations h; i.e., $(h \times h)^*\boldsymbol{\omega} = \boldsymbol{\omega}$. For $n = 2$ we have $\boldsymbol{\omega} = \Re\mathfrak{e}\, \omega_{cr}$. More generally, if $I : \mathbb{R}^2 \to S^2$ is a conformal mapping with image in a round sphere $S^2 \subset \mathbb{R}^n$, then $(I \times I)^*\boldsymbol{\omega} = \Re\mathfrak{e}\, \omega_{cr}$.

We give two additional interpretations of $\boldsymbol{\omega}$. For the first one, let $\Psi : \mathbb{S}^n \times \mathbb{S}^n \setminus \Delta \to T^*\mathbb{S}^n$ be the bijection given by $\Psi(x, y) = (x, \Psi_x(y))$, where $\Psi_x : \mathbb{S}^n \setminus \{x\} \to (x)^\perp \equiv T_x^*\mathbb{S}^n$ is the stereographic projection. Then $\boldsymbol{\omega}$ is essentially equal to the pull-back of the canonical symplectic form $\omega_{T^*\mathbb{S}^n}$ of $T^*\mathbb{S}^n$ through Ψ. To be precise, if $I : \mathbb{S}^n \to \mathbb{R}^n$ is the inverse of a stereographic projection, then

$$(I \times I)^* \Psi^* \omega_{T^*\mathbb{S}^n} = -2\boldsymbol{\omega}\,.$$

For the second interpretation, let $\mathcal{G}$ be the space of unitary geodesics in $\mathbb{H}^{n+1}$, considered modulo shifts in their arc-length parametrization. The tangent space $T_\ell \mathcal{G}$ is the space of Jacobi fields along the geodesic ℓ that are orthogonal to it. If $(\;,\;)$ denotes the Riemann metric of $\mathbb{H}^{n+1}$ and ∇ is the corresponding Levi-Civita connection, then

$$\omega_g(\xi, \eta) = (\xi(t), \nabla_{\dot{\ell}}\, \eta(t)) - (\eta(t), \nabla_{\dot{\ell}}\, \xi(t))$$

is independent of $t \in \mathbb{R}$. Indeed, the derivative with respect to t is easily seen to vanish. This defines an isometry-invariant symplectic form on $\mathcal{G}$. Using the Poincaré half-space model for $\mathbb{H}^{n+1}$ we can relate ω_g with $\boldsymbol{\omega}$ as follows. Given $(w, z) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ we let $\ell \in \mathcal{G}$ be the oriented geodesic with ideal endpoints w, z at $-\infty, \infty$ respectively. Given a local choice of a point $\ell(t)$, we define a 1-form ω_1 by $\omega_1(\eta) = (\eta(t), \dot{\ell}(t))$. Application of the invariant formula for exterior derivatives shows that $d\omega_1 = \omega_g$. By taking $\ell(t) = (1/2)(z + w, |z - w|) \in \mathbb{H}^{n+1}$ (and $\dot{\ell}(t) = (1/2)(z - w, 0) \in T_o\mathbb{H}^{n+1}$), one checks that $d\omega_1 = 2\boldsymbol{\omega}$.

4.1.2. The imaginary part. It was shown (in a more general context) in [9] that $\boldsymbol{\omega}$ is the unique (up to normalization) Möbius invariant 2-form in the space of point pairs $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ for $n \geq 3$. Still, the imaginary part of ω_{cr} can be generalized to higher dimensions as a differential form, not in the space of point pairs, but in the space of codimension 2-subspheres. We describe this differential form next, altough it will not be used in this paper. Let $\mathcal{S}(n-2, n)$ be the set of oriented codimension 2 subspheres in $\mathbb{S}^n$. We can realize $\mathbb{S}^n$ in the Minkowski space $\mathbb{R}^{n+2}_1$ as the intersection of the light cone and a space-like affine hyperplane. The action of the orientation preserving Möbius transformations on $\mathbb{S}^n$ corresponds to the linear action of the Lorentz group $SO(n+1, 1)$. Therefore $\mathcal{S}(n-2, n)$ can be identified with the set G of oriented timelike codimension 2 subspaces of $\mathbb{R}^{n+2}_1$. This space is a non-compact Grassmannian manifold $G = SO(n+1, 1)/SO(2) \times SO(n-1, 1)$ with an indefinite pseudo inner product $\langle\;,\;\rangle$. Just like in compact case, G has a Kähler form ω_K defined by $\omega_K(u, v) = \langle Ju, v\rangle$ $(u, v \in T_\Pi G, \Pi \in G)$, where J is the complex structure given by a $90°$ degrees rotation which can be considered as an element of $SO(2)$. By construction, this Kähler form ω_K is invariant under orientation preserving Möbius transformations. This form generalizes the imaginary part of ω_{cr} in the following sense.

PROPOSITION 4.1. *When $n = 2$,*

$$\Im\mathfrak{m}\, \omega_{cr} = -\frac{1}{2}\omega_K\,.$$

To be precise, the right hand side should be understood to be $-(1/2)(f \times f)^*\omega_K$, where $f : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{\text{pt.}\}$ is the inverse of an orientation preserving stereographic projection.

PROOF. As both ω_{cr} and ω_K are invariant under Möbius transformations, we may fix a point Π in G. Suppose e_0, e_1, e_2, e_3 form a pseudo-orthonormal basis of $\mathbb{R}^4_1$ with $e_0 \cdot e_0 = -1$

and $e_i \cdot e_j = \delta_{ij}$ $((i, j) \neq (0, 0))$. Assume $\Pi = \mathrm{Span}\langle e_0, e_1\rangle$. Then $T_\Pi G \cong \mathrm{Hom}(\Pi, \Pi^\perp)$ is spanned by v_{ij} $(i = 0, 1,\ j = 2, 3)$, where $v_{ij} \in \mathrm{Hom}(\Pi, \Pi^\perp)$ is given by $v_{ij}(e_i) = e_j$ and $v_{ij}(e_{1-i}) = 0$. They form a pseudo-orthonormal basis of $T_\Pi G$ with $\langle v_{0j}, v_{0j}\rangle = -1$ and $\langle v_{1j}, v_{1j}\rangle = 1$ $(j = 2, 3)$.

Since the complex structure J is obtained by $90°$ degrees rotation in the e_2e_3-plane, namely, $J(v_{i2}) = v_{i3}$ $(i = 0, 1)$, we have $\omega_K(v_{02}, v_{03}) = -1$, $\omega_K(v_{12}, v_{13}) = 1$, and $\omega_K(v_{ij}, v_{kl}) = 0$ if $\{v_{ij}, v_{kl}\}$ is not equal to $\{v_{02}, v_{03}\}$ or $\{v_{12}, v_{13}\}$.

On the other hand, by a suitable identification, Π correspnds to $((u, v), (x, y)) = ((1, 0), (-1, 0))$ in $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta$ and v_{ij} correspond to

$$v_{02} = \frac{\partial}{\partial v} + \frac{\partial}{\partial y},\ v_{03} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial x},\ v_{12} = \frac{\partial}{\partial v} - \frac{\partial}{\partial y},\ v_{13} = -\frac{\partial}{\partial u} - \frac{\partial}{\partial x}.$$

One should take care not to use a stereographic projection form the north pole here as it is orientation reversing. Now a direct computation shows that $\omega_K = -2\,\mathfrak{Im}\,\omega_{cr}$. □

4.2. Mutual energies for space curves. Let $K_1, K_2 \subset \mathbb{R}^3$ be a pair of disjoint oriented space curves. Each of them is the boundary of an orientable surface Ω_i (Seifert surface), but we will need these surfaces to be disjoint. This is not possible if K_1, K_2 are linked. Hence we consider $K_1, K_2 \subset \mathbb{R}^3 \subset \mathbb{R}^n$ for $n \geq 5$. Then there exist disjoint orientable surfaces $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ with $\partial\Omega_i = K_i$. Now we can generalize the definition 2.1 to space curves.

DEFINITION 4.2. In the situation described above, we define the mutual energy of K_1, K_2 by

$$E(K_1, K_2) = \frac{1}{2}\int_{\Omega_1 \times \Omega_2} \boldsymbol{\omega} \wedge \boldsymbol{\omega}.$$

This definition does not depend on the choice of Ω_1, Ω_2 as shown by the following proposition.

PROPOSITION 4.3. *In the situation above,*

$$E(K_1, K_2) = -\frac{1}{2}\int_{K_1 \times K_2} \cos\theta_1 \cos\theta_2 \frac{dp_1 dp_2}{|p_2 - p_1|^2}, \tag{27}$$

where $\theta_i \in [0, \pi]$ is the angle between $\overrightarrow{dp_i}$ and $p_2 - p_1$.

PROOF. The proof of (6) also works here but using

$$\lambda = \frac{\sum(z_i - w_i)dz_i}{|z - w|^2}, \qquad \rho = \frac{\sum(z_i - w_i)dw_i}{|z - w|^2},$$

so that $d\lambda = d\rho = \boldsymbol{\omega}$ (cf. (26)). □

4.3. Linking with circles. Next we give an interpretation of $E(K_1, K_2)$ as the average of some linking numbers with circles. Recall the following result of Banchoff and Pohl [5]: given two disjoint oriented curves $K_1, K_2 \subset \mathbb{R}^3$,

$$\int_{A(1,3)} \lambda(\ell, K_1) \cdot \lambda(\ell, K_2) d\ell = \int_{K_1 \times K_2} \cos\theta_1 \cos\theta_2 dp_1 dp_2$$

where $A(1,3)$ is the space of lines $\ell \subset \mathbb{R}^3$, endowed with a (suitably normalized) invariant measure $d\ell$, and λ denotes the linking number. Note that the integrand on the left hand side is independent of the orientation of ℓ. It changes sign when we change the orientation of K_1 or K_2.

We now look for an analogue of the previous result in the realm of Möbius geometry. The role of lines will be played by circles. Let us denote the set of all oriented circles $\gamma \subset \mathbb{R}^3$ by $\mathcal{S}(1,3)$. This is a homogeneous space of the Möbius group Möb_3 with isotropy group $\mathbb{S}^1 \times \text{Möb}_1$. Since these are unimodular groups, the space of circles $\mathcal{S}(1,3)$ admits a measure $d\gamma$ invariant under Möb_3 (cf. [16, p.168 (c)]). Let us describe this measure explicitly. Each circle $\gamma \subset \mathbb{R}^3$ is uniquely determined by its center $c \in \mathbb{R}^3$, the radius $r > 0$, and a unit vector $u \in \mathbb{S}^2$ orthogonal to the plane containing γ. Then the (unique up to a constant factor) Möbius invariant measure on the space of circles is

$$d\gamma = \frac{1}{r^4} dr dc du \tag{28}$$

where dc is the volume element of $c \in \mathbb{R}^3$, and du denotes the area element of $u \in \mathbb{S}^2$. Indeed, the latter measure is clearly invariant under the group Sim_3 generated by rigid motions and homotheties of $\mathbb{R}^3$. Such transformations act transitively on the space of circles. Hence, every two measures on $\mathcal{S}(1,3)$ that are invariant under Sim_3 must be a constant multiple of each other. But clearly the measures invariant under Möb_3, which we know exist, are also invariant under Sim_3.

Our next goal is to compute

$$I_3(K_1, K_2) = \int_{\mathcal{S}(1,3)} \lambda(\gamma, K_1) \cdot \lambda(\gamma, K_2) d\gamma\,.$$

It will be useful to take Ω_1, Ω_2 disjoint surfaces with $\partial \Omega_i = K_i$. This is not possible if the curves are linked. To solve this we consider again $K_1, K_2 \subset \mathbb{R}^n$ with $n \geq 5$, and we consider the general problem of determining

$$I_n(K_1, K_2) = \int_{\mathcal{S}(n-2,n)} \lambda(\xi, K_1) \cdot \lambda(\xi, K_2) d\xi\,,$$

where $d\xi$ is the conformally invariant measure in the space of oriented codimension 2 spheres $\mathcal{S}(n-2,n)$. Just like in the case $n = 3$, this space admits a Möbius invariant measure given in terms of the radius r, the center $c \in \mathbb{R}^n$ and a normal direction $u \in \mathbb{S}^{n-1}$ by

$$d\xi = \frac{1}{r^{n+1}} dr dc du\,.$$

Note that considering $K_1, K_2 \subset \mathbb{R}^n \subset \mathbb{R}^{n+p}$ one has $I_n(K_1, K_2) = c_{n,p} I_{n+p}(K_1, K_2)$ for a constant $c_{n,p}$ to be computed. Therefore, it is enough to consider the problem for $n \geq 5$.

Let $\mathcal{S}(0,n) = \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ denote the space of point pairs (oriented 0-spheres). We consider the flag space

$$\mathcal{F} = \{(w, z; \xi) \in \mathcal{S}(0,n) \times \mathcal{S}(n-2,n) \,:\, w, z \in \xi\}\,. \tag{29}$$

There is a natural double fibration

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{S}(0,n) & & \mathcal{S}(n-2,n) \end{tikzcd}$$

with π_1, π_2 the obvious maps. Note that $\mathcal{F}$ can be identified with $\mathcal{S}(0,n) \times G^+(2,n)$ where $G^+(2,n)$ denotes the Grassmannian of oriented planes in $\mathbb{R}^n$. This way π_1 is just the projection on the first factor. Note that the dimensions of $\mathcal{F}$, $\mathcal{S}(0,n)$, and $\mathcal{S}(n-2,n)$ are given by $4n-4$, $2n$, and $2n$ respectively.

PROPOSITION 4.4. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be disjoint surfaces with boundary K_1, K_2 respectively. Then*

$$I_n(K_1,K_2) = \int_{\Omega_1\times\Omega_2} ((\pi_1)_* \circ \pi_2^*)(d\xi) = -\frac{2\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{vol}(\mathbb{S}^{n-2})}{n(n-1)\pi} \cdot \int_{\Omega_1\times\Omega_2} \boldsymbol{\omega}\wedge\boldsymbol{\omega}$$

where $(\pi_1)_*$ denotes integration along the fibers of π_1.

PROOF. We show first the second equality. Again, it will be useful to consider $\mathbb{R}^n$ as the ideal boundary of $\mathbb{H}^{n+1}$. Given an orthonormal frame $o; e_1, \dots, e_{n+1}$ of $\mathbb{H}^{n+1}$, we consider the geodesic $\ell(t) = \exp_o(te_1)$, and the codimension 2 geodesic plane $L = \exp_o(e_n \wedge e_{n+1})^\perp$. This defines an element $(w, z; \xi) \in \mathcal{F}$, where $z = \lim_{t\to-\infty} \ell(t)$, $w = \lim_{t\to+\infty} \ell(t)$, and $\xi \subset \mathbb{S}^n$ is the set of ideal points of L. Using such local frames one can write

$$d\xi = \omega_{n+1} \wedge \omega_{1,n+1} \wedge \cdots \wedge \omega_{n-1,n+1} \wedge \omega_n \wedge \omega_{1,n} \wedge \dots \omega_{n-1,n}\,, \tag{30}$$

where $\omega_i = \langle do, e_i\rangle$, and $\omega_{ij} = \langle \nabla e_i, e_j\rangle$, where ∇ is the riemannian connection of $\mathbb{H}^{n+1}$. Indeed, the right hand side is a common expression of the isometry invariant measure of (codimension 2) geodesic planes of $\mathbb{H}^{n+1}$ (cf. [16, (17.35)]). Hence both sides coincide except for a constant factor. To find this factor, we assume by invariance that $o = (0, \dots, 0, 1)$ in the half-space model, and e_i is the canonical basis. Then $r = 1$, and $dr = \omega_{n+1}$, $dc = \omega_{1,n+1} \wedge \cdots \wedge \omega_{n-1,n+1} \wedge \omega_n$, $du = \omega_{1,n} \wedge \cdots \wedge \omega_{n-1,n}$. This shows (30).

Now, given $(w,z) \in \mathcal{S}(0,n)$ we take a frame $p; u_1, u_2, \dots, u_{n+1}$ defining an element $(w, z; \xi_0) \in \mathcal{F}$ as explained above. Then, for any other point $(w, z; \xi) \in \pi_1^{-1}(w,z)$ in the fiber we can choose a frame $p; e_1, e_2, \dots, e_{n+1}$ with $e_1 = u_1$. Note that,

$$\omega_n \wedge \omega_{n+1} = \langle e_n \wedge e_{n+1}, dp \wedge dp\rangle = \langle \sum_{2\le i<j\le n+1} p_{ij} u_i \wedge u_j, dp \wedge dp\rangle\,,$$

$$\omega_{1,n} \wedge \omega_{1,n+1} = \langle e_n \wedge e_{n+1}, \nabla e_1 \wedge \nabla e_1\rangle = \langle \sum_{2\le i<j\le n+1} p_{ij} u_i \wedge u_j, \nabla e_1 \wedge \nabla e_1\rangle\,,$$

where p_{ij} are the Plücker coordinates of $e_n \wedge e_{n+1}$ in $\bigwedge^2 (e_1)^\perp \subset \bigwedge^2 T_p\mathbb{H}^{n+1}$, i.e.,

$$p_{ij} = \begin{vmatrix} \langle e_n, u_i\rangle & \langle e_n, u_j\rangle \\ \langle e_{n+1}, u_i\rangle & \langle e_{n+1}, u_j\rangle \end{vmatrix}.$$

This way, the fiber $\pi_1^{-1}(w, z)$ is identified with a submanifold P (given by the Plücker relations) of the unit sphere $\mathbb{S}^{N-1}$ with $N = \binom{n}{2}$. Thus

$$d\xi = \sum_{i<j, r<s} p_{ij} p_{rs} \langle u_i, dp\rangle \wedge \langle u_j, dp\rangle \wedge \langle u_i, \nabla e_1\rangle \wedge \langle u_j, \nabla e_1\rangle \wedge dP\,, \tag{31}$$

where dP is the volume element on P induced by the metric of $\mathbb{S}^{N-1}$. Now, since $P \subset \mathbb{S}^{N-1}$ and P is isometric to the Grassmannian of oriented 2-planes in $\mathbb{R}^n$,

$$\int_P p_{ij}^2 dP = \binom{n}{2}^{-1} \int_P \sum_{2\le r<s\le n+1} p_{rs}^2 dP = \binom{n}{2}^{-1} \operatorname{vol}(P) = \binom{n}{2}^{-1} \frac{\operatorname{vol}(\mathbb{S}^{n-1})\operatorname{vol}(\mathbb{S}^{n-2})}{2\pi}\,.$$

Let this constant be denoted by β. On the other hand, the function p_{ij} is odd with respect to the symmetry of $\mathbb{S}^{N-1}$ fixing $(u_i)^\perp$. Hence,

$$\int_P p_{ij} p_{rs} dP = 0 \quad \text{for } \{i, j\} \neq \{r, s\}\,.$$

Therefore

$$\pi_{1*}\pi_2^* d\xi = \int_{\pi_1^{-1}(w,z)} \pi_2^* d\xi = \beta \sum_{2\le i<j\le n+1} \langle u_i, dp\rangle \wedge \langle u_j, dp\rangle \wedge \langle u_i, \nabla e_1\rangle \wedge \langle u_j, \nabla e_1\rangle$$

$$= \beta \sum_{2\le i<j\le n+1} \omega_i \wedge \omega_j \wedge \omega_{1i} \wedge \omega_{1j} = -\frac{\beta}{2} d\omega_1 \wedge d\omega_1 = -2\beta \cdot \boldsymbol{\omega} \wedge \boldsymbol{\omega}$$

since $d\omega_1 = 2\boldsymbol{\omega}$.

In order to show the first equality, let us consider the region $U = \pi_1^{-1}(\Omega_1 \times \Omega_2) \subset \mathcal{F}$, and the mapping $\phi = \pi_2|_U : U \to \mathcal{S}(n-2, n)$. By (31), one can check that the multiplicity of $\xi \in \mathcal{S}(n-2, n)$ as an image value of ϕ (taking orientations into account) is given by

$$\nu(\xi) = \sum_{z_1 \in \xi\cap\Omega_1, z_2 \in \xi\cap\Omega_2} \epsilon(z_1)\epsilon(z_2) = (\xi \cdot \Omega_1)(\xi \cdot \Omega_2) = \lambda(\xi, K_1)\lambda(\xi, K_2)\,,$$

where $\varepsilon(z_i)$ is the contribution of z_i to the algebraic intersection $\xi \cdot \Omega_i$. Here $S(n-2, n)$ was oriented by $d\xi$, and we used the orientation in $U \equiv \Omega_1 \times \Omega_2 \times P$ given by $d\Omega_1 \wedge d\Omega_2 \wedge dP$. Finally, the coarea formula and integration along the fibers yield

$$I_n(K_1, K_2) = \int_{\mathcal{S}(n-2,2)} \nu(\xi) d\xi = \int_U \pi_2^*(d\xi) = \int_{\pi_1^{-1}(\Omega_1\times\Omega_2)} \pi_2^*(d\xi) = \int_{\Omega_1\times\Omega_2} (\pi_1)_* \pi_2^* d\xi\,.$$

□

In particular, the constant $c_{n,p}$ determined by $I_n(K_1, K_2) = c_{n,p} I_{n+p}(K_1, K_2)$ is given by

$$c_{n,p} = \frac{\operatorname{vol}(\mathbb{S}^{n+p})\operatorname{vol}(\mathbb{S}^{n+p-1}) n(n-1)}{\operatorname{vol}(\mathbb{S}^n)\operatorname{vol}(\mathbb{S}^{n-1})(n+p)(n+p-1)}\,.$$

COROLLARY 4.5. *The mutual energy of a pair of disjoint space curves is given by*

$$E(K_1, K_2) = -\frac{3}{16\pi} \int_{\mathcal{S}(1,3)} \lambda(\gamma, K_1)\lambda(\gamma, K_2) d\gamma\,.$$

REMARK 4.6. Note that $E(K_1, K_2) = 0$ does not imply that every circle is trivially linked with K_1 or with K_2. An example where the mutual energy vanishes is given by a pair of circles K_1, K_2 such that K_1 is orthogonal to every sphere containing K_2. For instance, K_1 can be the line of points $p \in \mathbb{R}^3$ such that $|q - p|$ is constant for all $q \in K_2$.

4.4. Renormalized measure of circles linking a space curve. Let us now consider a single space curve $K \subset \mathbb{R}^3$ which is assumed to be closed but not necessarily connected. We will define a functional $E(K)$ such that $E(K_1 \cup K_2) = E(K_1) + E(K_2) + 2E(K_1, K_2)$ whenever K_1, K_2 are disjoint. Our results are closely analogous to the following formula due to Banchoff and Pohl (cf.[5])

$$\int_{A(1,3)} \lambda(\ell, K)^2 d\ell = -\int_{K\times K} \cos\tau \sin\theta_p \sin\theta_q dpdq = \int_{K\times K} \cos\theta_p \cos\theta_q dpdq\,, \tag{32}$$

where $\theta_p \in [0, \pi]$ (resp. $\theta_q \in [0, \pi]$) is the angle between $\overrightarrow{dp}$ (resp. $\overrightarrow{dq}$) and $q - p$, and τ is the angle between the two oriented planes through p, q tangent to K at p and q respectively. These planes are oriented by $\overrightarrow{dp} \wedge (q - p)$ and $\overrightarrow{dq} \wedge (q - p)$ respectively. In order to define $E(K)$ it would be natural to consider

$$\int_{\mathcal{S}(1,3)} \lambda(\gamma, K)^2 d\gamma.$$

However this integral diverges due to the blow up of the density $d\gamma = r^{-4} drdcdu$ when the radius r goes to 0. Hence we take the following renormalization.

DEFINITION 4.7. Let $K \subset \mathbb{R}^3$ be a smooth, closed (maybe non-connected) space curve. We define

$$E(K) = \lim_{\varepsilon\to 0}\left(\frac{3\pi L(K)}{8\varepsilon} - \frac{3}{16\pi}\int_{\mathcal{S}_\varepsilon(1,3)} \lambda(\gamma, K)^2 d\gamma\right),$$

where $\mathcal{S}_\varepsilon(1, 3)$ is the subset of $\mathcal{S}(1, 3)$ containing the circles of radius $r > \varepsilon$.

The following proposition gives two expressions of $E(K)$ which involve no renormalization.

PROPOSITION 4.8. *The previous limit exists, and coincides with the following integral*

$$E(K) = \frac{3}{16\pi}\int_{\mathcal{S}(1,3)} (\#(K \cap [\gamma]) - \lambda(\gamma, K)^2) d\gamma\,,$$

where $[\gamma]$ denotes the disk with boundary γ. The previous integral converges and coincides with

$$E(K) = -\frac{1}{2}\int_{K\times K} \cos\tau \sin\theta_p \sin\theta_q \frac{dpdq}{|q - p|^2}\,, \tag{33}$$

where θ_p, θ_q and τ are as in (32).

PROOF. Let $A(2, 3)$ be the space of oriented affine planes of $\mathbb{R}^3$, which are given by a direction $u \in \mathbb{S}^2$ and a signed distance ρ from the origin. Let

$$d\wp = d\rho du$$

which defines an invariant measure on $A(2,3)$. From the expression (28) of $d\gamma$,

(34)
$$\int_{\mathcal{S}_\varepsilon(1,3)} \#(K\cap[\gamma])d\gamma = \int_{A(2,3)}\int_\varepsilon^\infty \int_\wp \#(B_c(r)\cap\wp\cap K)\frac{dcdrd\wp}{r^4} = \frac{\pi}{\varepsilon}\int_{A(2,3)} \#(\wp\cap K)d\wp\,,$$

where dc denotes the area element of c inside $\wp$. From this, Crofton's formula (cf. [16, (14.73)]) yields the first equation, except for the convergence of the integral.

To see the second part we start with the following equality, which can be proved like (21):

$$\#([\gamma]\cap K) - \lambda^2(\gamma,K) = \sum_{x,y}\epsilon(x)\epsilon(y)\,, \tag{35}$$

where the sum runs over the pairs $x, y \in K\cap[\gamma]$, and $\varepsilon(x)$, $\varepsilon(y)$ are the intersection signs of K and $[\gamma]$. Integrating (35) with respect to γ and using (28) yields

$$\int_{\mathcal{S}(1,3)} (\#([\gamma]\cap K) - \lambda^2(\gamma,K))d\gamma = -\int_{A(2,3)}\int_0^\infty \frac{1}{4}\int_\wp \sum_{x,y}\epsilon(x)\epsilon(y)\frac{1}{r^4}dcdrd\wp\,,$$

where the sum runs over the pairs $x, y \in K\cap B_c(r)\cap\wp$. Now, an elementary computation shows

$$E(K) = -\frac{1}{\pi}\int_{A(2,3)} \sum_{x,y\in\wp\cap K}\frac{\epsilon(x)\epsilon(y)}{|y-x|}d\wp\,.$$

Finally, by the results of Pohl [15, Section 2] we get

$$E(K) = -\frac{1}{2}\int_{K\times K}\cos\tau\sin\theta_p\sin\theta_q\frac{dpdq}{|q-p|^2}\,.$$

We can now check the convergence of the integrals since

$$\sin\theta_p = \frac{\kappa(p)}{2}|q-p| + O(|q-p|^2) \tag{36}$$

where κ denotes the curvature of K. Indeed, let $f : (0,\varepsilon)\to\mathbb{R}^3$ be an embedding with $f((0,\varepsilon))\subset K$ and $|f'(s)| = 1$ for all $s\in(0,\varepsilon)$. Then, for $p = f(s), q = f(t)$

$$|\sin\theta_p| = \left|f'(s)\times\frac{(f(t)-f(s))}{|f(t)-f(s)|}\right| = \frac{|f'(s)\times(f'(s)(t-s)+\frac{1}{2}f''(s)(t-s)^2+O(|t-s|^3)|}{|f(t)-f(s)|}$$
$$= \frac{1}{2}|f''(s)||t-s| + O((t-s)^2)\,.$$

□

By Proposition 3.13 we have

COROLLARY 4.9. *When K is a planar curve, Definitions* 3.11 *and* 4.7 *of the energy $E(K)$ coincide.*

In particular $E(K) > 0$ for K planar and convex. This explains the choice of the sign in the definition of E. However, for space curves there is no lower (nor upper) bound of E. Indeed, if two arcs of K come close to each other (not orthogonally) then $E(K)$ blows up to $\pm\infty$.

REMARK 4.10. The functional E is continuous with respect to the topology of uniform C^2 convergence. Even more, suppose a sequence of closed curves $K_n \subset \mathbb{R}^3$ converging pointwise to a closed embedded curve $K \subset \mathbb{R}^3$ in the C^1 topology, and with uniformly bounded curvature. Then $\lim_{n\to\infty} E(K_n) = E(K)$. This follows from (33), and Lebesgue's dominated convergence theorem, which applies here in virtue of (36).

It is interesting to recall that the *writhe* of K is given by

$$W(K) = \frac{1}{4\pi}\int_{K\times K} \sin\tau \sin\theta_p \sin\theta_q \frac{dpdq}{|q-p|^2} \, .$$

Also, $W(K)$ is the average of signed self-intersections of projections of K. A remarkable fact is that the writhe is invariant under orientation preserving Möbius transformations (cf. [6]). We will see below that $E(K)$ not only has an integral expression similar to $W(K)$, but it shares also this invariance (cf. Corollary 4.17).

PROPOSITION 4.11.

$$E(K) = \lim_{\varepsilon\to 0}\left(\frac{L(K)}{\varepsilon} - \frac{1}{2}\int_{K\times K\setminus\Delta_\varepsilon} \cos\theta_p \cos\theta_q \frac{dpdq}{|q-p|^2}\right) . \tag{37}$$

PROOF. We use intergation by parts, as in Proposition 5 of [5]. Given $p, q \in K\times K\setminus\Delta$ let e_1, e_2, e_3 be an orthonormal moving frame (locally defined on $K\times K\setminus\Delta$ with $e_1 = (q-p)/|q-p|$, and $e_3 \perp T_pK$. As usual let $\omega_i = dp\cdot e_i$, and $\omega_{ij} = de_i\cdot e_j$. Then

$$\begin{aligned}
\cos\theta_p\cos\theta_q\frac{dp\wedge dq}{|q-p|^2} &= -\frac{d(|q-p|)\wedge\omega_1}{|q-p|^2} = d\left(\frac{1}{|q-p|}\right)\wedge\omega_1 \\
= d\left(\frac{\omega_1}{|q-p|}\right) - \frac{1}{|q-p|}d\omega_1 &= d\left(\frac{\omega_1}{|q-p|}\right) - \frac{\omega_{12}\wedge\omega_2}{|q-p|} - \frac{\omega_{13}\wedge\omega_3}{|q-p|} \\
= d\left(\frac{\omega_1}{|q-p|}\right) &+ \cos\tau\sin\theta_p\sin\theta_q\frac{dp\wedge dq}{|q-p|^2} \, ,
\end{aligned}$$

since $\omega_2 = \sin\theta_p dp$, $\omega_3 = 0$, and $\omega_{12} = \cos\tau\sin\theta_q|q-p|^{-1}dq$. On the other hand

$$\int_{K\times K\setminus\Delta_\varepsilon} d\frac{\omega_{12}}{|q-p|} = \int_{\partial\Delta_\varepsilon}\frac{\omega_{12}}{|q-p|} = 2\int_K \frac{1}{\varepsilon}dq + O(\varepsilon) = \frac{2}{\varepsilon}L(K) + O(\varepsilon) \, .$$

□

Propositions 4.3 and 4.11 imply

COROLLARY 4.12. *Let K_1, K_2 be a pair of disjoint oriented curves. Then*

$$E(K_1\cup K_2) = E(K_1) + E(K_2) + 2E(K_1, K_2) \, . \tag{38}$$

With the equation above and Definition 4.7 we recover Corollary 4.5.

PROPOSITION 4.13. *We have*

$$E(K) = \lim_{\varepsilon\to 0}\left(\frac{L(K)}{2\varepsilon} - \frac{1}{4}\int_{K\times K\setminus\Delta_\varepsilon}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}\right) ,$$

where $\overrightarrow{dp} \cdot \overrightarrow{dq} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3$.

PROOF. It is elementary to see

$$\overrightarrow{dp} \cdot \overrightarrow{dq} = (\cos\theta_p \cos\theta_q + \cos\tau \sin\theta_p \sin\theta_q) dp dq \,.$$

Now averaging (33) and (37) gives the result. □

COROLLARY 4.14. *Let K_1, K_2 be a pair of disjoint curves. Then*

$$E(K_1, K_2) = -\frac{1}{4} \int_{K_1 \times K_2} \frac{\overrightarrow{dp} \cdot \overrightarrow{dq}}{|q-p|^2} \,.$$

It is interesting to remark that if the power of the denominator is replaced by 1 then the previous integral becomes von Neumann's formula for the mutual inductance, up to a constant factor.

4.5. Gauss-Bonnet theorem for complete surfaces in hyperbolic space. Next we show the Möbius invariance of $E(K)$. To this end we will use a Gauss-Bonnet formula for complete surfaces in hyperbolic space.

Let $S \subset \mathbb{H}^4$ be a surface in hyperbolic 4-space (Poincaré model) with cone-like ends on the curve $K \subset \mathbb{R}^3 = \partial_\infty \mathbb{H}^4$ (recall Definition 3.16). Given an element $(x, e) \in N^1 S$, the unit normal bundle of S, the Lipschitz-Killing curvature $\kappa(x, e)$ is defined as the determinant of the endomorphism $de_{(x,e)}$ of $T_{(x,e)}(N^1 S)$. We are interested in the integral of $\kappa(x, e)$ along the fibers $N^1_x S$ of $N^1 S$. Using Gauss equation one gets easily

$$\frac{1}{\pi} \int_{N^1_x S} \kappa(x, e) de = \kappa_i(x) + 1 \,, \tag{39}$$

where de is the volume element on $N^1_x S$, and κ_i denotes the Gauss (intrinsic) curvature of S. The additive constant 1 comes from the sectional curvature of the ambient space $\mathbb{H}^4$. Given $\varepsilon > 0$ put $S_\varepsilon = \{x \in S \,:\, x^4 \geq \varepsilon\}$. Then, the classical intrinsic Gauss-Bonnet formula gives

$$\int_{S_\varepsilon} (\kappa_i + 1) dS = 2\pi \chi(S_\varepsilon) + A(S_\varepsilon) - \int_{\partial S_\varepsilon} k_g = 2\pi \chi(S_\varepsilon) + A(S_\varepsilon) - \frac{L(K)}{\varepsilon} + O(\varepsilon) \,,$$

where k_g is the geodesic curvature in S_ε. We used $k_g = 1 + O(\varepsilon^2)$, and the fact that the euclidean lengths of ∂S_e and K have a difference of order ε^2. Taking $\varepsilon \to 0$ we get

$$\frac{1}{\pi} \int_{N^1 S} \kappa(x, e) de dS = \int_S (\kappa_i(x) + 1) dS = 2\pi \chi(S) + \lim_{\varepsilon \to 0} \left(A(S_\varepsilon) - \frac{L(K)}{\varepsilon} \right) . \tag{40}$$

The convergence of the integrals follows from the hypothesis that S has cone-like ends by the same arguments as Proposition 7 in [17]. This formula appeared in a more general setting in [2]. The limit in (40) was called the *renormalized area* of S. Here we will use a different renormalization that leads to the same value.

PROPOSITION 4.15. *Let $\mathcal{L}_2$ denote the space of oriented 2-dimensional geodesic planes in $\mathbb{H}^4$. Let $\mathcal{L}_{2,\varepsilon}$ be the subset of $\mathcal{L}_2$ consisting of the planes which define a circle* *in $\mathbb{R}^3 = \partial_\infty \mathbb{H}^4$ of radius larger than ε. Then the renormalized area of S is given by*

$$\begin{aligned}\lim_{\varepsilon\to 0}\left(A(S_\varepsilon) - \frac{L(K)}{\varepsilon}\right) &= \lim_{\varepsilon\to 0}\left(\frac{3}{4\pi^2}\int_{\mathcal{L}_2} \#(\ell \cap S_\varepsilon)d\ell - \frac{L(K)}{\varepsilon}\right) \\ &= \lim_{\varepsilon\to 0}\left(\frac{3}{4\pi^2}\int_{\mathcal{L}_{2,\varepsilon}} \#(\ell \cap S)d\ell - \frac{3L(K)}{2\varepsilon}\right),\end{aligned}$$

where $d\ell$ is the pull-back of $d\gamma$ through the map $\mathcal{L}_2 \to \mathcal{S}(1,3)$ given by $\ell \mapsto \gamma = \partial_\infty \ell$.

PROOF. The first equality follows immediately from the Crofton formula (cf. [16, p.245]). In order to check the second equality, we need the following claim: given two surfaces $R, S \subset \mathbb{H}^4$ with cone-like ends on the same ideal curve $K \subset \partial_\infty \mathbb{H}^4$, one has

$$\lim_{\varepsilon\to 0}\int_{\mathcal{L}_2} (\#(\ell \cap R_\varepsilon) - \#(\ell \cap S_\varepsilon))d\ell = \lim_{\varepsilon\to 0}\int_{\mathcal{L}_{2,\varepsilon}} (\#(\ell \cap R) - \#(\ell \cap S))d\ell\,. \tag{41}$$

To show this, we consider

$$F = \{(p,\ell) \in \mathbb{H}^4 \times \mathcal{L}_2 \;:\; p \in \ell\}$$

and the projections $p_1 \colon F \to \mathbb{H}^4$, $p_2 \colon F \to \mathcal{L}_2$. Denoting by h the last coordiante function on $\mathbb{H}^4$, and by r the radius function on $\mathcal{L}_2$, equation (41) is equivalent to

$$\lim_{\varepsilon\to 0}\int_{\{h\circ p_1>\varepsilon\}} (\mathbf{1}_R \circ p_1 - \mathbf{1}_S \circ p_1)\; p_2^* d\ell = \lim_{\varepsilon\to 0}\int_{\{r\circ p_1>\varepsilon\}} (\mathbf{1}_R \circ p_1 - \mathbf{1}_S \circ p_1)\;\; p_2^* d\ell$$

where $\mathbf{1}_R, \mathbf{1}_S$ are the indicator functions of S, R respectively. Hence, to prove (41) it is enough to show the absolute convergence of

$$\int_F (\mathbf{1}_R \circ p_1 - \mathbf{1}_S \circ p_1)\; p_2^* d\ell = \int_{\mathcal{L}_2} (\#(\ell \cap R) - \#(\ell \cap S))d\ell\,. \tag{42}$$

With the same arguments as in Section 3 of [17] one shows that

$$\int_{\mathcal{L}_2} (\#(\ell \cap R) - \lambda^2(\ell, K))d\ell$$

is absolutely convergent. Hence we have convergence in (42), and (41) follows.

Therefore, it is enough to prove the second equality of the statement in the particular case $S = K \times (0,\infty) \subset \mathbb{H}^4$. In this case, by the Crofton formula

$$\int_{\mathcal{L}_2} \#(\ell \cap S_\varepsilon)d\ell = \frac{4\pi^2}{3}A(S_\varepsilon) = \frac{4\pi^2}{3}L(K)\int_\varepsilon^\infty \frac{1}{t^2}dt = \frac{4\pi^2}{3\varepsilon}L(K)\,.$$

By (34) and [16, (14.73)] we have

$$\int_{\mathcal{L}_{2,\varepsilon}} \#(\ell \cap S)d\ell = \frac{2\pi^2}{\varepsilon}L(K)\,.$$

Hence, all the limits in the statement vanish trivially for $S = K \times (0,\infty)$. □

By the previous proposition, equation (40) becomes

$$\int_S (\kappa_i + 1)dS = 2\pi\chi(S) + \lim_{\varepsilon\to 0}\left(\frac{3}{4\pi^2}\int_{\mathcal{L}_{2,\varepsilon}} \#(\ell\cap S)d\ell - \frac{3}{2\varepsilon}L(K)\right).$$

Combining this with Definition 4.7 we get the following

COROLLARY 4.16. *For any surface $S \subset \mathbb{H}^4$ with cone-like ends*

$$\frac{1}{\pi}\int_{N^1S}\kappa(x,e)dedS=\int_S(\kappa_i(x)+1)dS=2\pi\chi(S)+\frac{3}{4\pi^2}\int_{\mathcal{L}_2}(\#(\ell\cap S)-\lambda^2(\ell,K))d\ell-\frac{4}{\pi}E(K),$$

where $d\ell$ is the invariant measure on $\mathcal{L}_2$ corresponding to $d\gamma$.

COROLLARY 4.17. *$E(K)$ is invariant under Möbius transformations.*

PROOF. All other terms in the equation above are invariant under isometries of $\mathbb{H}^4$. □

4.6. Expressions via parallel curves. Here we show the following

PROPOSITION 4.18. *Let K be a closed space curve with nowhere vanishing curvature κ. Let K_ε be an ε-parallel curve given by $K_\varepsilon = \{x + \varepsilon\, n(x) \ :\ x \in K\}$, where n is the unit principal normal vector to K. Then*

$$E(K) = \lim_{\varepsilon\to 0}\left(\frac{\pi}{4\varepsilon}L(K) + E(K, K_\varepsilon)\right) - \frac{\pi}{8}\int_K \kappa(p)dp. \tag{43}$$

REMARK 4.19. The hypothesis that κ is nowhere zero is no loss of generality: performing a Möbius transformation we can bring every space curve K to a position $\widetilde{K}$ with non-vanishing curvature. Moreover this transformation can be taken arbitrarily close to the identity. Indeed, let $C(K)$ denote the curvature tube of K, namely, $C(K) = \cup_{p\in K} C_O(p)$, where $C_O(p)$ denotes the osculating circle to K at p. Let us take the image of K after an inversion in a sphere with a sufficiently large radius r whose center does not belong to $C(K)$, and is at distance r from K. Then an orientation reversing isometry of $\mathbb{R}^3$ gives the desired $\widetilde{K}$.

We will need the following estimate.

LEMMA 4.20. *Let K be a simple smooth space curve with non-vanishing curvature. Let ε and δ be small positive numbers with $\delta \ll \varepsilon$. Then for any point p in K we have*

$$\int_{K_\delta\cap B_\varepsilon(p)} \frac{\boldsymbol{v}_p\cdot\overrightarrow{dq}}{|q-p|^2} = \frac{\pi}{\delta} - \frac{2}{\varepsilon} - \frac{\pi}{2}\kappa(p) + O(\varepsilon), \tag{44}$$

where $\boldsymbol{v}_p$ is the unit tangent vector to K at p.

PROOF. Suppose K can be expressed as $K = f(\mathbb{S}^1)$ by an embedding f which is parametrized by the arc-length. We can assume that $p = f(0) = 0$. Then K_δ is given by $K_\delta = f_\delta(\mathbb{S}^1)$, where $f_\delta = f + \delta\kappa^{-1} f''$. Note that $f'\cdot f' \equiv 1$ and $f''\cdot f'' = \kappa^2$ imply

$$f'\cdot f'' = 0,\ f'\cdot f''' = -\kappa^2,\ f''\cdot f''' = \kappa\kappa', \quad\text{and}\quad f'\cdot f^{(4)} = -3\kappa\kappa'.$$

The numerator inside the integral of (44) can be estimated as

$$f'(0)\cdot f_\delta'(s) = (1-\kappa(0)\delta) + O(1)\delta s + O(s^2) \tag{45}$$

since direct computation shows

$$\begin{aligned} f'(0)\cdot f_\delta'(0) &= 1-\kappa(0)\delta\,, \\ f'(0)\cdot f_\delta''(0) &= O(1)\delta\,. \end{aligned}$$

On the other hand, the denominator inside the integral of (44) can be estimated as

$$\begin{aligned} f_\delta(s)\cdot f_\delta(s) &= f(s)\cdot f(s) + 2\frac{\delta}{\kappa(s)} f(s)\cdot f''(s) + \delta^2 \\ &= \delta^2 + \left(s^2 + O(s^4)\right) + 2\delta\left(\frac{1}{\kappa(0)} + O(s)\right)\left(-\frac{\kappa^2(0)}{2}s^2 + O(s^3)\right) \\ &= \delta^2 + (1-\kappa(0)\delta)\,s^2 + \delta O(s^3) + O(s^4) \\ &= \left(\delta^2 + (1-\kappa(0)\delta)s^2\right)\left(1+O(1)s^2\right), \end{aligned} \tag{46}$$

since

$$\frac{s\delta}{\delta^2 + (1-\kappa(0)\delta)s^2} = O(1)\,, \qquad \frac{s^2}{\delta^2 + (1-\kappa(0)\delta)s^2} = O(1)\,.$$

Let us denote $\kappa(0)$ and $\kappa'(0)$ simply by κ and κ' in what follows. Let $s_- < 0$ and $s_+ > 0$ be parameters when $f_\delta(s)$ passes through $\partial B_\varepsilon(f(0))$. Then, since $s_\pm = O(\varepsilon)$, equation (46) implies

$$s_\pm = \pm\sqrt{\frac{\varepsilon^2-\delta^2}{1-\kappa\delta}} + O(\varepsilon^3)\,.$$

Therefore, by (46) and (45), the left hand side of (44) can be estimated as

$$\begin{aligned} &\int_{-\sqrt{\frac{\varepsilon^2-\delta^2}{1-\kappa\delta}}+O(\varepsilon^3)}^{\sqrt{\frac{\varepsilon^2-\delta^2}{1-\kappa\delta}}+O(\varepsilon^3)} \frac{\left\{(1-\kappa\delta) + O(1)\delta s + O(1)s^2\right\}\left(1+O(1)s^2\right)}{\delta^2 + (1-\kappa\delta)s^2}\,ds \\ &= \int_{-\sqrt{\frac{\varepsilon^2-\delta^2}{1-\kappa\delta}}}^{\sqrt{\frac{\varepsilon^2-\delta^2}{1-\kappa\delta}}} \frac{1-\kappa\delta}{\delta^2 + (1-\kappa\delta)s^2}\,ds + O(\varepsilon) \\ &= \frac{2}{\delta}\sqrt{1-\kappa\delta}\,\arctan\left(\frac{\sqrt{\varepsilon^2-\delta^2}}{\delta}\right) + O(\varepsilon) \\ &= \frac{2}{\delta}\left(1-\frac{\kappa}{2}\delta\right)\left(\frac{\pi}{2}-\frac{\delta}{\varepsilon}\right) + O(\varepsilon)\,, \end{aligned}$$

which coincides with the right hand side of (44). □

We remark that the same proof, with minor modifications, shows (44) when K and K_δ are planar (not necessarily convex) curves with $K = \partial\Omega$ and $K_\delta = \partial\Omega_\delta$. Proposition 4.18 is immediate from Proposition 4.13, Corollary 4.14, and the following

PROPOSITION 4.21. *Let K be a smooth simple space curve with non-vanishing curvature κ. Let K_δ be a δ-parallel of K in the principal normal direction. Then*

$$\lim_{\varepsilon\to 0}\left(\frac{L(K)}{2\varepsilon}-\frac{1}{4}\int_{K\times K\setminus\Delta_\varepsilon}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}\right)=\lim_{\delta\to 0}\left(\frac{\pi}{4\delta}L(K)-\frac{1}{4}\int_{K\times K_\delta}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}\right)-\frac{\pi}{8}\int_K\kappa(p)dp\,.$$

PROOF (OF PROPOSITION 4.21 AND THEOREM 3.8). We first fix ε so that $0<\varepsilon<1/\max_{p\in K}\kappa(p)$. Suppose $\delta<\varepsilon$. Then

$$-\frac{1}{4}\int_{K\times K_\delta}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}=-\frac{1}{4}\int_{p\in K}\left(\int_{q\in K_\delta\cap B_\varepsilon(p)}\frac{\boldsymbol{v}_p\cdot\overrightarrow{dq}}{|q-p|^2}+\int_{q\in K_\delta\setminus B_\varepsilon(p)}\frac{\boldsymbol{v}_p\cdot\overrightarrow{dq}}{|q-p|^2}\right)dp\,.$$

Clearly

$$\lim_{\delta\to 0}\int_{p\in K}\int_{q\in K_\delta\setminus B_\varepsilon(p)}\frac{\boldsymbol{v}_p\cdot\overrightarrow{dq}}{|q-p|^2}=\int_{p\in K}\int_{q\in K\setminus B_\varepsilon(p)}\frac{\boldsymbol{v}_p\cdot\overrightarrow{dq}}{|q-p|^2}\,.$$

Therefore, if $\delta\ll\varepsilon$ we have, by (44)

$$-\frac{1}{4}\int_{K\times K_\delta}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}=-\frac{1}{4}\int_{p\in K}\left(\frac{\pi}{\delta}-\frac{2}{\varepsilon}-\frac{\pi}{2}\kappa(p)+\int_{K\setminus B_\varepsilon(p)}\frac{\boldsymbol{v}_p\cdot\overrightarrow{dq}}{|q-p|^2}\right)dp+O(\varepsilon)$$

$$=-\frac{\pi}{4\delta}L(K)+\frac{L(K)}{2\varepsilon}+\frac{\pi}{8}\int_K\kappa(p)dp-\frac{1}{4}\int_{K\times K\setminus\Delta_\varepsilon}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}+O(\varepsilon)\,,$$

which implies

$$\lim_{\delta\to 0}\left(\frac{\pi}{4\delta}L(K)-\frac{1}{4}\int_{K\times K_\delta}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}\right)-\frac{\pi}{8}\int_K\kappa(p)dp=\lim_{\varepsilon\to 0}\left(\frac{L(K)}{2\varepsilon}-\frac{1}{4}\int_{K\times K\setminus\Delta_\varepsilon}\frac{\overrightarrow{dp}\cdot\overrightarrow{dq}}{|q-p|^2}\right)\,.$$

This completes the proof. □

5. Möbius invariant expressions. For a compact simply connected domain $\Omega\subset\mathbb{R}^2$ with smooth boundary $K=\partial\Omega$, Theorem 1 in [17] yields

$$E(\Omega)=\frac{\pi^2}{2}+\frac{1}{4}\int_{K\times K}\theta\sin\theta\frac{dpdq}{|q-p|^2} \tag{47}$$

where θ is the oriented angle at p between K (positively oriented), and the circle through p and q that is positively tangent to K at q. More precisely, $\theta\in\mathbb{R}$ is the unique continuous determination of this angle defined on $K\times K$ that vanishes on the diagonal. Note that, unlike the previous expressions we obtained, the integrand in (47) is Möbius invariant.

Next we generalize (47) to compact domains, not necessarily simply connected. By equation (38) it is enough to give analogous expressions for the mutual energy $E(\Omega_1,\Omega_2)$ of two disjoint simply connected domains Ω_1,Ω_2. To this end, we will work with the flag space

$$\mathcal{F}=\{(w,z;\xi)\in\mathcal{S}(0,2)\times\mathcal{S}(1,2)\ :\ w,z\in\xi\}$$

which has natural projections $\pi_1\colon\mathcal{F}\to\mathcal{S}(0,2)$ and $\pi_2\colon\mathcal{F}\to\mathcal{S}(1,2)$. By thinking of $\mathbb{R}^2$ as the ideal boundary of half-space model of $\mathbb{H}^3$, each element $(w,z;\gamma)\in\mathcal{F}$ corresponds to a pair $(\ell, \wp)$ where $\ell \subset \mathbb{H}^3$ is a geodesic line contained in the geodesic plane $\wp \subset \mathbb{H}^3$. Let us choose (locally) an orthonormal frame $(o; e_1, e_2, e_3)$ with $o \in \ell$, $e_1 \in T_o\ell$, $e_3 \perp T_o\wp$. It is easy to check that the 1-form $\omega_{23} = \langle \nabla e_2, e_3 \rangle$ is independent of this choice. Hence it defines a global 1-form φ on $\mathcal{F}$. By construction, φ is Möbius invariant, it vanishes on the fibers of π_2 and measures the oriented angle on the fibers of π_1. The interest of φ comes from the fact that $d\varphi = 2\pi_1^* \Im\mathfrak{m}\,(\omega_{cr})$ (cf. [17, Remark 3]).

LEMMA 5.1. *Let $c(t) = (z(t), w(t); \gamma(t))$ be a curve in $\mathcal{F}$, such that $z(t) \equiv z$ is constant and the circles $\gamma(t)$ are all mutually tangent at z. Then $\varphi(c'(0)) = 0$.*

PROOF. The curve $c(t)$ corresponds to a family of pairs $(\ell(t), \wp(t))$ with $\ell(t) \subset \wp(t) \subset \mathbb{H}^3$. By hypothesis, these geodesics $\ell(t)$ have a common ideal endpoint z. By a Möbius transformation we can send z to infinity. This way, the geodesics $\ell(t)$ become vertical lines in the model. Morevoer, by hypothesis the totally geodesic planes $\wp(t)$ are mapped to a family of vertical parallel affine planes in the model. Then we can choose a moving frame $o(t), e_1(t), e_2(t), e_3(t)$ adapted to (the image of) $(\ell(t), \wp(t))$ as above and such that $o(t) = (o^1(t), o^2(t), 1)$, and $e_1(t), e_2(t), e_3(t)$ are constant vectors, forming an orthonormal basis of $\mathbb{R}^3$. Then clearly $\varphi(c'(t)) = \langle e_2'(t), e_3(t) \rangle = 0$. □

PROPOSITION 5.2. *Let $p_i : \mathbb{S}^1 \to K_i$ be regular parametrizations, and let $\theta(s,t) = \theta(p_1(s), p_2(t)) \in [0, 2\pi)$ be the oriented angle between the circle through $p_2(t)$ that is positively tangent to K_1 at $p_1(s)$, and the circle through $p_1(t)$ that is positively tangent to K_2 at $p_2(t)$. Then*

$$E(\Omega_1, \Omega_2) = \frac{\pi^2}{2} - \frac{1}{8} \int_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{\partial \theta(s,t)}{\partial s} \frac{\partial \theta(s,t)}{\partial t} dt ds\,.$$

PROOF. Let us pick up a point $q_i \in \Omega_i$ $(i = 1, 2)$ in the interior of each region. We denote $\Omega_i^* = \Omega_i \setminus \{q_i\}$. For each region, we take an orientation preserving diffeomorphism

$$F_i : \mathbb{S}^1 \times [0, 1) \longrightarrow \Omega_i^* \qquad i = 1, 2\,,$$

such that $F_i(t, 0) = p_i(t)$. The vector field $X_i = \partial F_i(x,t)/\partial x$ is defined on Ω_i^* and vanishes nowhere. Let us define a section $s_1 : \Omega_1^* \times \Omega_2^* \to \mathcal{F}$ such that $s_1(w, z) = (w, z; \xi)$ with $X_1(w) \in T_w\xi$. Similarly, we define s_2 on $\Omega_1^* \times \Omega_2^*$ so that $X_2(z) \in T_z\xi$ if $s_2(w, z) = (w, z; \xi)$. Let $\Omega_{i,\varepsilon} = \Omega_i \setminus B_\varepsilon(q_i)$. Then

$$E(\Omega_1, \Omega_2) = \lim_{\varepsilon \to 0} \frac{1}{8} \int_{\Omega_{1,\varepsilon} \times \Omega_{2,\varepsilon}} d(s_1^* \varphi) \wedge d(s_2^* \varphi)\,.$$

By Stokes,

$$\int_{\Omega_{1,\varepsilon} \times \Omega_{2,\varepsilon}} d(s_1^* \varphi) \wedge d(s_2^* \varphi) = \int_{(\partial \Omega_{1,\varepsilon} \times \Omega_{2,\varepsilon}) \cup (\Omega_{1,\varepsilon} \times \partial \Omega_{2,\varepsilon})} s_1^* \varphi \wedge d(s_2^* \varphi)\,.$$

Integration on $\partial\Omega_{1,\varepsilon} \times \Omega_{2,\varepsilon}$ vanishes by Lemma 5.1. Using $d(s_1^*\varphi \wedge s_2^*\varphi) = ds_1^*\varphi \wedge s_2^*\varphi - s_1^*\varphi \wedge ds_2^*\varphi$ we get

$$\int_{\Omega_{1,\varepsilon}\times\partial\Omega_{2,\varepsilon}} s_1^*\varphi \wedge d(s_2^*\varphi) = -\int_{\Omega_{1,\varepsilon}\times\partial\Omega_{2,\varepsilon}} d(s_1^*\varphi \wedge s_2^*\varphi) + \int_{\Omega_{1,\varepsilon}\times\partial\Omega_{2,\varepsilon}} ds_1^*\varphi \wedge s_2^*\varphi$$

and the latter integral vanishes again by Lemma 5.1. Taking care of orientations we conclude

$$E(\Omega_1, \Omega_2) = \int_{K_1\times K_2} s_1^*\varphi \wedge s_2^*\varphi$$

$$- \lim_{\varepsilon\to 0}\left(\int_{(K_1\times\partial B_\varepsilon(q_2))\cup(\partial B_\varepsilon(q_1)\times K_2)} s_1^*\varphi \wedge s_2^*\varphi + \int_{\partial B_\varepsilon(q_1)\times\partial B_\varepsilon(q_2)} s_1^*\varphi \wedge s_2^*\varphi\right).$$

Clearly,

$$\int_{K_1\times K_2} s_1^*\varphi \wedge s_2^*\varphi = \int_{\mathbb{S}^1\times\mathbb{S}^1} \frac{\partial\theta(s,t)}{\partial s}\frac{\partial\theta(s,t)}{\partial t} dt ds\,.$$

Applying the latter to the pairs of curves $(K_1, \partial B_\varepsilon(q_2))$, $(\partial B_\varepsilon(q_1), K_2))$, $(\partial B_\varepsilon(q_1), \partial B_\varepsilon(q_2))$, and taking limits gives the result. □

PROPOSITION 5.3. *Assume $K_i = \partial\Omega_i$ is connected for $i = 1, 2$, and let $K_i^* = K_i \setminus \{p_i^0\}$ for some arbitrary point $p_i^0 \in K_i$. Then*

$$E(\Omega_1, \Omega_2) = \frac{1}{4}\int_{K_1^*\times K_2^*} \theta \sin\theta \frac{dp_1 dp_2}{|p_2 - p_1|^2}\,,$$

where $\theta(p_1, p_2)$ is any continuous determination on $K_1^* \times K_2^*$ of the oriented angle between the circle positively tangent to K_1 at p_1, and the circle positively tangent to K_2 at p_2.

PROOF. Let us use the notations from the previous proof, with the convention $\mathbb{S}^1 = [0, 1]/\sim$, where $0 \sim 1$. We can assume $p_i^0 = F_i(0, 0)$. Let $\Omega_i' = F_i\left((\mathbb{S}^1 \setminus \{0\}) \times [0, 1)\right)$. To simplify the notation we will identify $\Omega_i' \equiv (0, 1) \times [0, 1)$. We will also write $(x, t) \equiv F_1(x, t) = w$, $(y, u) \equiv F_2(y, u) = z$. Since $\Omega_1' \times \Omega_2'$ is (homotopically) contractible, the restriction $\mathcal{F}|_{\Omega_1'\times\Omega_2'}$ is a trivial bundle (i.e., there exists a bundle isomorphism $\tau : \mathcal{F}|_{\Omega_1'\times\Omega_2'} \to \Omega_1' \times \Omega_2' \times \mathbb{S}^1$). Moreover the row in the diagram below lifts

$$\begin{array}{ccccc} & & & & \Omega_1' \times \Omega_2' \times \mathbb{R} \\ & & & & \downarrow \\ \Omega_1' \times \Omega_2' & \xrightarrow{s_j} & \mathcal{F}|_{\Omega_1'\times\Omega_2'} & \xrightarrow[\cong]{\tau} & \Omega_1' \times \Omega_2' \times \mathbb{S}^1 \end{array}$$

i.e., there exist $f_j : \Omega_1' \times \Omega_2' \to \mathbb{R}$ such that $(w, z; \exp(if_j(w, z)) = \tau(s_j(w, z))$.

Let now $\rho : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ monotone function such that $\rho(x) = 0$ for $x \leq 0$ and $\rho(x) = 1$ for $x \geq 1$. Given $\varepsilon > 0$ we define

$$h_\varepsilon(x, t, y, u) = \rho(u/\varepsilon) f_1(x, t, y, u) + \rho(t/\varepsilon) f_2(x, t, y, u)\,.$$

Then $s_\varepsilon(x, t, y, u) = \tau^{-1}(w, z; \exp(ih_\varepsilon(x, t, y, u))$ defines a section s_ε of π over $\Omega_1' \times \Omega_2'$ which we identified to $(0, 1) \times [0, 1) \times (0, 1) \times [0, 1)$. Hence we have $s_\varepsilon^*\varphi$ defined on $(0, 1) \times$

$[0,1)\times(0,1)\times[0,1)$. In fact, it extends to $\mathbb{S}^1\times[0,1)\times\mathbb{S}^1\times[0,1)$. Next we take a small $\delta>0$ and we apply Stokes theorem to the manifold $U_\delta=\mathbb{S}^1\times[0,1-\delta]\times\mathbb{S}^1\times[0,1-\delta]$:

$$4\int_{U_\delta}\frac{dzdw}{|z-w|^4}=\int_{\partial U_\delta}s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}=\int_{\{t=1-\delta\}\cup\{u=1-\delta\}\cup\{t=0\}\cup\{u=0\}}s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}\,.$$

The norm $\|s_\varepsilon^*\varphi\|_\infty$ is bounded for a fixed $\varepsilon>0$. Besides, $\|i_{\partial/\partial x}(\Im\mathfrak{m}\,\omega_{cr})\|=O(\delta)$ for $t=1-\delta$ or $u=1-\delta$. Hence,

$$\lim_{\delta\to0}\int_{\{t=1-\delta\}}s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}=0$$

$$\lim_{\delta\to0}\int_{\{u=1-\delta\}}s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}=0\,.$$

On the other hand

$$\int_{\{t=0,u>\varepsilon\}}s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}=0,\qquad\int_{\{u=0,t>\varepsilon\}}s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}=0\,. \tag{48}$$

Indeed, for $t=0$, $u>\varepsilon$, $s_\varepsilon=s_1$. In this case, by Lemma 5.1

$$s_\varepsilon^*\varphi\frac{\partial}{\partial y}=s_\varepsilon^*\varphi\frac{\partial}{\partial u}=0\,.$$

Hence,

$$(s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr})\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial u}\right)=0\,.$$

This shows the first equation in (48). The second one follows by symmetry. We have shown so far that

$$E(\Omega_1,\Omega_2)=\frac14\int_{\{0<x,y<1,t=0,0<u<\varepsilon\}\cup\{0<x,y<1,u=0,0<t<\varepsilon\}}s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}\,. \tag{49}$$

To compute the latter integral we take the limit as ε goes down to 0.

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_{\{0<x,y<1,t=0,0<u<\varepsilon\}}s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}\\
&=\lim_{\varepsilon\to0}\int_0^1\int_0^1\int_0^\varepsilon\left(s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}\right)_{(x,0,y,u)}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial u}\right)dudydx\\
&=\lim_{\varepsilon\to0}\int_0^1\int_0^1\int_0^1\varepsilon\left(s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}\right)_{(x,0,y,\varepsilon v)}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial u}\right)dvdydx\,.
\end{aligned}$$

Since the norm of $\varepsilon s_{\varepsilon *}$ is uniformly bounded, we may apply Lebesgue's dominated convergence theorem to put the limit inside the integral. Since $s_\varepsilon^*\varphi(\partial/\partial x)$, $s_\varepsilon^*\varphi(\partial/\partial y)$ are uniformly bounded,

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1\lim_{\varepsilon\to0}\varepsilon\left(s_\varepsilon^*\varphi\wedge\Im\mathfrak{m}\,\omega_{cr}\right)_{(x,0,y,\varepsilon v)}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial u}\right)dvdydx\\
&=\int_0^1\int_0^1\left(\int_0^1\lim_{\varepsilon\to0}\varepsilon s_\varepsilon^*\varphi\left(\frac{\partial}{\partial u}\right)dv\right)\Im\mathfrak{m}\,\omega_{cr}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)dydx\,.
\end{aligned}$$

Finally, by continuity, and since $\lim_{\varepsilon\to 0}\varepsilon s_{\varepsilon *}(\frac{\partial}{\partial u})$ is tangent to the fibers of π_1,

$$\lim_{\varepsilon\to 0}\varepsilon\varphi\left(s_{\varepsilon *}\left(\frac{\partial}{\partial u}\right)\right)=\varphi\left(\lim_{\varepsilon\to 0}\varepsilon s_{\varepsilon *}\left(\frac{\partial}{\partial u}\right)\right)=\lim_{\varepsilon\to 0}\frac{\partial h_\varepsilon}{\partial u}(x,0,y,\varepsilon v)=\rho'(v)f_1(x,0,y,0)\,.$$

Therefore

$$\lim_{\varepsilon\to 0}\int_{\{(x,y)\}}\int_{\{0<u<\varepsilon\}}\left(s_\varepsilon^*\varphi\wedge\Im\mathrm{m}\,\omega_{cr}\right)_{(x,0,y,u)}=\int_{\{(x,y)\}}f_1(x,0,y,0)\Im\mathrm{m}\,\omega_{cr}\,.$$

Similarly,

$$\lim_{\varepsilon\to 0}\int_{\{(x,y)\}}\int_{\{0<t<\varepsilon\}}\left(s_\varepsilon^*\varphi\wedge\Im\mathrm{m}\,\omega_{cr}\right)_{(x,t,y,0)}=-\int_{\{(x,y)\}}f_2(x,0,y,0)\Im\mathrm{m}\,\omega_{cr}\,.$$

This, together with (49), completes the proof since $\theta=f_2-f_1$, and

$$\Im\mathrm{m}\,\omega_{cr}=\sin\theta\,\frac{dx_1dx_2}{|x_2-x_1|^2}\,.$$

The renormalization in this article is also called Hadamard finite-part integrals or Hadamard regularization. □

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