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VARIATIONAL INEQUALITIES FOR PERTURBATIONS OF MAXIMAL MONOTONE OPERATORS IN REFLEXIVE BANACH SPACES

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Abstract. Let X be a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space X^*, and let K be a nonempty, closed and convex subset of X with 0 in its interior. Let T be maximal monotone and S a possibly unbounded pseudomonotone, or finitely continuous generalized pseudomonotone, or regular generalized pseudomonotone operator with domain K. Let ϕ be a proper, convex and lower semicontinuous function. New results are given concerning the solvability of perturbed variational inequalities involving the operator $T+S$ and the function ϕ. The associated range results for nonlinear operators are also given, as well as extensions and/or improvements of known results of Kenmochi, Le, Browder, Browder and Hess, De Figueiredo, Zhou, and others.

1. Introduction–Preliminaries. In what follows, X is a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space X^*. The norm of the space X, and any other normed spaces herein, will be denoted by $\|\cdot\|$. For $x \in X$ and $x^* \in X^*$, the pairing $\langle x^*, x\rangle$ denotes the value $x^*(x)$. Let X and Y be real Banach spaces. For a multivalued mapping $T : X \to 2^Y$, we define the domain $D(T)$ of T by $D(T) = \{x \in X; Tx \neq \emptyset\}$, and the range $R(T)$ of T by $R(T) = \bigcup_{x\in D(T)} Tx$. We also use the symbol $G(T)$ for the graph of T, i.e., $G(T) = \{(x, Tx); x \in D(T)\}$. A mapping $T : X \supset D(T) \to Y$ is "demicontinuous" if it is continuous from the strong topology of $D(T)$ to the weak topology of Y. A multi-valued mapping $T : X \supset D(T) \to 2^Y$ is "bounded" if it maps bounded subsets of $D(T)$ to bounded subsets of Y. It is "compact" if it is strongly continuous and maps bounded subsets of $D(T)$ to relatively compact subset of Y. It is "finitely continuous" if it is upper semicontinuous from each finite dimensional subspace F of X to the weak topology of Y. It is "quasibounded" if for every $M > 0$, there exists $K(M) > 0$ such that $[x, w^*] \in G(T)$ with $\|x\| \leq M$ and $\langle w^*, x\rangle \leq M\|x\|$ imply $\|w^*\| \leq K(M)$. It is "strongly quasibounded" if for every $M > 0$ there exists $K(M) > 0$ such that $[x, w^*] \in G(T)$ with $\|x\| \leq M$ and $\langle w^*, x\rangle \leq M$ imply $\|w^*\| \leq K(M)$. In what follows, a mapping will be called "continuous" if it is strongly continuous.

Let $\psi : [0,\infty) \to [0,\infty)$ be continuous strictly increasing function such that $\psi(0) = 0$ and $\psi(t) \to \infty$ as $t \to \infty$. The duality mapping corresponding to ψ denoted by $J_\psi : X \to$

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2^{X^*} is defined by

$$J_\psi(x) = \{x^* \in X^*;\ \langle x^*, x\rangle = \|x^*\|\|x\|,\ \|x^*\| = \psi(\|x\|)\}\,.$$

An application of the Hahn-Banach Theorem implies that $J_\psi(x) \neq \emptyset$ for each $x \in X$. Since X and X^* are locally uniformly convex, J_ψ is single valued and bicontinuous. If $\psi(t) = t$ for $t \geq 0$, then J_ψ is denoted by J, it is called the "normalized duality mapping" and is given by

$$J(x) = \{x^* \in X^*;\ \langle x^*, x\rangle = \|x\|^2,\ \|x^*\| = \|x\|\}\,.$$

An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone" if for every $x, y \in D(T)$ and every $u^* \in Tx$, $v^* \in Ty$, we have

$$\langle u^* - v^*, x - y\rangle \geq 0\,.$$

If T is monotone, we see that for any sequence $\{x_n\}$ in $D(T)$ with $x_n \rightharpoonup x_0 \in D(T)$, $v_n^* \in Tx_n$ for all n and some $v_0^* \in Tx_0$, we have

$$\liminf_{n\to\infty} \langle v_n^*, x_n - x_0\rangle = \liminf_{n\to\infty} [\langle v_n^* - v_0^*, x_n - x_0\rangle + \langle v_0^*, x_n - x_0\rangle] \geq 0\,.$$

A monotone mapping $T : X \supset D(T) \to 2^{X^*}$ is "maximal monotone" if $R(T + \lambda J) = X^*$ for every $\lambda > 0$. This is equivalent to saying that T is maximal monotone if and only if T is monotone and $\langle u^* - u_0^*, x - x_0\rangle \geq 0$ for every $(x, u^*) \in G(T)$ implies $x_0 \in D(T)$ and $u_0^* \in Tx_0$. Since X and X^* are locally uniformly convex, it follows that J is single valued, bounded, bicontinuous, maximal monotone and of type (S_+). If T is maximal monotone, the operator $T_t : X \to X^*$, $t \in (0, \infty)$, defined by $T_t x = (T^{-1} + tJ^{-1})^{-1}x$, is bounded, continuous, maximal monotone and such that $T_t x \rightharpoonup T^{(0)}x$ as $t \to 0^+$ for every $x \in D(T)$, where $\|T^{(0)}x\| = \inf\{\|y^*\|;\ y^* \in Tx\}$. The "resolvent" $J_t : X \to D(T)$, defined by $J_t x = x - tJ^{-1}(T_t x)$, is continuous and $T_t x \in T(J_t x)$ for every $x \in X$. Moreover, $\lim_{t\to 0} J_t x = x$ for all $x \in \overline{coD(T)}$, where coA denotes the convex hull of the set A. An operator $A : X \supseteq D(A) \to 2^{X^*}$ is called "coercive" if either $D(A)$ is bounded or there exists a function $\psi : [0, \infty) \to (-\infty, \infty)$ such that $\psi(t) \to \infty$ as $t \to \infty$ and

$$\langle y^*, x\rangle \geq \psi(\|x\|)\|x\|$$

for all $x \in D(A)$ and $y^* \in Ax$.

The following pseudomonotonicity definition may be found in Browder and Hess [1].

Definition 1.1. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "pseudomonotone" if

(i) for every $x \in D(T)$, Tx is nonempty, closed, convex and bounded subset of X^*;
(ii) T is finitely continuous, i.e., T is "weakly upper semicontinuous" on each finite-dimensional subspace F of X, i.e., for every $x_0 \in D(T) \cap F$ and every weak neighborhood V of Tx_0 in X^*, there exists a neighborhood U of x_0 in F such that $TU \subset V$;

(iii) for every sequence $\{x_n\} \subset D(T)$ and every sequence $\{y_n^*\}$ with $y_n^* \in Tx_n$ such that $x_n \rightharpoonup x_0 \in D(T)$ and

$$\limsup_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \le 0\,,$$

we have that for every $x \in D(T)$ there exists $y^*(x) \in Tx_0$ such that

$$\langle y^*(x), x_0 - x\rangle \le \liminf_{n\to\infty} \langle y_n^*, x_n - x\rangle\,.$$

Let $T : X \supseteq D(T) \to 2^{X^*}$ be pseudomonotone and the sequences $\{x_n\}$, $\{y_n^*\}$ as in Definition 1.1. Then letting x_0 in place of x in (iii) of Definition 1.1 we get

$$\liminf_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \ge 0\,.$$

DEFINITION 1.2. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "generalized pseudomonotone" if

(i) for each $x \in D(T)$, Tx is nonempty, closed, convex and bounded subset of X^*;

(ii) for every sequence $\{x_n\} \subset D(T)$ and every sequence $\{y_n^*\}$ with $y_n^* \in Tx_n$ such that $x_n \rightharpoonup x_0 \in D(T)$, $y_n^* \rightharpoonup y_0^* \in X^*$ and

$$\limsup_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \le 0\,,$$

we have $y_0^* \in Tx_0$ and $\langle y_n^*, x_n\rangle \to \langle y_0^*, x_0\rangle$ as $n \to \infty$.

If T is generalized pseudomonotone and $\{x_n\}$ and $\{y_n^*\}$ are as in Definition 1.2, we have

$$\liminf_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \ge 0\,.$$

Otherwise, by passing to a subsequence if necessary, the generalized pseudomonotonicity property of T gets violated.

An operator T_0 with $D(T_0) = X$ is called smooth if T_0 is bounded, maximal monotone and coercive. A generalized pseudomonotone operator T is called "regular" if $R(T + T_0) = X^*$ for any smooth operator T_0.

Browder and Hess [1, Proposition 3, p. 258] showed that a pseudomonotone operator T with $D(T) = X$ is generalized pseudomonotone. However, this fact might not be true if $D(T) \neq X$.

DEFINITION 1.3. An operator $T : X \supseteq D(T) \to 2^{X^*}$ is said to be of "type (S_+)" if

(i) for every $x \in D(T)$, Tx is a nonempty, closed, convex and bounded subset of X^*;

(ii) T is finitely continuous, i.e., T is weakly upper semicontinuous on each finite dimensional subspace of X(see Definition 1.1);

(iii) for every sequence $\{x_n\} \subset D(T)$ and every $y_n^* \in Tx_n$ with $x_n \rightharpoonup x_0 \in X$ and

$$\limsup_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \le 0\,,$$

we have $x_n \to x_0 \in D(T)$ and $\{y_n^*\}$ has a subsequence which converges weakly to $y_0^* \in Tx_0$. A mapping $T : X \supset D(T) \to 2^{X^*}$ is said to be of "type (S)" if (i) and (ii) hold and the inequality in (iii) is replaced by an equality.

For basic definitions and further properties of mappings of monotone type, the reader is referred to Barbu [2], Brézis, Crandall and Pazy [3], Browder and Hess [1], Browder [4], Cioranescu [5], Naniewicz and Panagiotopoulos [6], Pascali and Sburlan [7] and Zeidler [8]. A function $\phi : X \to (-\infty, \infty]$ is called "proper" if ϕ is not identically $+\infty$. It is "convex" if

$$\phi((1-\lambda)x + \lambda y) \le (1-\lambda)\phi(x) + \lambda\phi(y)$$

for all $x \in X$, $y \in X$ and all $\lambda \in [0, 1]$. Furthermore, ϕ is called "lower semicontinuous" if

$$\phi(x) \le \liminf_{y \to x} \phi(y), \quad x \in X,$$

or, equivalently, for each $\lambda > 0$ the level set $\{x \in X; \phi(x) \le \lambda\}$ is closed.

Let K denote a nonempty, closed and convex subset of a reflexive Banach space X and let I_K be the indicator function of K given by

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{if } x \in X \setminus K. \end{cases}$$

It is known that I_K is proper, convex and lower semicontinuous on X. The subdifferential of I_K at $x \in X$ is defined by

$$\partial I_K(x) = \{x^* \in X^*; \langle x^*, x - y \rangle \ge 0 \text{ for all } y \in K\}.$$

Here, $D(\partial I_K) = D(I_K) = K$ and $\partial I_K(x) = \{0\}$ for every $x \in \mathring{K}$. Let $\phi : X \supseteq D(\phi) \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function on X with $D(\phi) = \{x \in X; \phi(x) < +\infty\}$. For each $x \in X$, we denote by $\partial\phi(x)$ the set

$$\partial\phi(x) = \{x^* \in X^*; \langle x^*, x - y \rangle \ge \phi(x) - \phi(y) \text{ for all } y \in X\}.$$

It is known that $\partial\phi : X \supseteq D(\partial\phi) \to 2^{X^*}$ is maximal monotone and such that $D(\partial\phi)$ is a dense subset of $D(\phi)$. Furthermore, we have $\phi(x) = \min\{\phi(y); y \in X\}$ if and only if $0 \in \partial\phi(x)$. Other relevant properties may be found in Barbu [9].

Fix $f^* \in X^*$ and $A : X \supseteq D(A) \to 2^{X^*}$. We denote by VIP$(A, K, \phi, f^*)$ the variational inequality problem

$$\langle w^* - f^*, y - x \rangle \ge \phi(x) - \phi(y), \quad y \in K,$$

with the unknown vector $x \in D(A) \cap D(\phi) \cap K$ and $w^* \in Ax$. Since $D(\partial\phi) \subset D(\phi)$, it is not hard to see that the solvability of the inclusion

$$\partial\phi(x) + Ax \ni f^*$$

in $D(A) \cap D(\partial\phi) \cap K$ implies the solvability of the problem VIP(A, K, ϕ, f^*) in $D(A) \cap D(\phi) \cap K$, and equivalence holds if $D(\phi) = D(\partial\phi) = K$. In particular, if $\phi = I_K$, we denote the problem VIP(A, K, I_K, f^*) just by VIP(A, K, f^*), and we see that its solvability is equivalent to the solvability of the inclusion

$$\partial I_K(x) + Ax \ni f^*$$

in $D(A) \cap K$.

For basic results involving variational inequalities and monotone type mappings, the reader is referred to Barbu [2], Brézis [10], Browder and Hess [1], Browder [11], [4], Browder and Brézis [12], Hartman and Stampacchia [13], Kenmochi [14], Kinderlehrer and Stampacchia [15], Kobayashi and Otani [16], Lions and Stampacchia [17], Minty [19], [20], Moreau [21], Naniewicz and Panagiotopoulos [6], Pascali and Sburlan [7], Rockafellar [22], Stampacchia [18], Ton [23], Zeidler [8] and the references therein. A study of pseudomonotone operators and nonlinear elliptic boundary value problems may be found in Kenmochi [24]. For a survey of maximal monotone and pseudomonotone operators and perturbation results, we cite the handbook of Kenmochi [25]. Nonlinear perturbation results of monotone type mappings, variational inequalities and their applications may be found in Guan, Kartsatos and Skrypnik [26], Guan and Kartsatos [27], Le [28], Zhou [29] and the references therein. Variational inequalities for single single-valued pseudomonotone operators in the sense of Brézis may be found in Kien, Wong, Wong and Yao [30]. Existence results for multivalued quasilinear inclusions and variational-hemivariational inequalities may be found in Carl, Le and Motreanu [31], Carl [32] and Carl and Motreanu [33] and the references therein. Recently, Asfaw and Kartsatos [34] developed a new degree theory for multivalued pseudomonotone perturbations of maximal monotone operators. The authors also demonstrated there the applicability of the theory in solving nonlinear problems involving monotone type operators.

In this paper we study the solvability of variational inequalities, where the relevant operator A could be, e.g., the sum $T+S$ with $T : X \supseteq D(T) \to 2^{X^*}$ maximal monotone and $S : K \to 2^{X^*}$ at least pseudomonotone. The main reasons for studying the solvability of such perturbed inequalities and equations are the following.

(1) As mentioned above, the solvability of the problem

$$\partial\phi(x) + T(x) + S(x) \ni f^*$$

in $D(T) \cap D(\partial\phi) \cap K$ implies the solvability of the problem

$$\mathrm{VIP(T + S, K, \phi, f^*)}$$

in $D(T) \cap D(\phi) \cap K$, and the two problems are equivalent if $D(\phi) = D(\partial\phi) = K$. Therefore, the solvability of the problem $\mathrm{VIP}(T+S, K, \phi, f^*)$ in $D(T) \cap D(\phi) \cap K$ may be covered by range results for the sum of three monotone-type operators. However, as far as the authors can tell, there are no range results involving such operators.

(2) If $\phi \neq I_K$, the solvability of the inclusion $Tx + Sx \ni f^*$ does not necessarily imply the solvability of the problem

$$\mathrm{VIP(T + S, K, \phi, f^*)}\,.$$

In fact, if for some $x_0 \in D(T) \cap D(\phi) \cap K$, $v_0^* \in Tx_0$ and $z_0^* \in Sx_0$ the equation $v_0^* + z_0^* = f^*$ is satisfied, we do not necessarily have the solvability of the inequality

$$\langle v_0^* + z_0^* - f^*, x - x_0\rangle \geq \phi(x_0) - \phi(x)$$

for all $x \in K$ unless $\phi(x_0) = \min_{x\in K} \phi(x)$.

(3) It is known from Browder and Hess [1, Proposition 3, p. 258] that every pseudomonotone operator with effective domain all of X is generalized pseudomonotone. However, this fact is unknown if the domain is different from X. Because of this, we have treated the solvability of variational inequalities and equations separately for pseudomonotone and generalized pseudomonotone operators with domain a closed convex subset of X. Browder and Hess [1] mentioned the difficulty of treating generalized pseudomonotone operators which are not defined everywhere on X or on a dense linear subspace. A surjectivity result for single quasibounded coercive generalized pseudomonotone operator whose domain contains a dense linear subspace of X may be found in Browder and Hess [1, Theorem 5, p. 273]. Existence results for densely defined finitely continuous generalized pseudomonotone perturbations of maximal monotone operators may be found in Guan, Kartsatos and Skrypnik [26, Theorem 2.1, p. 335]. We should mention that there are no range results known to the authors for the sum $T + S$, where T is maximal monotone and S either pseudomonotone or generalized pseudomonotone with domain just K, where K is a nonempty, closed and convex subset of X.

In Section 2, we study the solvability of variational inequalities for bounded pseudomonotone perturbations of one or two maximal monotone operators. As a result, a new characterization for the maximality of the sum of two maximal monotone operators is given.

Section 3 contains existence results for the solvability of variational inequalities for finitely continuous generalized pseudomonotone perturbations of maximal monotone operators.

Section 4 contains results about possibly unbounded pseudomonotone or finitely continuous generalized pseudomonotone perturbations of maximal monotone operators.

In Section 5, we give new results for regular generalized pseudomonotone perturbations. In each of these sections the corresponding range results are discussed.

In Section 6, we give examples of single-valued as well as multivalued pseudomonotone operators which are suitable for the applicability of our theory.

The following lemma is due to Brézis, Crandall and Pazy in [3, Lemmas 1.2 and 1.3].

LEMMA 1.4. *Let B be a maximal monotone set in $X \times X^*$. If (u_n, u_n^*) is an element of B such that $u_n \rightharpoonup u$ and $u_n^* \rightharpoonup u^*$ and either*

$$\limsup_{n,m\to\infty} \langle u_n^* - u_m^*, u_n - u_m\rangle \le 0$$

or

$$\limsup_{n\to\infty} \langle u_n^* - u^*, u_n - u\rangle \le 0\,,$$

then $(u, u^*) \in B$ and $(u_n^*, u) \to (u^*, u)$ as $n \to \infty$.

The following lemma is a version of Lemma 1.4. Its proof in its present form may be found in Adhikari and Kartsatos [35, Lemma 1, p. 1244].

LEMMA 1.5. *Assume that the operators* $T : X \supseteq D(T) \to 2^{X^*}$ *and* $S : X \supseteq D(S) \to 2^{X^*}$ *are maximal monotone with* $0 \in T(0) \cap S(0)$. *Assume, further, that* $T + S$ *is maximal monotone. Assume there are a positive sequence* $\{t_n\}$ *such that* $t_n \downarrow 0^+$ *and a sequence* $\{x_n\}$ *in* $D(S)$ *such* $x_n \rightharpoonup x_0 \in X$ *and* $T_{t_n}x_n + w_n^* \rightharpoonup y_0^* \in X^*$, *where* $w_n^* \in Sx_n$. *Then the following are true.*

(i) *The inequality*

$$\lim_{n\to\infty} \langle T_{t_n}x_n + w_n^*, x_n - x_0\rangle < 0$$

is impossible;

(ii) *if*

$$\lim_{n\to\infty} \langle T_{t_n}x_n + w_n^*, x_n - x_0\rangle = 0\,,$$

then $x_0 \in D(T) \cap D(S)$ *and* $y_0^* \in (T + S)x_0$.

Browder and Hess [1] proved that a monotone mapping T with $0 \in \mathring{D}(T)$ is strongly quasibounded. The following lemma is due to Browder and Hess [1].

LEMMA 1.6. *Let* $X \supseteq D(T) \to 2^{X^*}$ *be strongly quasibounded maximal monotone such that* $0 \in T(0)$ *and* $\{t_n\}$ *be a sequence in* $(0, \infty)$ *and* $\{x_n\} \subseteq X$ *be such that*

$$\|x_n\| \le S, \left\langle T_{t_n}x_n, x_n\right\rangle \le S$$

for all n, *where* S *is a positive constant. Then there exists* $K = K(S) > 0$ *such that* $\|T_{t_n}x_n\| \le K$ *for all* n.

In what follows, we make frequent use of the following basic result of Browder and Hess [1, Proposition 15, p. 289].

LEMMA 1.7. *Let* K *be a compact convex subset of* X *and* $T : K \to 2^{X^*}$ *an operator such that for every* $x \in K$, Tx *is a nonempty, closed, convex and bounded subset of* X^*. *Assume that* T *is upper semicontinuous, with* X^* *being given its weak topology. Let* $f^* \in X^*$. *Then there exist elements* $x_0 \in K$ *and* $y_0^* \in Tx_0$ *such that*

$$\left\langle y_0^* - f^*, x - x_0\right\rangle \le 0$$

for all $x \in K$.

We observe that, for every $f^* \in X^*$, $-T + f^*$ is upper semicontinuous whenever T is upper semicontinuous. Under the hypothesis of the above lemma, we have the existence of $x_0^* \in K$ and $v_0^* \in -Tx_0 + f^*$ (i.e., $v_0^* = -w_0^* + f^*$ for some $w_0^* \in Tx_0$) such that $\left\langle -w_0^* + f^*, x - x_0\right\rangle \le 0$ for all $x \in K$. This implies $\left\langle w_0^* - f^*, x - x_0\right\rangle \ge 0$ for all $x \in K$, i.e., the problem VIP(T, K, f^*) is solvable in K.

The following lemma, which is an easy application of the uniform boundedness principle, may be found in Browder [36, Lemma 1].

LEMMA 1.8. *Let* X *be a Banach space,* $\{x_n\}$ *a sequence in* X *and* $\{\alpha_n\}$ *a sequence of positive numbers such that* $\alpha_n \to 0^+$ *as* $n \to \infty$. *For a fixed* $r > 0$, *assume that for every*

$h^* \in X^*$ with $\|h^*\| \le r$, there exists a constant C_{h^*} such that

$$\langle h^*, x_n \rangle \le \alpha_n \|x_n\| + C_{h^*}$$

for all n. Then the sequence $\{x_n\}$ is bounded.

The next lemma can be found in Browder [4, Proposition 7.2, p. 81].

LEMMA 1.9. *Let X be a reflexive Banach space, A a bounded subset of X and $x_0 \in \overline{A^w}$, where $\overline{A^w}$ is the weak closure of A in X. Then there exists a sequence $\{x_n\}$ in A such that $x_n \rightharpoonup x_0$ in X as $n \to \infty$.*

The following existence result for the solvability of a variational inequality for a single multivalued pseudomonotone operator is due to Browder and Hess [1, Theorem 15, p. 289].

LEMMA 1.10. *Let K be a nonempty, closed and convex subset of X with $0 \in K$. Let $S : K \to 2^{X^*}$ be pseudomonotone and coercive. Then for each $g^* \in X^*$ there exist $x_0 \in K$ and $w_0 \in Sx_0$ such that*

$$\langle w_0 - g^*, x - x_0 \rangle \ge 0$$

for all $x \in K$.

2. Variational inequalities for pseudomonotone perturbations. In this section we give some existence results for the problem VIP($T+S$, K, ϕ, f^*), where T is maximal monotone and S is bounded pseudomonotone. We begin with the definition of the solvability of a variational inequality over a given set.

DEFINITION 2.1. Let B be a subset of X. Let K be a nonempty subset of X and $A : X \supseteq D(A) \to 2^{X^*}$. Let $\phi : X \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous, and fix $f^* \in X^*$. We say that the variational inequality problem VIP(A, K, ϕ, f^*) is solvable in $D(A) \cap D(\phi) \cap B$ if there exist $x_0 \in D(A) \cap D(\phi) \cap B$ and $w_0^* \in Ax_0$ such that

$$\langle w_0^* - f^*, x - x_0 \rangle \ge \phi(x_0) - \phi(x)$$

for all $x \in K$.

Using this definition, it follows that the problem VIP(A, K, ϕ, f^*) has no solution in $D(A) \cap D(\phi) \cap \partial K$ if and only if there exists $u_0 \in K$ such that

$$\langle w^* - f^*, u_0 - x \rangle < \phi(x) - \phi(u_0)$$

for all $x \in D(A) \cap D(\phi) \cap \partial K$, $w^* \in Ax$.

In what follows, we make frequent use of the following useful lemma. A version of this lemma is due to Lions and Stampacchia [17] when X is a Hilbert space. Another version of it is due to Hartman and Stampacchia [13] and involves monotone finitely continuous operators defined on a closed convex subset of X. For further reference, we cite the book of Kinderlehrer and Stampacchia [15, Theorem 1.7, pp. 85–87, and Theorem 2.3, p. 91].

LEMMA 2.2. *Let K be a nonempty, closed and convex subset of X and $A : X \supseteq D(A) \to 2^{X^*}$. Let G be an open convex subset of X. Then the problem* VIP(A, K, ϕ, f^*)

is solvable in $D(A) \cap D(\phi) \cap K \cap G$ *provided that the problem* VIP$(A, K \cap \overline{G}, \phi, f^*)$ is solvable in $D(A) \cap D(\phi) \cap K \cap G$.

PROOF. Suppose that $x_0 \in D(A) \cap D(\phi) \cap K \cap G$ is a solution of the problem VIP$(A, K \cap \overline{G}, \phi, f^*)$, i.e., there exists $u_0^* \in Ax_0$ such that

$$\langle u_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x) \tag{1}$$

for all $x \in K \cap \overline{G}$. It suffices to show that x_0 solves the inequality VIP(A, K, ϕ, f^*). We observe that, by the convexity of K, for any $t \in (0, 1)$ and for any $x \in K$, we have $tx + (1-t)x_0 \in K$. For each $x \in K$, we claim that there exists $t_0 = t_0(x) \in (0, 1)$ such that $t_0 x + (1 - t_0)x_0 \in G$. Suppose there exists $y \in K$ such that $ty + (1-t)x_0 \notin G$ for all $t \in (0, 1)$, i.e., $ty + (1-t)x_0 \in X \backslash G$ for all $t \in (0, 1)$. Since G is open, letting $t \downarrow 0^+$, we obtain that $x_0 \notin G$. But this is a contradiction as $x_0 \in G$. Thus our claim follows, i.e., for every $x \in K$, there exists $t_0 = t_0(x) \in (0, 1)$ such that $y = t_0 x + (1 - t_0)x_0 \in K \cap G$. Replacing x by y in (1) and using the convexity of ϕ, we see that

$$\begin{aligned} t_0 \langle u_0^* - f^*, x - x_0 \rangle &= \langle u_0^* - f^*, y - x_0 \rangle \\ &\geq \phi(x_0) - \phi(y) \\ &\geq \phi(x_0) - [t_0 \phi(x) + (1 - t_0)\phi(x_0)] \\ &= t_0(\phi(x_0) - \phi(x)) . \end{aligned}$$

Since $t_0 \in (0, 1)$, we conclude that

$$\langle u_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x)$$

for all $x \in K$, i.e., the problem VIP(A, K, ϕ, f^*) is solvable by $x_0 \in D(A) \cap D(\phi) \cap K \cap G$. □

The following theorem will be used frequently in the sequel. For related results, the reader is referred to Browder [4, Theorem 7.8, pp. 92–96] $(D(T) = D(S) = K)$, Kenmochi [14, Theorem 5.2, p. 236] $(D(S) = X)$ and Le [28] $(D(S) = X)$.

THEOREM 2.3. *Let* K *be a nonempty, closed and convex subset of* X *with* $0 \in \mathring{K}$. *Let* $T : X \supseteq D(T) \to 2^{X^*}$ *be maximal monotone with* $0 \in T(0)$ *and* $S : K \to 2^{X^*}$ *pseudomonotone. Fix* $f^* \in X^*$. *Assume, further, that either* S *is bounded or* T *is strongly quasibounded and there exists* $k > 0$ *such that* $\langle w^*, x \rangle \geq -k$ *for all* $x \in K$ *and* $w^* \in Sx$.

(i) *If* K *is bounded, then the problem* VIP$(T + S, K, f^*)$ is solvable in $D(T) \cap K$.

(ii) If K is unbounded and there exists an open, convex and bounded subset G of X with $0 \in G$ such that the problem VIP$(T + S, K \cap \overline{G}, f^*)$ has no solution in $D(T) \cap K \cap \partial G$, then the problem VIP$(T + S, K, f^*)$ is solvable in $D(T) \cap K \cap G$.

PROOF. We first prove (i) and (ii) assuming the boundedness of S.

(i) Suppose K is bounded. Let $t > 0$ and T_t be the Yosida approximant of T. We notice that, for every $t > 0$, the operator $T_t + S$ is bounded and pseudomonotone on K. Using the boundedness of K, instead of the coercivity of the pseudomonotone operator $T_t + S$ in Lemma

1.10, we see that VIP($T_t + S, K, f^*$) is solvable in K. Thus, for every $t_n \downarrow 0^+$ there exists $x_n \in K$ and $w_n^* \in Sx_n$ such that

$$\langle T_{t_n}x_n + w_n^* - f^*, x - x_n\rangle \geq 0$$

for all n and all $x \in K$. Since the solvability of VIP($T_{t_n} + S, K, f^*$), with solution $x_n \in K$, is equivalent to the solvability of the inclusion

$$\partial I_K(x_n) + T_{t_n}x_n + w_n^* \ni f^*$$

for every n, there exists $v_n^* \in \partial I_K(x_n)$ such that

$$v_n^* + T_{t_n}x_n + w_n^* = f^*$$

for all n. Since $\{x_n\}$ and S are bounded, we have the boundedness of the sequence $\{w_n^*\}$. Since $0 \in T(0)$, we have $T_{t_n}(0) = 0$ for all n and hence $\langle v_n^*, x_n\rangle \leq \|w_n^*\|\|x_n\|$. The boundedness of $\{v_n^*\}$ follows from the fact that ∂I_K is strongly quasibounded. As a result, the sequence $\{T_{t_n}x_n\}$ is also bounded. Assume, by passing to subsequences if necessary, that $x_n \rightharpoonup x_0$, $v_n^* \rightharpoonup v_0^*$, $w_n^* \rightharpoonup w_0^*$ and $T_{t_n}x_n \rightharpoonup z_0^*$ as $n \to \infty$. Since K is closed and convex, it is weakly closed and hence $x_0 \in K$. Since S is pseudomonotone and ∂I_K is maximal monotone, we have

$$\liminf_{n\to\infty} \langle w_n^*, x_n - x_0\rangle \geq 0 \text{ and} \liminf_{n\to\infty} \langle v_n^*, x_n - x_0\rangle \geq 0\,.$$

Let J_{t_n} be the Yosida resolvent of T. It is well known that, for every n, $J_{t_n}x_n \in D(T)$, $J_{t_n}x_n = x_n - t_n J^{-1}(T_{t_n}x_n)$, $T_{t_n}x_n \in T(J_{t_n}x_n)$ for all n and $J_{t_n}x_n \rightharpoonup x_0$ and $x_n - J_{t_n}x_n \to 0$ as $n \to \infty$. Therefore, we have

$$\limsup_{n\to\infty} \langle T_{t_n}x_n, J_{t_n}x_n - x_0\rangle \leq 0\,.$$

Using Lemma 1.4, we conclude that $x_0 \in D(T)$, $z_0^* \in Tx_0$ and $\langle T_{t_n}x_n, J_{t_n}x_n\rangle \to \langle z_0^*, x_0\rangle$ as $n \to \infty$. Similarly, using the maximality of ∂I_K and Lemma 1.4, we can show that $v_0^* \in \partial I_K(x_0)$ and $\langle v_n^*, x_n\rangle \to \langle v_0^*, x_0\rangle$ as $n \to \infty$. On the other hand, we have

$$\limsup_{n\to\infty} \langle w_n^*, x_n - x_0\rangle \leq 0\,.$$

Since S is pseudomonotone, for every $x \in K$ there exists $y^*(x) \in Sx_0$ such that

$$\langle y^*(x), x_0 - x\rangle \leq \liminf_{n\to\infty} \langle w_n^*, x_n - x\rangle = -\langle v_0^* + z_0^* - f^*, x_0 - x\rangle$$

for all n. Since $v_0^* \in \partial I_K(x_0)$, we have $\langle v_0^*, x_0 - x\rangle \geq 0$ for all $x \in K$. Therefore,

$$\langle y^*(x), x_0 - x\rangle \leq \liminf_{n\to\infty} \langle w_n^*, x_n - x\rangle = -\langle z_0^* - f^*, x_0 - x\rangle$$

for all n. Since S is pseudomonotone, for every $x \in K$ there exists $y^*(x) \in Sx_0$ such that

$$\langle y^*(x) + z_0^* - f^*, x - x_0\rangle \geq 0\,.$$

By the Hahn-Banach separation theorem, using $f^* - z_0^*$ in place of g^*, there exists $y_0^* \in Sx_0$ such that

$$\langle y_0^* - (f^* - z_0^*), x - x_0\rangle \geq 0$$

for all $x \in K$, which implies

$$\langle y_0^* + z_0^* - f^*, x - x_0 \rangle \geq 0$$

for all $x \in K$. This implies that VIP$(T + S, K, f^*)$ is solvable in $D(T) \cap K$.

(ii) Suppose K is unbounded and the hypothesis in (ii) holds true. Since $K \cap \overline{G}$ is a nonempty closed, convex and bounded subset of X with $0 \in \overbrace{K \cap \overline{G}}^{\circ}$, we apply the conclusion of (i) using the closed, convex and bounded subset $K \cap \overline{G}$ in place of K, to obtain the solvability of the problem VIP$(T + S, K \cap \overline{G}, f^*)$ in $D(T) \cap K \cap \overline{G}$. Since the problem VIP$(T + S, K \cap \overline{G}, f^*)$ has no solution in $D(T) \cap K \cap \partial G$, we use Lemma 2.2 (with $\phi = I_K$) to conclude that the variational inequality VIP$(T + S, K, f^*)$ is solvable in $D(T) \cap K \cap G$.

Next we assume that T is strongly quasibounded and there exists $k > 0$ such that $\langle w^*, x \rangle \geq -k$ for all $x \in K$ and $w^* \in Sx$. We prove the result in (i). Since $T_t + S$ is pseudomonotone on K, Lemma 1.10 says that for every $t > 0$, the problem VIP$(T_t + S, K, f^*)$ is solvable in K. This is equivalent to the solvability of the inclusion $\partial I_K(x) + T_t x + Sx \ni f^*$ in K. Thus, for every $t_n \downarrow 0^+$, there exists $x_n \in K$, $v_n^* \in \partial I_K(x_n)$ and $w_n^* \in Sx_n$ such that

$$v_n^* + T_{t_n} x_n + w_n^* = f^* \tag{2}$$

for all n. Since $0 \in T(0)$, we see that $T_{t_n}(0) = 0$ for all n. Since T_{t_n} is monotone for all n, we have $\langle v_n^*, x_n \rangle \leq k + \|f^*\| \|x_n\| \leq Q$, where Q is an obvious upper bound. Since ∂I_K is strongly quasibounded, it follows that $\{v_n^*\}$ is bounded. Using a similar argument along with the strong quasiboundedness of T and Lemma 1.6, we obtain the boundedness of $\{T_{t_n} x_n\}$ and, subsequently, the boundedness of $\{w_n^*\}$ from (2). Following the argument of the proof of (i) with S bounded, we obtain the solvability of the problem VIP$(T + S, K, f^*)$ in $D(T) \cap K$. The proof of (ii) under this case can be completed as in (ii) with S bounded. The detail is omitted. □

Le [28] gave a range result for bounded pseudomonotone perturbation S (with $D(S) = X$) of maximal monotone operators satisfying an inner product condition as in the following corollary for the case $G = B_R(0)$. We give an analogous result below, where G is a bounded, open and convex subset of X with $0 \in G$, and $D(S) = K$, with K a nonempty, closed and convex subset of X.

COROLLARY 2.4. *Let K be a nonempty, closed and convex subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone with $0 \in T(0)$ and $S : K \to 2^{X^*}$ pseudomonotone. Assume, further, that either S is bounded or T is strongly quasibounded and there exists $k > 0$ such that $\langle w^*, x \rangle \geq -k$ for all $x \in K$ and $w^* \in Sx$. Fix $f^* \in X^*$. Let G be an open, convex and bounded subset of X with $0 \in G$ such that, for some $u_0 \in K \cap \overline{G}$, we have*

$$\langle v^* + w^* - f^*, x - u_0 \rangle > 0 \tag{3}$$

for all $x \in D(T) \cap \partial(K \cap \overline{G})$, $v^* \in Tx$ and $w^* \in Sx$. Then the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T) \cap K \cap G$.

PROOF. We first observe that $0 \in \overbrace{K \cap \overline{G}}^{\circ}$. By Theorem 2.3, the problem VIP($T + S$, $K \cap \overline{G}$, f^*) is solvable in $D(T) \cap K \cap \overline{G}$. By (3), the problem VIP($T + S$, $K \cap \overline{G}$, f^*) has no solution in $D(T) \cap \partial(K \cap \overline{G})$. Since the solvability of the inclusion

$$\partial I_{K\cap\overline{G}}(x) + Tx + Sx \ni f^*$$

is equivalent to the solvability of the variational inequality VIP($T + S$, $K \cap \overline{G}$, f^*), it follows that the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T) \cap \overbrace{K \cap \overline{G}}^{\circ}$. □

Browder [4, Theorem 7.12, pp. 100–101] showed the existence of a solution to the problem VIP(S, K, ϕ, f), where S is bounded pseudomonotone and coercive with $D(S) = K$, $0 \in K$ and $\phi : K \to (-\infty, \infty]$ is proper, convex and lower semicontinuous having 0 as its minimum on K. Furthermore, Kenmochi [14] proved the existence of a solution to the problem VIP(S, K, ϕ, f), where S is pseudomonotone on K satisfying the (pm_4)-condition (see Definition 4.1 below) along with a coercivity-type condition involving S and ϕ.

The following theorem gives a new existence result for solutions of the problem VIP($T + S$, K, ϕ, f^*), where T is maximal monotone and S is bounded pseudomonotone. We remark that, using the definition of $\partial\phi$, it is not hard to see that the solvability of the problem VIP($\partial\phi + T + S$, K, f^*) in $D(\partial\phi) \cap D(T) \cap K$ implies the solvability of the inequality VIP($T + S$, K, ϕ, f^*) in $D(T) \cap D(\phi) \cap K$. Furthermore, using Lemma 2.2, the solvability of VIP($\partial\phi + T + S$, K, f^*) in $D(\partial\phi) \cap D(T) \cap K$ is achieved by solving the local problem VIP($\partial\phi + T + S$, $K \cap \overline{B}_R(0)$, f^*) in $D(T) \cap K \cap B_R(0)$.

THEOREM 2.5. *Let K be a nonempty, closed and convex subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and $S : K \to 2^{X^*}$ bounded pseudomonotone. Let $\phi : X \to (-\infty, \infty]$ be proper, convex and lower semicontinuous and such that $0 \in D(\phi)$ and there exists $k > 0$ such that $\phi(x) \geq -k$ for all $x \in X$. Fix $f^* \in X^*$. Then*

(i) *if K is bounded, then the problem* VIP($T + S$, K, ϕ, f^*) *is solvable in $D(T) \cap K \cap D(\phi)$,*

(ii) *if K is unbounded and there exists a bounded open convex subset G of X with $0 \in G$ such that the problem* VIP($T + S$, $K \cap \overline{G}$, ϕ, f^*) has no solution in $D(T) \cap D(\phi) \cap K \cap \partial G$, then the problem VIP($T + S$, K, ϕ, f^*) is solvable in $D(T) \cap K \cap D(\phi) \cap G$.

PROOF. (i) Suppose that K is bounded. We first prove the solvability of the problem VIP($\partial\phi + T + S$, K, f^*) in $D(T) \cap D(\partial\phi) \cap K$. To this end, we notice that the solvability of the problem VIP($\partial\phi + T + S$, K, f^*) in $D(\partial\phi) \cap D(T) \cap K$ is equivalent to the solvability of the inclusion

$$\partial I_K(x) + \partial\phi(x) + Tx + Sx \ni f^* \text{ in } D(\partial\phi) \cap D(T) \cap K\,.$$

Since $D(\partial I_K) = K$ and $0 \in \mathring{K} \cap D(T)$, it follows that $\partial I_K + T$ is maximal monotone. Let $A := \partial\phi$ and, for every $t > 0$, let A_t be the Yosida approximant of A. Since $A_t + S$ is bounded pseudomonotone, using the argument in the proof of Theorem 2.3 with K bounded, the maximal monotone operator T and the bounded pseudomonotone operator $A_t + S$, we obtain that the problem VIP$(T + A_t + S, K, f^*)$ is solvable in $D(T) \cap K$, which is equivalent to the solvability of the inclusion

$$\partial I_K(x) + Tx + A_t + Sx \ni f^* \text{ in } D(T) \cap K\,.$$

Thus, for every $t_n \downarrow 0^+$ there exist $x_n \in D(T) \cap K$, $u_n^* \in \partial I_K(x_n)$, $v_n^* \in Tx_n$ and $w_n^* \in Sx_n$ such that

$$u_n^* + v_n^* + A_{t_n}x_n + w_n^* = f^* \tag{4}$$

for all n. Next we see that

$$\begin{aligned}\langle A_{t_n}x_n, x_n\rangle &= \langle A_{t_n}x_n, x_n - J_{t_n}^A x_n\rangle + \langle A_{t_n}x_n, J_{t_n}^A x_n\rangle \\ &= t_n\langle A_{t_n}x_n, J^{-1}(A_{t_n}x_n)\rangle + \langle A_{t_n}x_n, J_{t_n}^A x_n\rangle \\ &= t_n\|A_{t_n}x_n\|^2 + \langle A_{t_n}x_n, J_{t_n}^A x_n\rangle\end{aligned}$$

for all n. Using the properties of the Yosida resolvent of A, we see that $J_{t_n}^A x_n \in D(A)$ and $A_{t_n}x_n \in A(J_{t_n}^A x_n) = \partial\phi(J_{t_n}^{\partial\phi} x_n)$ for all n. On the other hand, by the definition of $\partial\phi$ and the assumption $\phi(x) \geq -k$, we have

$$\langle A_{t_n}x_n, J_{t_n}^A x_n\rangle \geq \phi(J_{t_n}^A x_n) - \phi(0) \geq -k - \phi(0) \tag{5}$$

for all n. Since $\{x_n\}$ and S are bounded, we have the boundedness of $\{w_n^*\}$. From (4) we get

$$\langle u_n^*, x_n\rangle \leq (\|w_n^*\| + \|f^*\|)\|x_n\| + k + \phi(0)\,.$$

Since $0 \in D(\phi)$, we have that $\phi(0) < +\infty$. The boundedness of the sequence $\{u_n^*\}$ follows from the fact that ∂I_K is strongly quasibounded and maximal monotone with the domain K. Since $u_n^* \in \partial I_K(x_n)$, we have $\langle u_n^*, x_n\rangle \geq 0$ for all n. Combining (4) and (5), we have

$$\langle v_n^*, x_n\rangle \leq \|f\|\|x_n\| + \phi(0) + k$$

for all n. As a result, the boundedness of the sequence $\{v_n^*\}$ follows because the sequence $\{x_n\}$ is bounded and T is strongly quasibounded maximal monotone. Consequently, using the equality (4), we obtain the boundedness of $\{A_{t_n}x_n\}$. Assume, by passing to subsequences if necessary, that $x_n \rightharpoonup x_0$, $u_n^* \rightharpoonup u_0^*$, $v_n^* \rightharpoonup v_0^*$, $w_n^* \rightharpoonup w_0^*$ and $A_{t_n}x_n \rightharpoonup z_0^*$ as $n \to \infty$. Since K is closed and convex, it is weakly closed and hence $x_0 \in K$. By using the property of pseudomonotonicity of S, it is easy to see that

$$\liminf_{n\to\infty}\langle w_n^*, x_n - x_0\rangle \geq 0\,.$$

We claim that

$$d := \liminf_{n\to\infty}\langle u_n^* + v_n^*, x_n - x_0\rangle \geq 0\,.$$

In fact, if this is not true, there exists a subsequence, denoted again by $\{\langle u_n^* + v_n^*, x_n - x_0\rangle\}$, such that

$$\lim_{n\to\infty}\langle u_n^* + v_n^*, x_n - x_0\rangle < 0\,.$$

Since $\partial I_K + T$ is maximal monotone, we use Lemma 1.4 to obtain $x_0 \in D(\partial I_K + T)$, $u_0^* + v_0^* \in (\partial I_K + T)(x_0)$ and $\langle u_n^* + v_n^*, x_n\rangle \to \langle u_0^* + v_0^*, x_0\rangle$ as $n \to \infty$. This implies $d = 0$, which is a contradiction. As a result, (4) implies

$$\limsup_{n\to\infty} \langle A_{t_n} x_n, x_n - x_0\rangle \le 0 .$$

Let $J_{t_n}^A$ be the Yosida resolvent of A. We know that $J_{t_n}^A x_n \in D(A)$, $J_{t_n}^A x_n = x_n - t_n J^{-1}(A_{t_n} x_n)$, $A_{t_n} x_n \in A(J_{t_n} x_n)$ for all n and $J_{t_n}^A x_n \rightharpoonup x_0$ and $x_n - J_{t_n}^A x_n \to 0$ as $n \to \infty$. Therefore, we have

$$\limsup_{n\to\infty} \langle A_{t_n} x_n, J_{t_n}^A x_n - x_0\rangle \le 0 .$$

Using Lemma 1.4 again, we conclude that $x_0 \in D(A)$, $z_0^* \in Ax_0$ and $\langle A_{t_n} x_n, J_{t_n}^A x_n\rangle \to \langle z_0^*, x_0\rangle$ as $n \to \infty$. Thus, (4) implies

$$\limsup_{n\to\infty} \langle u_n^* + v_n^*, x_n - x_0\rangle \le 0 .$$

From Lemma 1.4, we obtain $x_0 \in D(T) \cap K$, $u_0^* + v_0^* \in (\partial I_K + T)(x_0)$ and $\langle u_n^* + v_n^*, x_n\rangle \to \langle u_0^* + v_0^*, x_0\rangle$ as $n \to \infty$. Consequently, $x_0 \in D(A) \cap D(T) \cap K$ and

$$\limsup_{n\to\infty} \langle w_n^*, x_n - x_0\rangle = 0 .$$

Since S is pseudomonotone, for every $x \in K$ there exists $y^*(x) \in Sx_0$ such that

$$\langle y^*(x), x_0 - x\rangle \le \liminf_{n\to\infty} \langle w_n^*, x_n - x\rangle = -\langle u_0^* + v_0^* + z_0^* - f^*, x_0 - x\rangle ,$$

where the equality follows from (4). Thus, for every $x \in K$ there exists $y^*(x) \in Sx_0$ such that

$$\langle y^*(x) + u_0^* + v_0^* + z_0^* - f^*, x - x_0\rangle \ge 0 .$$

Following the proof of Theorem 2.3, we see that there exists a unique $y_0^* \in Sx_0$ such that

$$\langle y_0^* + u_0^* + v_0^* + z_0^* - f^*, x - x_0\rangle \ge 0$$

for all $x \in K$. Using the definition of ∂I_K and $\partial\phi$, since $u_0^* \in \partial I_K(x_0)$ and $z_0^* \in \partial\phi(x_0)$, we see that $\langle u_0^*, x_0 - x\rangle \ge 0$ for all $x \in K$, and $\langle z_0^*, x_0 - x\rangle \ge \phi(x_0) - \phi(x)$ for all $x \in X$. As a consequence, we get

$$\begin{aligned}\langle v_0^* + y_0^* - f^*, x - x_0\rangle &\ge \langle u_0^* + z_0^*, x_0 - x\rangle \\ &\ge \phi(x_0) - \phi(x)\end{aligned}$$

for all $x \in K$. Since $D(\partial\phi) \subset D(\phi)$, it follows that $x_0 \in D(T) \cap D(\phi) \cap K$. Therefore, the problem VIP$(T + S, K, \phi, f^*)$ is solvable in $D(T) \cap D(\phi) \cap K$.

(ii) Suppose that (ii) holds. Since $K \cap \overline{G}$ is a nonempty, closed, convex and bounded subset of X, using $K \cap \overline{G}$ in place of K in the argument of (i), we conclude that the problem VIP$(T + S, K \cap \overline{G}, \phi, f^*)$ is solvable in $D(T) \cap D(\phi) \cap K \cap \overline{G}$. Since the problem VIP$(T + S, K \cap \overline{G}, \phi, f)$ has no solution in $D(T) \cap D(\phi) \cap K \cap \partial G$, we use Lemma 2.2 to conclude that VIP$(T + S, K, \phi, f^*)$ is solvable in $D(T) \cap D(\phi) \cap K \cap G$. □

We remark that Theorem 2.5 extends the result of Kenmochi [14, Theorem 4.1, p. 254] to the effect that we consider the operator $T + S$ instead of the single pseudomonotone operator S.

In the following corollary we use a coercivity-type condition involving the operator $T+S$ and the function ϕ.

COROLLARY 2.6. *Let K be a nonempty, closed and convex subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in D(T)$ and $S : K \to 2^{X^*}$ bounded pseudomonotone. Let $\phi : X \to (-\infty, \infty]$ be proper, convex lower semicontinuous with $0 \in D(\phi)$ and there exists a real number $k > 0$ such that $\phi(x) \geq -k$ for all $x \in X$. Assume, further, that there exists $u_0 \in K$ with $\phi(u_0) < \infty$ satisfying*

$$\inf_{v^* \in Tx, w^* \in Sx,\ x \in D(T) \cap K} \frac{\langle v^* + w^*, x - u_0 \rangle + \phi(x)}{\|x\|} \to \infty \text{ as } \|x\| \to \infty .$$

Then for every $f^* \in X^*$, the problem VIP$(T + S, K, \phi, f^*)$ is solvable in $D(T) \cap D(\phi) \cap K$.

PROOF. Since $\phi(u_0) < \infty$, for every $f^* \in X^*$ there exists $R = R(f^*) > 0$, which can be chosen so that $u_0 \in \overline{B_R(0)}$, such that

$$\langle v^* + w^* - f^*, x - u_0 \rangle + \phi(x) > \phi(u_0)$$

for all $x \in D(T) \cap K \cap \partial B_R(0)$. This is equivalent to saying that the problem VIP$(T + S, K \cap \overline{B_R(0)}, \phi, f^*)$ has no solution in $D(T) \cap D(\phi) \cap K \cap \partial B_R(0)$. On the other hand, using the closed, convex and bounded set $K \cap \overline{B_R(0)}$ and applying (i) of Theorem 2.5, we see that VIP$(T + S, K \cap \overline{B_R(0)}, \phi, f^*)$ is solvable in $D(T) \cap K \cap \overline{B_R(0)}$, which implies that VIP$(T + S, K \cap \overline{B_R(0)}, \phi, f^*)$ is solvable in $D(T) \cap K \cap B_R(0)$. Applying Lemma 2.2, we conclude that VIP$(T + S, K, \phi, f^*)$ is solvable in $D(T) \cap K \cap B_R(0)$. □

The following theorem gives a new existence result for the solvability of the problem VIP$(T + S + P, K, f^*)$ and the inclusion problem $Tx + Sx + Px \ni f^*$, where both T and S are maximal monotone and P is bounded pseudomonotone.

THEOREM 2.7. *Let K be nonempty, closed and convex subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone and such that there exists $k_1 > 0$ with $\langle u^*, x \rangle \geq -k_1$ for all $x \in D(T)$ and $u^* \in Tx$. Let $S : X \supseteq D(S) \to 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in S(0)$. Suppose that $P : K \to 2^{X^*}$ is bounded pseudomonotone. Assume, further, that there exist $R > 0$, $u_0 \in D(T) \cap D(S) \cap K \cap B_R(0)$ and $k_2 > 2R|Tu_0|$ such that*

$$\langle w^* + z^* - f^*, x - u_0 \rangle \geq k_2$$

for all $x \in D(T) \cap D(S) \cap K \cap \partial B_R(0)$, $w^* \in Sx$ and $z^* \in Px$. Then the following are true.

(i) *The problem* VIP$(T + S + P, K, f^*)$ *is solvable in $D(T) \cap D(S) \cap K \cap B_R(0)$.*

(ii) *If $K = X$, the inclusion $Tx + Sx + Px \ni f^*$ is solvable in $D(T) \cap D(S) \cap B_R(0)$.*

PROOF. We first prove (i). Let $\partial I_K : K \to 2^{X^*}$ be the subdifferential of the indicator function on K. It is well-known that $D(\partial I_K) = K$ and $\partial I_K(x) = \{0\}$ for all $x \in \mathring{K}$. Since

$0 \in \mathring{K}$, we have $0 \in \partial I_K(0)$ and ∂I_K is strongly quasibounded and maximal monotone. Let T_t be the Yosida approximant of T. Since $u_0 \in D(T)$, we have $\|T_t u_0\| \leq |Tu_0|$, where $|Tu_0| = \inf\{\|x^*\|; x^* \in Tu_0\}$ for all $t > 0$. Thus, for every $t > 0$, $T_t + P$ is bounded, pseudomonotone and such that

$$\begin{aligned}\langle w^* + z^* + T_t x - f^*, x - u_0\rangle &= \langle w^* + z^* + T_t x - T_t u_0 + T_t u_0 - f^*, x - u_0\rangle \\ &\geq k_2 - |Tu_0|\|x - u_0\| \\ &\geq k_2 - 2R|Tu_0| > 0\end{aligned}$$

for all $x \in D(S) \cap K \cap \partial B_R(0)$, $w^* \in Sx$ and $z^* \in Px$. Since $u_0 \in K \cap \overline{B_R(0)}$, it follows that VIP$(P + T_t + S, K \cap \overline{B_R(0)}, f^*)$ has no solution in $D(S) \cap K \cap \partial B_R(0)$. Since $T_t + P$ is bounded and pseudomonotone, we use Theorem 2.3 with the operators S and $T_t + P$ to conclude that VIP$(S + T_t + P, K, f^*)$ is solvable in $D(S) \cap K \cap B_R(0)$. Thus, for every $t_n \downarrow 0^+$, there exist $x_n \in D(S) \cap K \cap B_R(0)$, $v_n^* \in \partial I_K(x_n)$, $w_n^* \in Sx_n$ and $z_n^* \in Px_n$ such that

$$v_n^* + w_n^* + z_n^* + T_{t_n} x_n = f^* \tag{6}$$

for all n. Since $\{x_n\}$ and P are bounded, we see that the sequence $\{z_n^*\}$ is bounded. Next, since $0 \in K$, we get from the definition of ∂I_K that $\langle v_n^*, x_n\rangle \geq 0$ for all n. Thus, using (6), we obtain

$$\begin{aligned}\langle w_n^*, x_n\rangle &\leq -\langle z_n^* - f^*, x_n\rangle - \langle T_{t_n} x_n, x_n - J_{t_n} x_n\rangle - \langle T_{t_n} x_n, J_{t_n} x_n\rangle - \langle v_n^*, x_n\rangle \\ &\leq (\|z_n^*\| + \|f^*\|)\|x_n\| - \langle T_{t_n} x_n, t_n J^{-1}(T_{t_n} x_n)\rangle + k_1 \\ &= (\|z_n^*\| + \|f^*\|)\|x_n\| - t_n\|T_{t_n} x_n\|^2 + k_1 \leq M\,,\end{aligned}$$

where M is an upper bound for the sequence $\{(\|z_n^*\| + \|f^*\|)\|x_n\| + k_1\}$. Therefore, the strong quasiboundedness of S implies the boundedness of the sequence $\{w_n^*\}$. Similarly, we get

$$\begin{aligned}\langle v_n^*, x_n\rangle &\leq -\langle T_{t_n} x_n, x_n - J_{t_n} x_n\rangle - \langle T_{t_n} x_n, J_{t_n} x_n\rangle \\ &\quad + (\|w_n^*\| + \|z_n^*\| + \|f^*\|)\|x_n\| \\ &\leq k_1 + (\|w_n^*\| + \|z_n^*\| + \|f^*\|)\|x_n\| \leq N\,,\end{aligned}$$

where N is an upper bound for the sequence $\{k_1 + (\|w_n^*\| + \|z_n^*\| + \|f^*\|)\|x_n\|\}$. Using the strong quasiboundedness of ∂I_K, it follows that the sequence $\{v_n^*\}$ is bounded, which implies in turn the boundedness of the sequence $\{T_{t_n} x_n\}$. Assume that $x_n \rightharpoonup x_0$, $v_n^* \rightharpoonup v_0^*$, $w_n^* \rightharpoonup w_0^*$, $z_n^* \rightharpoonup z_0^*$ and $T_{t_n} x_n \rightharpoonup u_0^*$ as $n \to \infty$. Since K is closed and convex, it is weakly closed and hence $x_0 \in K$. Since P is pseudomonotone and S and ∂I_K are monotone, we have

$$\liminf_{n\to\infty}\langle w_n^*, x_n - x_0\rangle \geq 0,\ \liminf_{n\to\infty}\langle v_n^*, x_n - x_0\rangle \geq 0 \text{ and } \liminf_{n\to\infty}\langle z_n^*, x_n - x_0\rangle \geq 0\,.$$

Thus, we obtain

$$\limsup_{n\to\infty}\langle T_{t_n} x_n, x_n - x_0\rangle \leq 0\,.$$

Using the maximality of T and Lemma 1.4, we conclude that $x_0 \in D(T)$, $u_0^* \in Tx_0$ and $\langle T_{t_n} x_n, x_n\rangle \to \langle u_0^*, x_0\rangle$ as $n \to \infty$. Similarly, we see that $x_0 \in D(S) \cap K$, $v_0^* + w_0^* \in (\partial I_K +$

$S)x_0$ and $\langle v_n^* + w_n^*, x_n\rangle \to \langle v_0^* + w_0^*, x_0\rangle$ as $n \to \infty$. Finally, by the pseudomonotonicity of P, for every $x \in K$ there exists $y^*(x) \in Sx_0$ such that

$$\langle y^*(x) + w_0^* + u_0^* - f^*, x - x_0\rangle \geq 0\,.$$

As in the argument of the last part of the proof of Theorem 2.3, there exists $y_0^* \in Sx_0$ such that

$$\langle y_0^* + w_0^* + u_0^* - f^*, x - x_0\rangle \geq 0$$

for all $x \in K$. This shows that the problem VIP$(T + S + P, K, f^*)$ is solvable in $D(T) \cap D(S) \cap K$. The proof of (i) is complete.

(ii) Using (i) with $K = X$, we see that the inequality VIP$(T + S + P, X, f^*)$ is solvable in $D(T) \cap D(S) \cap B_R(0)$. Using the definition of the solvability of a variational inequality, it is easy to see that the inclusion

$$Tx + Sx + Px \ni f^*$$

is solvable in $D(T) \cap D(S) \cap B_R(0)$. The proof is complete. □

As an application of Theorem 2.7, the following corollary gives a maximality criterion for the sum of two maximal monotone operators. Basic maximality criteria can be found in Browder and Hess [1] and Rockaffelar [37].

COROLLARY 2.8. *Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone and such that there exists $k_1 > 0$ satisfying $\langle u^*, x\rangle \geq -k_1$ for all $x \in D(T)$ and $u^* \in Tx$. Let $S : X \supseteq D(S) \to 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in S(0)$ such that $D(T) \cap D(S) \neq \emptyset$. Then $T + S$ is maximal monotone.*

PROOF. Choose $u_0 \in D(T) \cap D(S)$. Choose $r > 0$ such that $u_0 \in D(T) \cap D(S) \cap B_r(0)$. Then, for $w_0^* \in Su_0$, using the monotonicity of J and S, we have

$$\begin{aligned}\langle w^* + Jx - f^*, x - u_0\rangle &\geq \langle w_0^* + Jx - f^*, x - u_0\rangle \\ &\geq \|x\|^2 - \|u_0\|\|x\| \\ &\quad -(\|w_0^*\| + \|f^*\|)\|x - u_0\| \to \infty \text{ as } \|x\| \to \infty\,.\end{aligned}$$

Therefore, for any $k_2 > 0$, there exists $R_1 > 0$ such that

$$\langle w^* + Jx - f^*, x - u_0\rangle > k_2$$

for all $\|x\| \geq R_1$, $w^* \in Sx$. We choose $R = \max\{r, R_1\}$ so that $u_0 \in D(T) \cap D(S) \cap B_R(0)$ and

$$\langle w^* + Jx - f^*, x - u_0\rangle > k_2$$

for all $x \in D(S) \cap \partial B_R(0)$ and $w^* \in Sx$. Using J in place of P in (ii) of Theorem 2.7, we conclude that the inclusion $Tx + Sx + Jx \ni f^*$ is solvable in $D(T) \cap D(S) \cap B_R(0)$. Since $f^* \in X^*$ is arbitrary, it follows that $R(T + S + J) = X^*$. This complete the maximality of $T + S$. □

3. Variational inequalities for generalized pseudomonotone perturbations. In this section we give some results about the solvability of variational inequalities involving perturbations which are generalized pseudomonotone operators. Browder and Hess [1, Proposition 4, p. 258] showed that a bounded generalized pseudomonotone operator S is pseudomonotone if $D(S) = X$. However, this fact is unknown if $D(S) \neq X$. Because of this, we study the solvability of variational inequality problems separately for bounded pseudomonotone and bounded generalized pseudomonotone perturbations. A range result for single multivalued, densely defined, quasibounded, finitely continuous generalized pseudomonotone operator may be found in Browder and Hess [1, Theorem 5, p. 273]. Furthermore, range results for quasibounded, finitely continuous, generalized pseudomonotone perturbations S of maximal monotone operators, with S either densely defined or $D(S) = X$, under weaker coercivity assumptions on $T + S$, may be found in Guan, Kartsatos and Skrypnik [26] and Guan and Kartsatos [27], respectively. Variational inequality results of the type $\mathrm{VIP}(T + S, K, f^*)$, where T is maximal monotone with $D(T) = X$, S is bounded, finitely continuous and generalized pseudomonotone with $D(S) = K$ (with K closed and convex with $0 \in K$) may be found in Zhou [29].

We now give the following existence result concerning the solvability of a variational inequality involving finitely continuous generalized pseudomonotone perturbations of a maximal monotone operator with $D(T)$ not necessarily all of X.

THEOREM 3.1. *Let K be a nonempty, closed and convex subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and $S : K \to 2^{X^*}$ finitely continuous generalized pseudomonotone such that there exists $k > 0$ satisfying $\langle w^*, x\rangle \geq -k$ for all $x \in K$ and $w^* \in Sx$. Fix $f^* \in X^*$.*

(i) *If K is bounded, then the problem $\mathrm{VIP}(T + S, K, f^*)$ is solvable in $D(T) \cap K$.*

(ii) *If K is unbounded and there exists an open, bounded and convex subset G of X with $0 \in G$ such that the problem $\mathrm{VIP}(T + S, K \cap \overline{G}, f^*)$ has no solution in $D(T)\cap K\cap\partial G$, then the problem $\mathrm{VIP}(T+S, K, f^*)$ is solvable in $D(T)\cap K\cap G$.*

Furthermore, if either the hypothesis of (i) *or* (ii) *holds and the inclusion $Tx + Sx \ni f^*$ has no solution in $D(T) \cap \partial K$, then the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T) \cap \mathring{K}$.*

PROOF. (i) Assume that K is bounded. For each $t > 0$, let T_t be the Yosida approximant of T. It is known that T_t is bounded, continuous and maximal monotone with domain all of X. We follow in part Browder and Hess [1, Theorem 15, p. 289] who considered a single multivalued pseudomonotone operator.

Let Λ be the collection of all finite dimensional subspaces of X. For each $F \in \Lambda$, let $j_F : F \to X$ be the inclusion mapping and $j_F^* : X^* \to F^*$ the dual projection mapping of X^* onto F^*. Let $K_F := K \cap F$. Since K is bounded, K_F is a compact subset of F for every $F \in \Lambda$. Since S is pseudomonotone, $-(j_F^*(T_t + S))$ is upper semicontinuous with nonempty, closed, convex and bounded values in X^*. Thus, the operator $j_F^*(T_t + S)j_F : K_F \to F^*$ is upper semicontinuous. Using Lemma 1.7, there exist $x_F \in K_F$ and $w_F^* \in Sx_F$ such that

$$\langle j_F^*(T_t x_F + w_F^* - f^*), x - x_F \rangle \geq 0$$

for all $x \in K_F$, which is equivalent to saying that

$$\langle T_t x_F + w_F^*, x - x_F \rangle \geq \langle f^*, x - x_F \rangle$$

for all $x \in K_F$. Since K is closed convex and bounded, the family $\{x_F\}_{F \in \Lambda}$ is uniformly bounded and K is a weakly compact subset of X. For each $F \in \Lambda$, we define

$$V_F := \bigcup_{F \subset F'} \{x_{F'}\} .$$

We observe that, for every F, $\overline{V_F}^w$ is a weakly closed subset of the weakly compact subset K. Furthermore, the family $\{\overline{V_F}^w\}$ satisfies the finite intersection property. Therefore, we have

$$V := \bigcap_{F \in \Lambda} \overline{V_F}^w \neq \emptyset .$$

Fix $x \in K$ and choose $x_0 \in V$ and a subspace F_0 of X such that $x_0,\ x \in F_0$. Using Lemma 1.9, we choose a sequence $\{x_n\}$ in V_{F_0} such that $x_n \rightharpoonup x_0$ as $n \to \infty$. By the definition of V_{F_0}, for every n we choose F_n such that $F_0 \subseteq F_n$ and $x_n \in K_{F_n}$. Since K is closed and convex, it is weakly closed and hence $x_0 \in K$. From the definition of x_n, it follows that

$$\langle T_t x_n + w_n^*, u - x_n \rangle \geq \langle f^*, u - x_n \rangle$$

for all $u \in K_{F_n}$ for some $w_n^* \in Sx_n$, where $K_{F_n} = K \cap F_n$. From the definition of V_{F_0}, we have $x \in K_{F_n}$ for all n, which implies

$$\langle T_t x_n + w_n^*, x - x_n \rangle \geq \langle f^*, x - x_n \rangle$$

for all n and all $x \in K$. Using the definition of ∂I_K, there exists $u_n^* \in \partial I_K(x_n)$ such that

$$u_n^* + T_t x_n + w_n^* = f^*$$

for all n.

Next, we show that $x_0 \in K$ is a solution of the problem VIP$(T_t + S, K, f^*)$. Since $\{x_n\}$ is bounded and T_t is bounded, it follows that $\{T_t x_n\}$ is bounded. Since $\langle w_n^*, x_n \rangle \geq -k$ for all n, we use the monotonicity of T_t and $T_t(0) = 0$ to obtain $\langle u_n^*, x_n \rangle \leq k + \|f^*\| \|x_n\| \leq C$ for all n, where C is an appropriate upper bound. Since $0 \in \mathring{K}$ and $D(\partial I_K) = K$, ∂I_K is strongly quasibounded maximal monotone. As a result, $\{u_n^*\}$ is bounded, which implies the boundedness of $\{w_n^*\}$. Assume, by passing to a subsequence if necessary, that $w_n^* \rightharpoonup w_0^*$ and $T_t x_n \rightharpoonup u_0^*$ as $n \to \infty$. Since S is generalized pseudomonotone, we see from Definition 1.2 that

$$\liminf_{n \to \infty} \langle w_n^*, x_n - x_0 \rangle \geq 0 .$$

Thus,

$$\limsup_{n \to \infty} \langle T_t x_n, x_n - x_0 \rangle \leq 0 .$$

The maximal monotonicity of T_t along with Lemma 1.4 implies $u_0^* = T_t x_0$ and $\langle T_t x_n, x_n\rangle \to \langle T_t x_0, x_0\rangle$ as $n \to \infty$. Consequently, we have

$$\limsup_{n\to\infty} \langle w_n^*, x_n - x_0\rangle \le 0\,.$$

Since S is generalized pseudomonotone, it follows that $w_0^* \in Sx_0$ and $\langle w_n^*, x_n\rangle \to \langle w_0^*, x_0\rangle$ as $n \to \infty$. Letting $n \to \infty$ in the last inequality above involving $T_t x_n$, we obtain

$$\langle T_t x_0 + w_0^*, x - x_0\rangle \ge \langle f^*, x - x_0\rangle$$

for all $x \in K$. Thus, the problem VIP$(T_t + S, K, f^*)$ is solvable in K.

Thus, for every $t_n \downarrow 0^+$, the problem VIP$(T_{t_n} + S, K, f^*)$ is solvable in K, i.e., there exists $y_n \in K$, $w_n^* \in Sy_n$ and $v_n^* \in \partial I_K(y_n)$ such that

$$v_n^* + T_{t_n} y_n + w_n^* = f^* \tag{7}$$

for all n. It is well known that $D(\partial I_K) = K$. Since $0 \in \mathring{K}$, the mapping ∂I_K is strongly quasibounded maximal monotone from K into X^*. Since $0 \in T(0)$, we have $T_{t_n}(0) = 0$ and the assumption $0 \in K$ implies $\langle v_n^*, y_n\rangle \ge 0$ for all n. Therefore, using (7), we see that

$$\langle v_n^*, y_n\rangle \le k + \|f^*\|\|y_n\| \le Q$$

for all n, where Q is an upper bound for the sequence $\{k + \|f\|\|x_n\|\}$. The boundedness of the sequence $\{v_n^*\}$ follows from the strong quasiboundedness of ∂I_K. In addition, using (7), we get

$$\langle T_{t_n} y_n, y_n\rangle \le Q$$

for all n, where Q is as above. Thus, the boundedness of the sequence $\{T_{t_n} y_n\}$ follows from Lemma 1.6. As a result, the sequence $\{w_n^*\}$ is bounded. Assume without loss of generality that $y_n \rightharpoonup y_0 \in K$, $v_n^* \rightharpoonup v_0^*$ and $T_{t_n} y_n \rightharpoonup z_0^*$ as $n \to \infty$. Using the monotonicities of T_{t_n} and ∂I_K, we see that

$$\limsup_{n\to\infty} \big\langle w_n^*, y_n - y_0\big\rangle \le 0\,.$$

Since S is generalized pseudomonotone, we have $w_0^* \in Sy_0$ and $\langle w_n^*, y_n\rangle \to \langle w_0^*, y_0\rangle$ as $n \to \infty$. Using this and the monotonicity of ∂I_K, we get

$$\liminf_{n\to\infty} \langle v_n^* + w_n^*, y_n - y_0\rangle \ge 0\,,$$

which implies

$$\limsup_{n\to\infty} \langle T_{t_n} y_n, y_n - y_0\rangle \le 0\,.$$

Let J_{t_n} be the Yosida resolvent of T. We know that $J_{t_n} y_n = y_n - t_n J^{-1}(T_{t_n} y_n)$, $J_{t_n} y_n \in D(T)$ and $T_{t_n} y_n \in T(J_{t_n} y_n)$ for all n. Since $\{T_{t_n} y_n\}$ is bounded, $t_n \downarrow 0^+$ as $n \to \infty$ and $y_n \rightharpoonup y_0$, it follows that $J_{t_n} y_n \rightharpoonup y_0$ as $n \to \infty$. Consequently, (7) implies

$$\limsup_{n\to\infty} \langle T_{t_n} y_n, J_{t_n} y_n - y_0\rangle \le 0\,.$$

The maximality of T and Lemma 1.4 imply $y_0 \in D(T)$, $z_0^* \in Ty_0$ and $\langle T_{t_n} y_n, J_{t_n} y_n \rangle \to \langle z_0^*, y_0 \rangle$ as $n \to \infty$. Applying a similar argument for the mapping ∂I_K, we see that $v_0 \in \partial I_K$. Finally, taking the limit as $n \to \infty$ in (7), we conclude that

$$v_0^* + z_0^* + w_0^* = f^* .$$

Therefore, the problem VIP($T + S, K, f^*$) is solvable in $D(T) \cap K$.

(ii) Suppose that the hypothesis in (ii) holds. Using the closed, convex and bounded set $K \cap \overline{G}$ instead of K in (i), we obtain the solvability of the problem VIP($T + S, K \cap \overline{G}, f^*$) in $D(T) \cap K \cap \overline{G}$. Since the problem VIP($T+S, K \cap \overline{G}, f^*$) has no solution in $D(T) \cap K \cap \partial G$, we may use Lemma 2.2 to conclude that the problem VIP($T+S, K, f^*$) is solvable in $D(T) \cap K$.

It is known that the solvability of the problem VIP($T + S, K, f^*$) is equivalent to the solvability of the inclusion $\partial I_K(x) + Tx + Sx \ni f^*$ in $D(T) \cap K$. Therefore, if either the hypothesis of (i) or (ii) holds and the inclusion $\partial I_K(x) + Tx + Sx \ni f^*$ has no solution in $D(T) \cap \partial K$, then the solution lies in $D(T) \cap \mathring{K}$. Since $\partial I_K(x) = \{0\}$ for all $x \in \mathring{K}$, we obtain the solvability of the inclusion $Tx + Sx \ni f^*$ in $D(T) \cap \mathring{K}$. □

We note that if $K = \overline{B_R(0)}$ and T and S are as in Theorem 3.1, part (i) of Theorem 3.1 implies that VIP($T + S, K, f^*$) is solvable in $D(T) \cap K$. This is equivalent to saying that the inclusion

$$\partial I_K(x) + Tx + Sx \ni f^*$$

is solvable in $D(T) \cap K$. Thus, the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T) \cap B_R(0)$ provided that the inclusion in the above display has no solution in $D(T) \cap \partial K$, which is equivalent to saying that $Tx + Sx + \lambda Jx \ni f^*$ has no solution in $D(T) \cap \partial B_R(0)$ for any $\lambda \geq 0$. This is because the subdifferential $\partial I_K(x)$ is now given by

$$\partial I_K(x) = \begin{cases} \{0\} & \text{if } x \in B_R(0), \\ \{\lambda Jx : \lambda \geq 0\} & \text{if } x \in \partial B_R(0), \\ \emptyset & \text{if } x \in X \setminus \overline{B_R(0)}. \end{cases}$$

Let Γ_β denote the set of all functions $\beta : R_+ \to R_+$ such that $\beta(t) \to 0$ as $t \to \infty$. A range result for densely defined quasibounded, finitely continuous generalized pseudomonotone perturbation of maximal monotone operator may be found in Guan, Kartsatos and Skrypnik [26]. A new variational inequality result in the spirit of [26], is given below.

THEOREM 3.2. *Let K be a nonempty, closed and convex subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and $S : K \to 2^{X^*}$ finitely continuous generalized pseudomonotone. Fix $f^* \in X^*$. Assume, further, the following conditions hold.*

(i) *There exists a strictly increasing continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ and $\psi(t) \to \infty$ as $t \to \infty$ satisfying $\langle w^*, x \rangle \geq -\psi(\|x\|)$, $x \in K$ and $w^* \in Sx$;*

(ii) *There exist $R > 0$, $u_0 \in K$ and $\beta \in \Gamma_\beta$ such that*

$$\langle v^* + w^* - (f^* + g^*), x - u_0 \rangle \geq -\beta(\|x\|)\|x\|$$

for all $g^* \in X^*$ *with* $\|g^*\| \le R$, $x \in D(T) \cap K$, $v^* \in Tx$ *and* $w^* \in Sx$.
Then the problem $\mathrm{VIP}(T+S, K, f^*)$ *is solvable in* $D(T) \cap K$.

PROOF. Fix $f^* \in X^*$ and suppose that K is bounded, i.e., for some $r > 0$, $K \subseteq B_r(0)$. Since ψ is strictly increasing, it follows that $\psi(\|x\|) \le \psi(r)$ for all $x \in K$. As a result, we see that $\langle w^*, x\rangle \ge -\psi(r) = -k_r$ for all $x \in K$ and $w^* \in Sx$. Applying (i) of Theorem 3.1, we obtain the solvability of the problem $\mathrm{VIP}(T+S, K, f^*)$ in $D(T) \cap K$. Assume K is unbounded. Let J_ψ be the duality mapping corresponding to the function ψ. For every $\varepsilon > 0$ and $x \ne 0$ we have

$$\langle v^* + w^* + \varepsilon J_\psi x - f^*, x\rangle \ge \psi(\|x\|)\|x\| \left(\varepsilon - \frac{1}{\|x\|} - \frac{\|f^*\|}{\psi(\|x\|)} \right) \to \infty \quad \text{as } \|x\| \to \infty$$

for all $v^* \in Tx$, $w^* \in Sx$. Consequently, there exists $R_\varepsilon = R(\varepsilon) > 0$ such that

$$\langle z^* + w^* + \varepsilon J_\psi x - f^*, x\rangle > 0 \tag{8}$$

for all $x \in D(T) \cap K \cap \partial B_{R_\varepsilon}(0)$, $z^* \in Tx$. Since $K \cap \overline{B_{R_\varepsilon}(0)}$ is bounded and $S + J_\psi$ is finitely continuous generalized pseudomonotone, we may apply (i) of Theorem 3.1 to conclude that the problem $\mathrm{VIP}(T + S + \varepsilon J_\psi, K \cap \overline{B_{R_\varepsilon}(0)}, f^*)$ is solvable in $D(T) \cap K \cap \overline{B_{R_\varepsilon}(0)}$. Since $0 \in K \cap \overline{B_{R_\varepsilon}(0)}$, (8) implies that the problem $\mathrm{VIP}(T + S + \varepsilon J_\psi, K \cap \overline{B_{R_\varepsilon}(0)}, f^*)$ has no solution in $D(T) \cap K \cap \partial B_{R_\varepsilon}(0)$, i.e., the problem $\mathrm{VIP}(T + S + \varepsilon J_\psi, K \cap \overline{B_{R_\varepsilon}(0)}, f^*)$ is solvable in $D(T) \cap K \cap B_{R_\varepsilon}(0)$. Thus, using Lemma 2.2, we get the solvability of the problem $\mathrm{VIP}(T + S + \varepsilon J_\psi, K, f^*)$ in $D(T) \cap K \cap B_{R_\varepsilon}(0)$, i.e., for $\varepsilon_n \downarrow 0^+$ there exist $x_n \in D(T) \cap K \cap B_{R_{\varepsilon_n}}(0)$, $w_n^* \in Sx_n$, and $z_n^* \in Tx_n$ such that

$$\langle z_n^* + w_n^* + \varepsilon_n J_\psi x_n - f^*, x - x_n\rangle \ge 0 \tag{9}$$

for all $x \in K$ and all n. Equivalently, there exists $v_n^* \in \partial I_K(x_n)$ such that

$$v_n^* + z_n^* + w_n^* + \varepsilon_n J_\psi x_n = f^* \tag{10}$$

for all n. Since $u_0 \in K$, we obtain from (9)

$$\begin{aligned} -\beta(\|x_n\|)\|x_n\| &\le \langle z_n^* + w_n^* - f^* - g^*, x_n - u_0\rangle \\ &\le -\varepsilon_n \psi(\|x_n\|)(\|x_n\| - \|u_0\|) - \langle g^*, x_n - u_0\rangle . \end{aligned}$$

If the sequence $\{x_n\}$ is unbounded, then $\|x_n\| \ge \|u_0\|$ for all large n and

$$\langle g^*, x_n\rangle \le \langle g^*, u_0\rangle + \beta(\|x_n\|)\|x_n\|$$

for all large n. Therefore, by Lemma 1.8, the sequence $\{x_n\}$ is bounded. Since $0 \in T(0)$, we have $\langle w_n^*, x_n\rangle \ge -\psi(\|x_n\|)$ and the boundedness of $\{v_n^*\}$ follows from (10). Using a similar argument, the boundedness of the sequence $\{z_n^*\}$ follows from the fact that T is strongly quasibounded. Consequently, we have the boundedness of the sequence $\{w_n^*\}$. Assume without loss of generality that $x_n \rightharpoonup x_0 \in K$, $v_n^* \rightharpoonup v_0^*$, $w_n^* \rightharpoonup w_0^*$ and $z_n^* \rightharpoonup z_0^*$ as $n \to \infty$. Since S is generalized pseudomonotone, it is easy to see that

$$\liminf_{n\to\infty} \langle w_n^*, x_n - x_0\rangle \ge 0 ,$$

which implies

$$\limsup_{n\to\infty} \langle v_n^* + z_n^*, x_n - x_0\rangle \le 0 .$$

Since $0 \in \mathring{K} \cap D(T)$, we see that $\partial I_K + T$ is maximal monotone. Thus, by Lemma 1.4, we have $x_0 \in D(T) \cap K$, $v_0^* + z_0^* \in (\partial I_K + T)(x_0)$ and $\langle v_n^* + z_n^*, x_n\rangle \to \langle v_0^* + z_0^*, x_0\rangle$ as $n \to \infty$. Consequently, (10) implies

$$\lim_{n\to\infty} \langle w_n^*, x_n - x_0\rangle = 0 .$$

The generalized pseudomonotonicity of S implies that $w_0^* \in Sx_0$ and $\langle w_n^*, x_n\rangle \to \langle w_0^*, x_0\rangle$ as $n \to \infty$. Finally, taking the limit as $n \to \infty$ in (10), we conclude that the problem VIP($T + S, K, f^*$) is solvable in $D(T) \cap K$. □

Zhou [29] proved a version of Theorem 3.2 with $D(T) = X$ using the fact that $T + S + \varepsilon J$ is of type (S_+) with S bounded. We remark that Theorem 3.2 improves the result of Zhou [29] in that the maximal monotone operator T may now be just strongly quasibounded with $D(T) \neq X$.

4. Variational inequalities for unbounded generalized pseudomonotone perturbations. Kenmochi [14] introduced the definition of multivalued operators of type (pm_4) as follows.

DEFINITION 4.1. An operator $S : X \to 2^{X^*}$ is said to satisfy "Condition (pm_4)" if for every $x \in X$ and every bounded subset B of X there exists a number $N(B, x)$ such that

$$\langle y^*, y - x\rangle \ge N(B, x)$$

for all $(y, y^*) \in G(S)$ with $y \in B$.

Kenmochi [14] showed that an operator S with $D(S) = X$ which satisfies (i) and (iii) of Definition 1.1 and Condition (pm_4) satisfies also (ii) of Definition 1.1, which implies that S is pseudomonotone. Furthermore, he gave various surjectivity results for perturbations of nonlinear maximal monotone operators.

In this section we give an existence result for the problem VIP($T + S, K, f^*$), where T is maximal monotone and S is finitely continuous generalized pseudomonotone, possibly unbounded, with $D(S) = X$ satisfying condition (pm_4). The following uniform boundedness result is important for our consideration.

LEMMA 4.2. *Assume that $S : X \to 2^{X^*}$ satisfies Condition (pm_4). Let $\{x_n\} \subset X$ be bounded and $w_n^* \in Sx_n$ be such that, for some $y_0 \in X$, the condition*

$$\limsup_{n\to\infty} \langle w_n^*, x_n - y_0\rangle < +\infty$$

is satisfied. Then the sequence $\{w_n^*\}$ is bounded in X^*.

PROOF. Assume that there exists a real number M such that

$$\limsup_{n\to\infty} \langle w_n^*, x_n - y_0\rangle \le M .$$

Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $x_n \in B_R(0) := B$ for all n. Using condition (pm_4), we see that for every $x \in X$ there exists $N(B, x)$ such that

$$\langle w_n^*, x_n - x\rangle \geq N(B, x)$$

for all n. Next, for every $x \in X$ we have

$$\langle w_n^*, y_0 - x\rangle = \langle w_n^*, x_n - x\rangle - \langle w_n^*, x_n - y_0\rangle$$

for all n, and hence

$$\liminf_{n\to\infty} \langle w_n^*, y_0 - x\rangle \geq N(B, x) - M\,.$$

Given $x \in X$ and letting $y_0 - x$ in place of x above, we know that there exists a number $N(B, y_0 - x)$ such that

$$\liminf_{n\to\infty} \langle w_n^*, x\rangle \geq N(B, y_0 - x) - M\,.$$

Letting $-x$ in place of x, there exists a number $N(B, y_0 + x)$ such that

$$\limsup_{n\to\infty} \langle w_n^*, x\rangle \leq -N(B, y_0 + x) + M\,.$$

Therefore, for every $x \in X$ the sequence $\{\langle w_n^*, x\rangle\}$ is bounded. By the uniform boundedness principle, it follows that $\{w_n^*\}$ is bounded. □

We give the following result for possibly unbounded generalized pseudomonotone perturbations.

THEOREM 4.3. *Let K be a nonempty, closed and convex subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone with $0 \in T(0)$. Assume that $S : X \to 2^{X^*}$ is finitely continuous generalized pseudomonotone which satisfies Condition (pm_4). Fix $f^* \in X^*$.*

(i) *If K is bounded, then the problem* VIP$(T + S, K, f^*)$ *is solvable in $D(T) \cap K$.*

(ii) *If K is unbounded and there exists a bounded open and convex subset G of X with $0 \in G$ such that the problem* VIP$(T + S, K \cap \overline{G}, f^*)$ *has no solution in $D(T) \cap K \cap \partial G$, then the problem* VIP$(T + S, K, f^*)$ *is solvable in $D(T) \cap K \cap G$.*

(iii) *Suppose that G is a bounded, open and convex subset of X with $0 \in G$ and there exists $u_0 \in \overline{G}$ such that*

$$\langle v^* + w^* - f^*, x - u_0\rangle > 0$$

for all $x \in D(T) \cap \partial G$, $v^* \in Tx$ and $w^* \in Sx$. Then the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T) \cap G$.

(iv) *Suppose that either K is bounded or the hypothesis in* (ii) *holds. If the inclusion*

$$\partial I_K(x) + Tx + Sx \ni f^*$$

has no solution in $D(T) \cap \partial K$, then the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T) \cap \mathring{K}$.

PROOF. (i) Let K be bounded. Since $T_t + S$ is finitely continuous, we follow the finite dimensional argument used in the proof of (i) of Theorem 3.1 to conclude that there exist $x_n \in K$, $w_n^* \in Sx_n$ and $v_n^* \in \partial I_K(x_n)$ such that

$$v_n^* + T_{t_n} x_n + w_n^* = f^* \tag{11}$$

for all n. Note that the above conclusion requires only the finite continuity of $T_t + S$ for each $t > 0$ and the generalized pseudomonotonicity of S. Since $0 \in T(0)$, it follows that $T_{t_n}(0) = 0$ for all n. Since $0 \in K$, we have $\langle v_n^*, x_n \rangle \geq 0$ for all n and

$$\limsup_{n\to\infty} \langle w_n^*, x_n \rangle \leq N ,$$

where N is an upper bound for the sequence $\{\|f^*\| \|x_n\|\}$. Applying Lemma 4.2 with $y_0 = 0$, we conclude that the sequence $\{w_n^*\}$ is bounded. Furthermore, we see that $\langle v_n^*, x_n \rangle \leq M$ where M is upper bound for the sequence $\{(\|w_n^*\| + \|f^*\|)\|x_n\|\}$. Since ∂I_K is strongly quasibounded, the sequence $\{v_n^*\}$ is bounded, and hence the sequence $\{T_{t_n} x_n\}$ is bounded. Assume there exist subsequences, denoted again by $\{x_n\}$, $\{w_n^*\}$ and $\{T_{t_n} x_n\}$, respectively, such that $x_n \rightharpoonup x_0 \in K$, $w_n^* \rightharpoonup w_0^*$, $v_n^* \rightharpoonup v_0^*$ and $T_{t_n} x_n \rightharpoonup z_0^*$ as $n \to \infty$. Since S is generalized pseudomonotone and ∂I_K is monotone, we have

$$\liminf_{n\to\infty} \langle w_n^*, x_n - x_0 \rangle \geq 0 \text{ and} \liminf_{n\to\infty} \langle v_n^*, x_n - x_0 \rangle \geq 0 .$$

Let J_{t_n} be the Yosida resolvent of T. We know that $J_{t_n} x_n \in D(T)$, $J_{t_n} x_n = x_n - t_n J^{-1}(T_{t_n} x_n)$ and $x_n - J_{t_n} x_n \to 0$ and $J_{t_n} x_n \rightharpoonup x_0$ as $n \to \infty$. From this we obtain

$$\limsup_{n\to\infty} \langle T_{t_n} x_n, J_{t_n} x_n - x_0 \rangle \leq 0 .$$

Using Lemma 1.4, we get $x_0 \in D(T)$, $v_0 \in T x_0$ and $\langle T_{t_n} x_n, x_n \rangle \to \langle z_0^*, x_0 \rangle$ as $n \to \infty$. On the other hand, we have

$$\limsup_{n\to\infty} \langle w_n^*, x_n - x_0 \rangle \leq 0 .$$

Since S is generalized pseudomonotone, $w_0^* \in Sx_0$ and $\langle w_n^*, x_n \rangle \to \langle w_0^*, x_0 \rangle$ as $n \to \infty$. Following a similar argument and Lemma 1.4, we see that the maximality of ∂I_K implies $v_0^* \in \partial I_K(x_0)$. Finally, taking the limit as $n \to \infty$ in (11), we obtain

$$v_0^* + z_0^* + w_0^* = f^* .$$

This shows that the problem VIP($T + S, K, f^*$) is solvable in $D(T) \cap K$.

(ii) Suppose (ii) holds. The conclusion follows via Lemma 2.2.

(iii) Using the hypothesis in (iii), we see that the problem VIP($T + S, \overline{G}, f^*$) has no solution in $D(T) \cap \partial G$. Then, by using (ii), X instead of K, we obtain that VIP($T + S, X, f^*$) is solvable in $D(T) \cap G$, i.e., there exists $x_0 \in D(T) \cap G$, $v_0^* \in T x_0$ and $w_0^* \in Sx_0$ such that $\langle u_0^* + w_0^* - f^*, x - x_0 \rangle \geq 0$ for all $x \in X$. Setting $x + x_0$ in place of x, we get $\langle u_0^* + w_0^* - f^*, x \rangle \geq 0$. Similarly, letting $-x + x_0$ in place of x, we obtain $\langle u_0^* + w_0^* - f^*, x \rangle \leq 0$. Combining these, we conclude that $u_0^* + w_0^* = f^*$.

(iv) Suppose the hypothesis in (iv) holds. Using either (i) or (ii), we see that VIP$(T + S, K, f^*)$ is solvable in $D(T) \cap K$, which is equivalent to the solvability of the inclusion

$$\partial I_K(x) + Tx + Sx \ni f^*$$

in $D(T) \cap K$. Since $\partial I_K(x) + Tx + Sx \ni f^*$ has no solution in $D(T) \cap \partial K$ and $\partial I_K(x) = \{0\}$ for all $x \in \mathring{K}$, we conclude that the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T) \cap \mathring{K}$. □

We also note that Le [28] proved (iii) of Theorem 4.3 for a bounded pseudomonotone operator S and $B_R(0)$ instead of a bounded, open and convex subset G. Since every bounded pseudomonotone operator trivially satisfies the Condition (pm_4), (iii) of Theorem 4.3 improves the result of Le [28]. Furthermore, Figueiredo [38] proved (iv) of Theorem 4.3 with $T = 0$, $K = \overline{B_R(0)}$, for some $R > 0$, S is pseudomonotone with $D(S) = X$ and λJ, for all $\lambda > 0$, instead of ∂I_K. Kenmochi [14] improved the result of Figueiredo [38], for a pseudomonotone mapping S with $D(S) = X$, by assuming a Leray-Schauder-type condition with ∂I_K in place of λJ for all $\lambda > 0$. Asfaw and Kartsatos [34] proved (iv) of Theorem 4.3 with S bounded and using $K = \overline{B_R(0)}$, λJ, $\lambda > 0$, instead of ∂I_K. For related results, the reader is also referred to Kartsatos and Quarcoo [39, Theorem 4] and Kartsatos and Skrypnik [40, Theorem 5.8].

We now give the following surjectivity result.

COROLLARY 4.4. *Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone with $0 \in D(T)$. Let $S : X \to 2^{X^*}$ be finitely continuous generalized pseudomonotone. Assume that S satisfies Condition (pm_4) and*

$$\inf_{w^* \in Sx,\ z^* \in Tx} \frac{\langle z^* + w^*, x\rangle}{\|x\|} \to \infty \ \text{as}\ \|x\| \to \infty\,.$$

Then $R(T + S) = X^*$.

PROOF. By the coercivity condition on $T + S$, there exists $R = R(f^*) > 0$ such that $\langle v^* + z^* + w^* - f^*, x\rangle > 0$ for all $x \in D(T) \cap \partial B_R(0)$, $v^* \in \partial I_{\overline{B_R(0)}}(x)$, $w^* \in Sx$ and $z^* \in Tx$. This says that the inclusion

$$\partial I_{\overline{B_R(0)}}(x) + Tx + Sx \ni f^*$$

has no solution in $D(T) \cap \partial B_R(0)$. Using Theorem 4.3, we conclude that $Tx + Sx \ni f^*$ is solvable in $D(T) \cap B_R(0)$. Since f^* is arbitrary, $T + S$ is surjective. □

We remark that Corollary 4.4 extends some results of Kenmochi [14] to unbounded generalized pseudomonotone perturbations of maximal monotone operators.

5. Variational inequalities for regular generalized pseudomonotone perturbations. In this section we give a result concerning the existence of a solution for a variational problem involving possibly unbounded regular generalized pseudomonotone perturbations of maximal monotone operators. We cite Browder and Hess [1] for properties and range results for single regular generalized pseudomonotone operators as well as their perturbations by maximal monotone operators. It is proved in [1, Theorem 4, p. 272] that a pseudomonotone operator

S with $D(S) = X$ is regular if there exists $k > 0$ satisfying the condition $\langle w^*, x\rangle \geq -k\|x\|$ for all $x \in X$ and $w^* \in Sx$. Browder and Hess [1, Theorem 8, p. 283] proved that the sum $T + S$ is regular generalized pseudomonotone provided that T is strongly quasibounded maximal monotone with $0 \in D(T)$ and S is regular generalized pseudomonotone with $D(S) = X$ satisfying $\langle w^*, x\rangle \geq -k\|x\|$ for all $x \in X$, $w^* \in Sx$ and some $k > 0$. A variational inequality result for single coercive regular generalized pseudomonotone operator may be found in Browder and Hess [1, Theorem 14, p. 288]. Kenmochi [14, Theorem 4.1, p. 254] studied the solvability of variational inequality problems of the type VIP(S, K, ϕ, f^*), where S is a multivalued pseudomonotone operator satisfying Condition (pm_4) and ϕ is proper, convex and lower semicontinuous, using coercivity-type assumptions involving S and ϕ.

THEOREM 5.1. *Let K be a nonempty, closed and convex subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone with $0 \in D(T)$ and $S : X \to 2^{X^*}$ regular generalized pseudomonotone satisfying Condition (pm_4). Let $\phi : X \to (\infty, \infty]$ be proper convex lower semicontinuous with $D(\phi) = K$. Assume, further, that there exists $u_0 \in D(T) \cap \mathring{K}$ such that*

$$\inf_{w^* \in Sx} \frac{\langle w^*, x - u_0\rangle}{\|x\|} \to \infty \text{ as } \|x\| \to \infty\,.$$

Then for every $f^* \in X^*$, the problem VIP($T + S, K, \phi, f^*$) *is solvable in $D(T) \cap K$. Furthermore, $Tx + Sx \ni f^*$ is solvable in $D(T)$ provided that $K = X$ and $\phi = 0$ on X.*

PROOF. Let $A = \partial\phi$. Using Barbu [9, Proposition 1.6, p. 9], we know that $\overline{D(A)} = K$ and $\mathring{K} \subseteq D(A)$. We first show that VIP($A + T + S, K, f^*$) is solvable in $D(A) \cap D(T)$, i.e., the inclusion $Ax + Tx + Sx \ni f^*$ is solvable in $D(A) \cap D(T)$. Since $0 \in \mathring{K} \subseteq D(A)$ and $0 \in D(T)$, we see that $0 \in \mathring{D(A)} \cap D(T)$. Hence, $B = A + T$ is a maximal monotone operator. Let B_t be the Yosida approximant of B for $t > 0$, and $\tilde{J}x = J(x - u_0)$, $x \in X$. Since the operator $B_t + \varepsilon\tilde{J}$ is smooth for all $t > 0$ and $\varepsilon > 0$ and S is regular, it follows that the operator $B_t + S + \varepsilon\tilde{J}$ is surjective for all $t > 0$ and $\varepsilon > 0$. Thus, for any $f^* \in X^*$ and every sequence $t_n \downarrow 0^+$ and $\varepsilon_n \downarrow 0^+$, there exist $x_n \in X$ and $w_n^* \in Sx_n$ such that

$$B_{t_n}x_n + w_n^* + \varepsilon_n\tilde{J}x_n = f^*. \tag{12}$$

Since $u_0 \in \mathring{K} \cap D(T) \subseteq D(B)$, we use the monotonicity of B and the fact that $\|B_{t_n}u_0\| \leq |Bu_0|$ for all n to arrive at

$$\langle w_n^*, x_n - u_0\rangle \leq \|f\|\|x_n\| + |Bu_0|\|x_n\| + (\|f\| + |Bu_0|)\|u_0\|$$

for all n, where $|Bu_0| = \inf\{\|x^*\|; x^* \in Bu_0\}$. The sequence $\{x_n\}$ is bounded. Otherwise, we get the contradiction

$$\lim_{\|x_n\| \to \infty} \frac{\langle w_n^*, x_n - u_0\rangle}{\|x_n\|} = \infty \leq \|f^*\| + |Bu_0|\,.$$

As a result, we have

$$\limsup_{n\to\infty} \langle w_n^*, x_n - u_0\rangle < +\infty\,.$$

Since $\tilde{J}$ is bounded, the sequence $\{\tilde{J}x_n\}$ is bounded. Since S satisfies Condition (pm_4), from Lemma 4.2, we conclude the boundedness of the sequence $\{w_n^*\}$. The boundedness of $\{B_{t_n}x_n\}$ follows from (12). Let $v_n^* = B_{t_n}x_n$ and assume that $x_n \rightharpoonup x_0$, $w_n^* \rightharpoonup w_0^*$ and $v_n^* \rightharpoonup v_0^*$ as $n \to \infty$. Using the operators $S = 0$ on X and B in place of T in Lemma 1.5, we conclude that $\liminf_{n\to\infty} \langle B_{t_n}x_n, x_n - x_0\rangle \geq 0$. Using this, the monotonicity of $\tilde{J}$ and (12), we get

$$\limsup_{n\to\infty} \langle w_n^*, x_n - x_0\rangle \leq 0\,.$$

Since S is generalized pseudomonotone, $w_0^* \in Sx_0$ and $\langle w_n^*, x_n\rangle \to \langle w_0^*, x_0\rangle$ as $n \to \infty$. Thus, we get

$$\limsup_{n\to\infty} \langle B_{t_n}x_n, x_n - x_0\rangle = 0\,.$$

Applying Lemma 1.5, it follows that $x_0 \in D(B) = D(T)\cap D(A) \subseteq D(T)\cap K$ and $v_0^* \in Bx_0$. Finally, taking the limit as $n \to \infty$ in (12), we get $v_0^* + w_0^* = f^*$. Thus, the problem VIP$(T + S, K, \phi, f^*)$ is solvable in $D(T) \cap K$. Furthermore, if $K = X$ and $\phi = 0$ on X, it is not hard to see that the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T)$. □

We mention here that Theorem 5.1 is a new variational inequality as well as range result for regular generalized pseudomonotone perturbations of maximal monotone operators.

For the sake of completeness, we give the proof of the following range result for the sum $T + S$ instead of a single regular generalized psuedomonotone operator S considered in Browder and Hess [1, Theorem 11, p. 285].

THEOREM 5.2. *Let K be a nonempty, closed, convex and bounded subset of X with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \to 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and $S : X \to 2^{X^*}$ be regular generalized pseudomonotone such that there exists a real number $k > 0$ satisfying $\langle w^*, x\rangle \geq -k\|x\|$ for all $x \in X$ and $w^* \in Sx$. Let $f^* \in X^*$ be fixed. Assume, further, that*

$$\partial I_K(x) + Tx + Sx \not\ni f^*$$

for all $x \in D(T) \cap \partial K$. Then the inclusion $Tx + Sx \ni f^*$ is solvable in $D(T) \cap K$.

PROOF. To complete the proof, it is sufficient to prove that the inclusion

$$\partial I_K(x) + Tx + Sx \ni f^*$$

is solvable in $D(T) \cap K$. To this end, we note that $D(\partial I_K) = K$ and $0 \in \mathring{K}$, and hence ∂I_K is strongly quasibounded maximal monotone operator. Furthermore, for each $t > 0$, it is not hard to show that $\partial I_K + T_t$ is strongly quasibounded maximal monotone. Using Browder and Hess [1, Theorem 8, p. 283], we conclude that $\partial I_K + T_t + S$ is regular generalized pseudomonotone with domain K, i.e., for each $t > 0$ and $\varepsilon > 0$, the operator $\partial I_K + T_t + S + \varepsilon J$ is surjective. As a result, for each $t_n \downarrow 0^+$ and $\varepsilon_n \downarrow 0^+$, there are $x_n \in K$, $v_n^* \in \partial I_K(x_n)$ and $w_n^* \in Sx_n$ such that

$$v_n^* + T_{t_n}x_n + w_n^* + \varepsilon_n Jx_n = f^* \tag{13}$$

for all n. Since K is bounded, the sequences $\{x_n\}$ and $\{\varepsilon_n J x_n\}$ are bounded. Using (13), we see that

$$\langle T_{t_n} x_n, x_n \rangle \leq (k + \|f^*\|)\|x_n\| \leq Q$$

for all n, where Q is an upper bound for the sequence $\{(k + \|f^*\|)\|x_n\|\}$. Since T is strongly quasibounded, by Lemma 1.6, we get the boundedness of the sequence $\{T_{t_n} x_n\}$. Using similar argument, it follows that the sequence $\{v_n^*\}$ is bounded because ∂I_K is strongly quasibounded maximal monotone with $0 \in \partial I_K(0)$. Finally, from (13), we get the boundedness of the sequence $\{w_n^*\}$. Assume that $x_n \rightharpoonup x_0 \in K$, $w_n^* \rightharpoonup w_0^*$ and $v_n^* + T_{t_n} x_n \rightharpoonup v_0^*$ as $n \to \infty$. Applying Lemma 1.5, we obtain that

$$\liminf_{n\to\infty} \langle v_n^* + T_{t_n} x_n, x_n - x_0 \rangle \geq 0\,. \tag{14}$$

As a consequence, using (13), we conclude that

$$\limsup_{n\to\infty} \langle w_n^*, x_n - x_0 \rangle \leq 0\,. \tag{15}$$

The generalized pseudomonotonicity of S gives that $w_0^* \in S x_0$ and $\langle w_n^*, x_n \rangle \to \langle w_0^*, x_0 \rangle$ as $n \to \infty$. Finally, combining (13) and (14), we conclude that

$$\lim_{n\to\infty} \langle v_n^* + T_{t_n} x_n, x_n - x_0 \rangle = 0\,.$$

Consequently, using Lemma 1.5, we obtain $x_0 \in D(T) \cap K$ and $v_0^* \in (\partial I_K + T)(x_0)$. In conclusion, letting $n \to \infty$ in (13), we have $v_0^* + w_0^* = f^*$, which implies the solvability of the inclusion

$$Tx + Sx + \partial I_K(x) \ni f^*$$

in $D(T) \cap K$. Since this inclusion has no solution in $D(T) \cap \partial K$ and $\partial I_K(x) = \{0\}$ for all $x \in \mathring{K}$, we conclude that $x_0 \in D(T) \cap \mathring{K}$ solves the inclusion $Tx + Sx \ni f^*$. □

We note that Theorem 5.2 is an extension of Browder and Hess [1, Theorem 11, p. 285] for the sum $T + S$ in place of S, and the fact that we have used a Leray-Schauder condition involving ∂I_K for any nonempty, closed, convex and bounded subset K of X instead of λJ for all $\lambda > 0$.

6. Possible applications. In order to demonstrate the applicability of our theory, we give below examples of maximal monotone and multivalued pseudomonotone operators.

Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary, p, p' such that $1 < p < \infty$ and $1/p + 1/p' = 1$, and $X = W_0^{1,p}(\Omega)$. For every $i = 1, 2, \ldots, N$, the function $a_i : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$ satisfies the following conditions.

(A1) $a_i(x, s, \xi)$ satisfies the Carathéodory conditions, i.e., it is measurable in $x \in \Omega$ for all $(s, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, and continuous in (s, ξ) a.a. with respect to $x \in \Omega$. Furthermore, there exist constants $c_0 > 0$ and $k_0 \in L^q(\Omega)$ such that

$$|a_i(x, s, \xi)| \leq k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1})\,,$$

a.a. for $x \in \Omega$, and for all $(s, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, where $|\xi|$ denotes the Euclidean norm of ξ in $\boldsymbol{R}^N$.

(A2) The functions a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi - \xi') > 0$$

for a.a. $x \in \Omega$, and all $(s, \xi),\ (s, \xi') \in \boldsymbol{R} \times \boldsymbol{R}^N$.

(A3) There exists $c_1 > 0$ and a function $k_1 \in L^1(\Omega)$ such that

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq c_1|\xi|^p - k_1(x)$$

for a.a. $x \in \Omega$ and all $(s, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$.

We consider a second-order quasilinear elliptic differential operator of the form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u(x)), \quad x \in \Omega, \ u \in X, \ \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right).$$

The operator A generates an operator $\tilde{A} : X \to X^*$ given by

$$\langle \tilde{A}u, \varphi \rangle = \int_{\Omega} \sum_{i=1}^{N} a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} dx, \quad u \in X, \ \varphi \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. It is well known that under the conditions (A1) through (A3) the operator $\tilde{A}$ is bounded, continuous and pseudomonotone.

For the function $j : \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ we assume the following conditions.

(J1) the function $x \to j(x, s)$ is measurable in Ω for all $s \in \boldsymbol{R}$, and $s \to j(x, s)$ is locally Lipschitz continuous a.a. $x \in \Omega$.

(J2) Let $\partial j(x, s)$ denote Clarke's generalized gradient of the function $s \to j(x, s)$ given by

$$\partial j(x, s) = \{\xi \in \boldsymbol{R};\ j^0(x, s; r) \geq \xi r, \text{ for all } r \in \boldsymbol{R}\}$$

for a.a. $x \in \Omega$, where $j^0(x, s; r)$ is the generalized directional derivative of the function $s \to j(x, s)$ at s in the direction r given by

$$j^0(x, s; r) = \limsup_{y \to s,\ t \downarrow 0} \frac{j(x, y + tr) - j(x, y)}{t}.$$

Furthermore, there exist $c > 0$, $q \in [p, p^*]$ and $k \in L^{q'}(\Omega)$ such that

$$\eta \in \partial j(x, s) : |\eta| \leq k(x) + c|s|^{q-1}$$

for a.a. $x \in \Omega$ and all $s \in \boldsymbol{R}$, where p^* denotes the critical Sobolev exponent with $p^* = Np/(N - p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$.

Let $\tilde{J} : L^q(\Omega) \to \boldsymbol{R}$ be defined by

$$\tilde{J}(u) = \int_{\Omega} j(x, u(x)) dx.$$

By $(J1)$ and $(J2)$, $\tilde{J}$ is well defined and Lipschitz continuous on bounded subsets of $L^q(\Omega)$. Moreover, Clarke's generalized gradient of $\tilde{J}$, $\partial\tilde{J} : L^q(\Omega) \to 2^{L^{q'}(\Omega)}$, is well defined and characterized by, for each $u \in L^q(\Omega)$,

$$\eta \in \partial\tilde{J}(u) \Rightarrow \eta \in L^{q'}(\Omega),\ \eta(x) \in \partial j(x, u(x)), \quad \text{for a.a. } x \in \Omega .$$

Let $i : X \hookrightarrow L^q(\Omega)$ be the natural embedding and $i^* : L^{q'}(\Omega) \hookrightarrow X^*$ the adjoint of i. Let $S : X \to 2^{X^*}$ be defined by

$$Su = (i^* \circ \partial\tilde{J} \circ i)(u),\ u \in X .$$

Carl and Motreanu [33, Lemma 3.1, p. 1109] showed that the operator S is bounded and pseudomonotone. By a result of Browder and Hess [1, Proposition 9, p. 267], the operator $\tilde{A} + S : X \to 2^{X^*}$ is also bounded and pseudomonotone. The theory developed in this paper may be applied in the solvability of variational inequalities as well as inclusion problems for operators of the type $T + \tilde{A} + S$, where $T : X \supseteq D(T) \to 2^{X^*}$ is an arbitrary maximal monotone operator, by using either inner product or Leray-Schauder conditions.

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