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A LIFTING FUNCTOR FOR TORIC SHEAVES

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Abstract. For a variety X which admits a Cox ring, we introduce a functor from the category of quasi-coherent sheaves on X to the category of graded modules over the homogeneous coordinate ring of X. We show that this functor is right adjoint to the sheafification functor and therefore left-exact. Moreover, we show that this functor preserves torsion-freeness and reflexivity. For the case of toric sheaves, we give a combinatorial characterization of its right derived functors in terms of certain right derived limit functors.

1. Introduction. Consider an affine normal variety $W = \mathrm{Spec}(S)$ over an algebraically closed field K, G a diagonalizable group scheme which acts on W, and $H \subseteq G$ a closed subgroup scheme. We denote T the quotient of diagonalizable group schemes G/H. Moreover, we assume the following.

- There exists a Zariski-open G-invariant subset $\hat{X}$ of W such that a good quotient $X = \hat{X}/\!/H$ exists. We denote $\pi : \hat{X} \to X$ the corresponding projection.
- X admits an affine T-invariant open covering (this is automatic if T is a torus, see [Sum74]).
- The complement $Z = W \setminus \hat{X}$ has codimension at least 2.

The actions of G and H on W induce gradings both on $\mathcal{O}_{\hat{X}}$ and S by the character groups $\mathcal{X}(G)$ and $\mathcal{X}(H)$, respectively, which are compatible via the surjection $\mathcal{X}(G) \twoheadrightarrow \mathcal{X}(H)$. In particular, $\mathcal{O}_{\hat{X}}$ decomposes as a direct sum of $\mathcal{O}_X$-modules $\mathcal{O}_{\hat{X}} \cong \bigoplus_{\alpha \in \mathcal{X}(H)} (\mathcal{O}_{\hat{X}})_\alpha$, where we can identify the structure sheaf $\mathcal{O}_X$ of X in a natural way with the zero component $(\mathcal{O}_{\hat{X}})_0$ or, equivalently, with the sheaf of rings of H-invariants $\mathcal{O}_{\hat{X}}^H$. With this, we require moreover the following.

- $\mathcal{X}(H) \cong A_{d-1}(X)$, the divisor class group, where $d = \dim X$.
- For every $\alpha \in \mathcal{X}(H)$ there exists for a suitable representative $D_\alpha \in A_{d-1}$ an isomorphism of $\mathcal{O}_X$-modules between $(\mathcal{O}_{\hat{X}})_\alpha$ and the divisorial sheaf $\mathcal{O}(D_\alpha)$ on X.
- These isomorphisms induce an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_{\hat{X}} \cong \bigoplus_{\alpha \in \mathcal{X}(H)} \mathcal{O}(D_\alpha)$. In particular, the latter carries an induced structure of a sheaf of $\mathcal{X}(H)$-graded rings.

These conditions essentially imply that X is a variety which admits a Cox ring (see Proposition 3.2), where we admit some possible further action by a diagonalizable group scheme on X. In particular, this class of varieties includes the Mori dream spaces. The main application we have in mind is the case where T is a torus and X a toric variety such that S is the

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associated homogeneous coordinate ring as defined in [Cox95]. It was shown in [PT10, §6] that by taking local invariants we obtain an exact and essentially surjective functor which maps an $\mathcal{X}(G)$-graded S-module E to a T-equivariant quasi-coherent sheaf $\widetilde{E}$ on X, the so-called *sheafification functor*. Conversely, there is a functor from the category of T-equivariant sheaves on X to the category of $\mathcal{X}(G)$-graded S-modules, mapping a quasi-coherent sheaf $\mathcal{E}$ to an $\mathcal{X}(G)$-graded S-module $\Gamma_*\mathcal{E} := \Gamma(\hat{X}, \pi^*\mathcal{E})$. This functor is right inverse to the sheafification functor, i.e., we have $\widetilde{\Gamma_*\mathcal{E}} \cong \mathcal{E}$ for any $\mathcal{E}$. However, the functor Γ_* in many cases is not very well behaved. So it usually does not preserve properties such as torsion-freeness and reflexivity. Also, by being the composition of the right-exact functor π^* with the left-exact global section functor, Γ_* does not have any exactness properties. In general, Γ_* is right-exact if $\hat{X} = W$ (and thus X is affine) and it is left-exact if π is a flat morphism.

The aim of this note is to construct an alternative functor to Γ_*, which we are going to call the *lifting functor*, which maps a quasi-coherent T-equivariant sheaf $\mathcal{E}$ to an $\mathcal{X}(G)$-graded S-module $\widehat{\mathcal{E}}$. We will show that the lifting functor has the following two general properties:

1. The lifting functor is right adjoint to the sheafification functor and therefore left-exact (Theorem 3.10).
2. Lifting preserves torsion-freeness and reflexivity. For torsion-free sheaves it preserves coherence (Theorem 4.4).

The lifting functor is an offspring of the author's recent work on toric sheaves [Per11]. Assume that X is a toric variety and $\hat{X}$ the standard quotient presentation as in [Cox95]. By results of Klyachko [Kly90], [Kly91], any coherent reflexive T-equivariant sheaf $\mathcal{E}$ can be described by a finite-dimensional vector space together with a family of filtrations parameterized by the rays of the fan associated to X. In order to represent $\mathcal{E}$ by an appropriate $\boldsymbol{Z}^n$-graded module over the homogeneous coordinate ring, it is a rather straightforward observation that, rather than taking $\Gamma_*\mathcal{E}$, we can choose a reflexive sheaf which is associated to precisely the same filtrations as $\mathcal{E}$ (this is possible because there is a one-to-one correspondence between the rays of the fans associated to X and $\hat{X}$, respectively). Our results show that this ad-hoc observation indeed has a functorial interpretation. In Section 5, we will see that the lifting functor has moreover a very nice interpretation in the combinatorial setting of [Per11].

2. Preliminaries.

2.1. Let A be any abelian group, S an A-graded K-algebra, and E an A-graded S-module. Then $E \cong \bigoplus_{\alpha \in A} E_\alpha$ and for any $\beta \in A$ we denote $E(\beta) = \bigoplus_{\alpha \in A} E_{\alpha+\beta}$ the *degree shift* of E by β.

2.2. For any two S-modules E and F, the tensor product $E \otimes_S F$ can be A-graded as follows. Consider first the K-vector space $E \otimes_K F$ and set $(E \otimes_K F)_\alpha = \bigoplus_{\beta \in A}(E_\beta \otimes_K F_{\alpha-\beta})$. Then for $\alpha \in A$ we form $(E \otimes_S F)_\alpha$ as the quotient of $(E \otimes_K F)_\alpha$ by the subvector space generated by $re \otimes f - e \otimes rf$ for $e \in E_\beta$, $f \in F_\gamma$, $r \in S_\delta$ with $\beta + \gamma + \delta = \alpha$. Note that $E(\alpha) \otimes_S F \cong E \otimes_S F(\alpha) \cong (E \otimes_S F)(\alpha)$.

2.3. For any A-graded S-modules E, F, the *graded* version of $\mathrm{Hom}_S(E, F)$ by definition is given by

$$\mathrm{HOM}_S^A(E, F) := \bigoplus_{\alpha \in A} \mathrm{Hom}_S^A(E, F(\alpha)),$$

where $\mathrm{Hom}_S^A(E, F(\alpha)) = \{f \in \mathrm{Hom}_S(E, F);\ f(E_\beta) \subseteq F_{\beta+\alpha}$ for every $\beta \in A\}$. We can consider in a natural way $\mathrm{HOM}_S^A(E, F)$ as a subset of $\mathrm{Hom}_S(E, F)$. Moreover, within the graded setting, HOM_S^A satisfies the same general functorial properties as the standard Hom (see [Nv04, §2]). Note that when we speak of the category of A-graded modules, the set of morphisms between modules E and F is given by $\mathrm{Hom}_S^A(E, F)$ and *not* by $\mathrm{HOM}_S^A(E, F)$.

2.4. We will deal with three gradings, given by the character groups $\mathcal{X}(T)$, $\mathcal{X}(G)$, and $\mathcal{X}(H)$, respectively. Any given $\mathcal{X}(G)$-graded ring S carries an $\mathcal{X}(H)$-grading as well via the surjection $\mathcal{X}(G) \twoheadrightarrow \mathcal{X}(H)$. To distinguish between these two gradings, we write the homogeneous components $S_{(\alpha)}$ and S_χ for the $\mathcal{X}(H)$- and the $\mathcal{X}(G)$-grading, respectively, where $\alpha \in \mathcal{X}(H)$ and $\chi \in \mathcal{X}(G)$. For $\chi \in \mathcal{X}(G)$ we may also write $S_{(\chi)}$ for the $\mathcal{X}(H)$-homogeneous component determined by the image of χ in $\mathcal{X}(H)$. Then $S_{(\chi)}$ has a natural $\mathcal{X}(T)$-grading which is given by $S_{(\chi)} \cong \bigoplus_{\eta \in \mathcal{X}(T)} (S_{(\chi)})_\eta$ with $(S_{(\chi)})_\eta = S_{\chi+\eta}$. We use the same conventions for $\mathcal{X}(G)$- and $\mathcal{X}(H)$-graded S-modules.

2.5. For any $\mathcal{X}(G)$-graded S-modules E, F, we have the two graded modules $\mathrm{HOM}_S^{\mathcal{X}(G)}(F, E)$ and $\mathrm{HOM}_S^{\mathcal{X}(H)}(F, E)$, together with the natural sequence of inclusions

$$\mathrm{HOM}_S^{\mathcal{X}(G)}(F, E) \subseteq \mathrm{HOM}_S^{\mathcal{X}(H)}(F, E) \subseteq \mathrm{Hom}_S(F, E)$$

(which even satisfy certain topological properties [Nv04, §2.4]).

2.6. The $\mathcal{X}(H)$-invariant subring $R = S_{(0)}$ is automatically $\mathcal{X}(T)$-graded. It is also $\mathcal{X}(G)$-graded by trivial extension, i.e., we set $R_\chi = 0$ for every $\chi \in \mathcal{X}(G) \setminus \mathcal{X}(T)$. Likewise, every $\mathcal{X}(T)$-graded R-module can be given an $\mathcal{X}(G)$-grading.

2.7. With the notation as in 2.3, note that we have $\mathrm{HOM}_S^A(F, E) = \bigoplus_{\alpha \in A} \mathrm{Hom}_S^A(F, E(\alpha)) = \bigoplus_{\alpha \in A} \mathrm{Hom}_S^A(F(-\alpha), E))$. That is, we identify the α-th graded component $\mathrm{HOM}_S^A(F, E)_\alpha$ with $\mathrm{Hom}_S^A(F(-\alpha), E)$. However, in order to avoid some cumbersome signs, we will usually write expressions like $\widehat{E} = \bigoplus_{\alpha \in A} \mathrm{Hom}_S^A(S(\alpha), E)$, where it is understood that the proper grading is given by $(\widehat{E})_\alpha = \mathrm{Hom}_S^A(S(-\alpha), E)$.

2.8. The sheafification functor as defined in [PT10] maps an $\mathcal{X}(H)$-graded (respectively $\mathcal{X}(G)$-graded) S-module E to a quasi-coherent sheaf $\widetilde{E}$ over X as follows. Let open affine covers $\{U_i = \mathrm{Spec}(R_i)\}_{i \in I}$ and $\{\hat{U}_i = \pi^{-1}(U_i) = \mathrm{Spec}(S_i)\}_{i \in I}$ on X and $\hat{X}$, respectively, be given, such that $U_i = \hat{U}_i /\!\!/ H$ (by our general assumptions, both covers can be chosen T- and G-invariant, respectively). Then $R_i = S_i^H = (S_i)_{(0)}$ for every $i \in I$ and we can associate to every U_i the R_i-module $\Gamma(\hat{U}_i, E)_{(0)}$, where by abuse of notation we identify E with its associated quasi-coherent sheaf over W. These glue naturally to give a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Moreover, if the U_i are chosen T-invariant, then the R_i are $\mathcal{X}(T)$-graded, and both the R_i and S_i are $\mathcal{X}(G)$-graded by 2.6. In this case, $\mathcal{E}$ has also an induced T-equivariant structure.

3. The right adjoint. For a given morphism of schemes $f : U \to V$ and a quasi-coherent sheaf $\mathcal{F}$ on V, it is standard to define the pullback $f^*\mathcal{F}$ as $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_V} \mathcal{O}_U$. This defines a right-exact functor from the category of (quasi-)coherent $\mathcal{O}_V$-modules to the category of (quasi-)coherent $\mathcal{O}_U$-modules. However, this is not the only conceivable way to define a pull-back functor; instead, one could consider the sheaf

$$f^{\wedge}\mathcal{F} := \mathcal{H}om_{f^{-1}\mathcal{O}_V}(\mathcal{O}_U, f^{-1}\mathcal{F}) .$$

Clearly, $f^{\wedge}$ is a left-exact functor from the category of quasi-coherent $\mathcal{O}_V$-modules to the category of quasi-coherent $\mathcal{O}_U$-modules. In the affine case, i.e., $U = \operatorname{Spec}(A)$, $V = \operatorname{Spec}(B)$ for some commutative rings A, B, and $\mathcal{F}$ the sheaf corresponding to a B-module F, $f^{\wedge}\mathcal{F}$ corresponds to the module $\operatorname{Hom}_B(A, F)$, where the A-module structure is given by $(rg)(r') = g(rr')$ for $r, r' \in A$ and $g \in \operatorname{Hom}_B(A, F)$.

However, the following example shows that $f^{\wedge}$ is not well-behaved as, for instance, in general it does not preserve finitely generatedness and torsion-freeness.

EXAMPLE 3.1. Assume $B = F = K$ and $A = K[x]$. Then we have isomorphisms of K-vector spaces

$$\operatorname{Hom}_K(K[x], K) \cong \operatorname{Hom}_K\left(\bigoplus_{i\geq 0} K, K\right) \cong \prod_{i\leq 0} \operatorname{Hom}_K(K, K) \cong \prod_{i\leq 0} K.$$

So we have created from a one-dimensional K-vector space a $K[x]$-module with an uncountable generating set. Also, the graded submodule $\bigoplus_{i\leq 0} \operatorname{Hom}_K(K[x]_{-i}, K)$ is the torsion submodule of $\operatorname{Hom}_K(K[x], K)$. To see this, denote $y^{-i} \in \operatorname{Hom}_K(K[x]_i, K)$ the basis element dual to x^i. Then it is easy to see that for any $j \geq 0$ and $i \leq 0$ we have $x^j y^i = y^{i+j}$ which becomes zero for $i + j > 0$. Moreover, note that this torsion module coincides with the injective hull of K as a $K[X]$-module (see [BH94, Corollary 3.6.7 & Proposition 3.6.16 (c)]).

These pathologies will be avoided by our general assumptions on X and $\hat{X}$. Let $\{U_i\}_{i\in I}$ and $\{\hat{U}_i = \pi^{-1}(U_i)\}_{i\in I}$ be affine T- and G-invariant covers, respectively, as in 2.8. By the general properties of good quotients, the $\hat{U}_i$ form an affine open covering of $\hat{X}$ such that $U_i = \hat{U}_i /\!/ H$ for every $i \in I$. We denote $U_i = \operatorname{Spec}(R_i)$ and $\hat{U}_i = \operatorname{Spec}(S_i)$; then $R_i = S_i^H$ for every $i \in I$. Moreover, both the R_i and S_i are $\mathcal{X}(G)$-graded by 2.6. Using the notation from the introduction, we observe the following properties of the homogeneous coordinate ring S.

PROPOSITION 3.2. (i) *There is an isomorphism of S_0-modules $S \cong \bigoplus_{\alpha\in X(H)} \Gamma(X, \mathcal{O}(D_\alpha))$ which is compatible with the $\mathcal{X}(H)$-grading of S. In particular, the latter carries an induced ring structure.*

(ii) *For any $i \in I$, the graded component $(S_i)_{(\alpha)}$ is isomorphic to $\Gamma(U_i, \mathcal{O}(D_\alpha))$.*

PROOF. (i) The $\mathcal{X}(H)$-graded isomorphism $\mathcal{O}_{\hat{X}} \to \bigoplus_{\alpha\in\mathcal{X}(H)} \mathcal{O}(D_\alpha)$ induces for every $i \in I$ and $\alpha \in \mathcal{X}(H)$ a natural isomorphism of R_i-modules $\Gamma(\hat{U}_i, (\mathcal{O}_{\hat{X}})_\alpha) \to \Gamma(U_i, \mathcal{O}(D_\alpha))$. By definition of divisorial sheaves, both $\Gamma(\hat{X}, (\mathcal{O}_{\hat{X}})_\alpha)$ and $\Gamma(X, \mathcal{O}(D_\alpha))$ can be constructed as intersection of the $\Gamma(\hat{U}_i, (\mathcal{O}_{\hat{X}})_\alpha)$ and $\Gamma(U_i, \mathcal{O}(D_\alpha))$, respectively, inside the fields of rational functions $K(\hat{X})$ and $K(X)$, respectively. As the isomorphisms naturally commute with restriction maps, they induce isomorphisms $S_{(\alpha)} = \Gamma(W, \mathcal{O}_W)_{(\alpha)} = \Gamma(\hat{X}, (\mathcal{O}_{\hat{X}})_{(\alpha)}) \to \Gamma(X, \mathcal{O}(D_\alpha))$ for every $\alpha \in \mathcal{X}(H)$ and hence we obtain the desired isomorphism (note that the second equality follows because $\hat{X}$ has codimension 2 in W).

(ii) Follows from the previous discussion and by remarking that $(S_i)_{(\alpha)} = \Gamma(\hat{U}_i, \mathcal{O}_{\hat{X}})_{(\alpha)}$. □

3.3. For any character (and thus divisor class of X) $\alpha \in \mathcal{X}(H)$, there is naturally associated the module $\mathcal{O}(\alpha) \cong \widetilde{S(\alpha)}$, which by Proposition 3.2 is reflexive and of rank one (see e.g. [CLS11, Proposition 8.0.4] for the correspondence between Weil divisors classes α and isomorphism classes of divisorial sheaves on X). This module is a distinguished representative for the isomorphism class of such sheaves associated to the class α. Similarly, if we choose some $\chi \in \mathcal{X}(G)$ which maps to α via the surjection $\mathcal{X}(G) \twoheadrightarrow \mathcal{X}(H)$, we obtain an induced T-equivariant structure on $\mathcal{O}(\alpha)$, which we denote by $\mathcal{O}(\chi) := \widetilde{S(\chi)}$. Moreover, we observe that the G-action on $\hat{X}$ induces a decomposition $\mathcal{O}_{\hat{X}} \cong \bigoplus_{\chi \in \mathcal{X}(G)} (\mathcal{O}_{\hat{X}})_\chi$ which is compatible with the $\mathcal{X}(H)$-grading via the surjection $\mathcal{X}(G) \twoheadrightarrow \mathcal{X}(H)$. Using the sheaves $\mathcal{O}(\chi)$, we can write this decomposition as $\mathcal{O}_{\hat{X}} \cong \bigoplus_{\chi \in \mathcal{X}(G)} \mathcal{O}(\chi)^T$. Also, for every $\alpha \in \mathcal{X}(H)$, we have a natural decomposition $\mathcal{O}(\alpha) \cong \bigoplus_\chi \mathcal{O}(\chi)^T$, where the direct sum runs over all χ which map to α. With this, the two gradings on S can be represented as $S \cong \bigoplus_{\alpha \in \mathcal{X}(H)} \Gamma(X, \mathcal{O}(\alpha)) \cong \bigoplus_{\chi \in \mathcal{X}(G)} \Gamma(X, \mathcal{O}(\chi)^T) \cong \bigoplus_{\alpha \in \mathcal{X}(G)} \Gamma(X, \mathcal{O}(\chi))^T$.

DEFINITION 3.4. Let $\mathcal{E}$ be a T-equivariant quasi-coherent sheaf on X. Then we set

$$\mathcal{E}_H := \bigoplus_{\alpha \in \mathcal{X}(H)} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E}),$$

and

$$\mathcal{E}_G := \bigoplus_{\chi \in \mathcal{X}(G)} \mathcal{H}om^T_{\mathcal{O}_X}(\mathcal{O}(\chi), \mathcal{E}).$$

Both $\mathcal{E}_H$ and $\mathcal{E}_G$ are graded sheaves where the graded pieces are given by $(\mathcal{E}_H)_\alpha = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(-\alpha), \mathcal{E})$ and $(\mathcal{E}_G)_\chi = \mathcal{H}om^T_{\mathcal{O}_X}(\mathcal{O}(-\chi), \mathcal{E})$, respectively (see 2.7).

PROPOSITION 3.5. *Let $\mathcal{E}$ be a T-equivariant quasi-coherent sheaf on X.*

(i) *Both $\mathcal{E}_H$ and $\mathcal{E}_G$ are quasi-coherent subsheaves of $\pi^*\mathcal{E}$, and $\mathcal{E}_H \cong \mathcal{E}_G$ as $\mathcal{O}_{\hat{X}}$-modules.*

(ii) *$(\mathcal{O}_X)_H$ (and therefore $(\mathcal{O}_X)_G$) is isomorphic to $\mathcal{O}_{\hat{X}}$.*

(iii) *If $\Gamma(U_i, \mathcal{E})$ is a first syzygy module for some i, then so is $\Gamma(\hat{U}_i, \mathcal{E}_H)$.*

(iv) *If $\mathcal{E}$ is coherent and torsion-free, then $\mathcal{E}_H$ and $\mathcal{E}_G$ are coherent and torsion-free as well.*

PROOF. (i) First note that for every $\chi \in \mathcal{X}(G)$ which maps to $\alpha \in \mathcal{X}(H)$, we have a natural inclusion of sheaves of K-vector spaces $\phi_\chi : \mathcal{H}om^T_{\mathcal{O}_X}(\mathcal{O}(\chi), \mathcal{E}) \hookrightarrow$

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E})$. Summing over all such characters, we get a map of sheaves

$$\phi_\alpha := \sum_{\eta \in \mathcal{X}(T)} \phi_{\chi+\eta} : \bigoplus_{\eta \in \mathcal{X}(T)} \mathcal{H}om^T_{\mathcal{O}_X}(\mathcal{O}(\chi + \eta), \mathcal{E}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E}).$$

Locally, we denote $E_i := \Gamma(U_i, \mathcal{E})$ for every U_i and this map translates to an isomorphism of R_i-modules $\bigoplus_{\eta \in \mathcal{X}(T)} \mathrm{Hom}^{\mathcal{X}(T)}_{R_i}((S_i)_{(\chi+\eta)}, E_i) \to \mathrm{HOM}^{\mathcal{X}(T)}_{R_i}((S_i)_{(\alpha)}, E_i)$. Because $(S_i)_{(\alpha)}$ is a finitely generated R_i-module by our general assumptions, the latter is isomorphic to $\mathrm{Hom}_{R_i}((S_i)_\alpha, E_i)$ (see [Nv04, Corollary 2.4.4]). So, ϕ_α is indeed an isomorphism and by summing over all $\chi \in \mathcal{X}(G)$, we get an isomorphism $\sum_{\chi \in \mathcal{X}(G)} \phi_\chi : \mathcal{E}_G \to \mathcal{E}_H$. Now, $\Gamma(\hat{U}_i, \mathcal{E}_H) \cong \bigoplus_{\alpha \in \mathcal{X}(H)} \mathrm{Hom}_{R_i}((S_i)_{(\alpha)}, E_i)$ and therefore $\mathcal{E}_H$ (and thus $\mathcal{E}_G$) is quasi-coherent. Moreover, observe that locally we have $\Gamma(\hat{U}_i, \pi^{\hat{}}\mathcal{E}) \cong \mathrm{Hom}_{R_i}(S_i, E) \cong \mathrm{Hom}_{R_i}(\bigoplus_{\alpha \in \mathcal{X}(H)} (S_i)_{(\alpha)}, E_i) \supseteq \bigoplus_{\alpha \in \mathcal{X}(H)} \mathrm{Hom}_{R_i}((S_i)_{(\alpha)}, E_i)$, so $\mathcal{E}_H$ (and thus $\mathcal{E}_G$) indeed is a subsheaf of $\pi^{\hat{}}\mathcal{E}$.

(ii) It suffices to show that for any i the module $\hat{R}_i := \bigoplus_{\alpha \in \mathcal{X}(H)} \mathrm{Hom}_{R_i}((S_i)_{(\alpha)}, R_i)$ is naturally isomorphic to S_i. For this, we have seen in Proposition 3.2 (ii) that $(S_i)_{(\alpha)} \cong \Gamma(U_i, \mathcal{O}(\alpha))$. So, by the fact that U_i is affine, this implies that $\mathrm{Hom}_{R_i}((S_i)_{(\alpha)}, R_i)$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{O}_{U_i}}(\mathcal{O}(\alpha)|_{U_i}, \mathcal{O}_{U_i})$ for any $\alpha \in \mathcal{X}(H)$. The well-known one-to-one correspondence between Weil divisors classes α and isomorphism classes of divisorial sheaves on X in particular entails that the $\mathcal{O}(\alpha)$ are reflexive sheaves of rank one (see e.g. [CLS11, Proposition 8.0.4]). Therefore, because U_i is affine, the global sections $(S_i)_{(\alpha)}$ correspond to reflexive R_i-modules of rank one. In particular, this correspondence is naturally compatible with dualizing, i.e., $\mathrm{Hom}_{R_i}((S_i)_{(\alpha)}, R_i) \cong (S_i)_{(-\alpha)}$ [CLS11, Proposition 8.0.6]. So we get $(\hat{R}_i)_{(\alpha)} \cong \mathrm{Hom}_{R_i}((S_i)_{(-\alpha)}, R_i)$ for every α and therefore we have natural graded isomorphisms $\hat{R}_i \cong \bigoplus_{\alpha \in \mathcal{X}(H)} (\hat{R}_i)_{(\alpha)} \cong \bigoplus_{\alpha \in \mathcal{X}(H)} (S_i)_{(\alpha)} \cong S_i$.

(iii) By assumption, we can represent E_i as a first syzygy $0 \to E_i \to R_i^{\oplus I}$, where I is some index set. Applying $\bigoplus_{\alpha \in \mathcal{X}(H)} \mathrm{Hom}_{R_i}((S_i)_{(\alpha)}, -)$ preserves left-exactness and direct sums in the right argument, and so we obtain an exact sequence $0 \to \hat{E}_i \to \hat{R}_i^{\oplus I} \cong S_i^{\oplus I}$, where $\hat{E}_i := \bigoplus_{\alpha \in \mathcal{X}(H)} \mathrm{Hom}_{R_i}((S_i)_{(\alpha)}, E_i) \cong \Gamma(\hat{U}_i, \mathcal{E}_H)$, and the latter isomorphism follows from (ii).

(iv) It suffices to show that for any i the module $\hat{E}_i := \bigoplus_{\alpha \in \mathcal{X}(H)} \mathrm{Hom}_{R_i}((S_i)_{(\alpha)}, E_i)$ is torsion-free. Because E_i is by assumption torsion-free and finitely generated, it can be represented as a first syzygy module $0 \to E_i \to R_i^{n_i}$ for some integer n_i. Applying (iii), we obtain an exact sequence $0 \to \hat{E}_i \to S_i^{n_i}$. Hence, $\hat{E}_i$ is finitely generated and torsion-free. □

We come now to our main definition.

DEFINITION 3.6. Let $\mathcal{E}$ be a T-equivariant quasi-coherent sheaf on X. Then we call the $\mathcal{X}(G)$-graded S-module

$$\widehat{\mathcal{E}} := \Gamma(\hat{X}, \mathcal{E}_G)$$

the *lifting* of $\mathcal{E}$.

REMARK 3.7. Note that the S-module $\widehat{\mathcal{E}}$ carries both an $\mathcal{X}(G)$-grading as well as an $\mathcal{X}(H)$-grading, which are given by

$$\widehat{\mathcal{E}} \cong \bigoplus_{\alpha \in \mathcal{X}(H)} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E}) \cong \bigoplus_{\chi \in \mathcal{X}(G)} \operatorname{Hom}^T_{\mathcal{O}_X}(\mathcal{O}(\chi), \mathcal{E})$$

(see 2.7 for our convention on the grading). In general, in the presence of a nontrivial T-action on X, it sometimes might be too restrictive or undesirable for technical reasons to consider only T-equivariant sheaves (such as in Section 4, see Remark 4.1). In such cases, one could consider a "coarsening" of the lifting functor, e.g. by assuming that $G = H$ (or by replacing G by any diagonalizable group sitting between H and G and thereby passing from the T-action to the action by some possibly non-trivial subgroup; we leave this generalization to the reader). Then the corresponding lifting procedure would result in a module, say, $\bar{\mathcal{E}} := \bigoplus_{\alpha \in \mathcal{X}(H)} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E})$. Apparently, $\bar{\mathcal{E}}$ then is defined for any quasicoherent sheaf on X. If $\mathcal{E}$ is T-equivariant, then above isomorphisms show that $\bar{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ are isomorphic as $X(H)$-graded modules, i.e., in this case, assuming $G = H$ is essentially equivalent to forgetting the T-equivariant structure.

Note morever that, by construction, the lifting is functorial and left-exact. Furthermore, if X is smooth, then every sheaf of the form $\mathcal{O}(\alpha)$ is invertible and we have natural isomorphisms $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}(-\alpha), \mathcal{E}) \cong \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}(\alpha))$ for every α (respectively $\operatorname{Hom}^T_{\mathcal{O}_X}(\mathcal{O}(-\chi), \mathcal{E}) \cong \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}(\chi))^T$ for every χ). In this case, our lifting functor is naturally equivalent to the usual lifting functor Γ_*.

PROPOSITION 3.8. *The sheafification functor is left-inverse to the lifting functor.*

PROOF. We show that $(\widehat{\mathcal{E}})\tilde{} \cong \mathcal{E}$ for any T-equivariant quasi-coherent sheaf on X. The corresponding statement about morphisms then will be evident. With notation as in the proof of Proposition 3.5, we have for every $i \in I$

$$\Gamma(U_i, (\widehat{\mathcal{E}})\tilde{}) = \operatorname{HOM}^{\mathcal{X}(G)}_{R_i}(S_i, E_i)_{(0)} = \operatorname{Hom}^{\mathcal{X}(G)}_{R_i}((S_i)_{(0)}, E_i) = \operatorname{Hom}^{\mathcal{X}(T)}_{R_i}(R_i, E_i) \cong E_i\,.$$

By naturality, the E_i glue to yield $\mathcal{E}$. □

Before we can prove our main result, we need to clarify how homomorphism spaces are related under going back and forth under lifting and sheafification.

LEMMA 3.9. (i) *For any $\mathcal{X}(G)$-graded S-module E, there exists a natural homomorphism of $\mathcal{X}(G)$-graded S-modules $E \to \widehat{\tilde{E}}$.*

(ii) *Let $\mathcal{E}, \mathcal{F}$ be T-equivariant quasi-coherent sheaves on X. Then the lifting induces a surjective homomorphism of K-vector spaces*

$$\operatorname{Hom}^{\mathcal{X}(G)}_S(\widehat{\mathcal{E}}, \widehat{\mathcal{F}}) \twoheadrightarrow \operatorname{Hom}^T_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})\,.$$

(iii) *Let E be an $\mathcal{X}(G)$-graded S-module and $\mathcal{F}$ a quasi-coherent sheaf on X. Then the sheafification and the lifting induce a surjective homomorphism of K-vector spaces*

$$\operatorname{Hom}^{\mathcal{X}(G)}_S(E, \widehat{\mathcal{F}}) \rightarrowtail \operatorname{Hom}^T_{\mathcal{O}_X}(\tilde{E}, \mathcal{F})\,.$$

PROOF. (i) Degree-wise we define a map $\phi_\chi : E_{(\chi)} \to \mathrm{Hom}_S^{\mathcal{X}(G)}(S_{(-\chi)}, E_{(0)})$ for $\chi \in \mathcal{X}(G)$ by setting $(\phi_\chi(e))(s) := s \cdot e$ for every $s \in S_{(-\chi)}$. We leave it to the reader to check that this indeed yields an $\mathcal{X}(G)$-homogeneous homomorphism of S-modules.

(ii) By revisiting the constructions of the proof of Proposition 3.8, we conclude that the functorially induced composition

$$\mathrm{Hom}_{\mathcal{O}_X}^T(\mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Hom}_S^{\mathcal{X}(G)}(\widehat{\mathcal{E}}, \widehat{\mathcal{F}}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}^T(\mathcal{E}, \mathcal{F})$$

is a natural isomorphism. In particular, the second homomorphism is surjective.

(iii) By (i), we obtain a homomorphism of S-modules $\mathrm{Hom}_S^{\mathcal{X}(G)}(\widehat{\tilde{E}}, \widehat{\mathcal{F}}) \to \mathrm{Hom}_S^{\mathcal{X}(G)}(E, \widehat{\mathcal{F}})$ which naturally commutes with the maps $\mathrm{Hom}_S^{\mathcal{X}(G)}(\widehat{\tilde{E}}, \widehat{\mathcal{F}}) \to \mathrm{Hom}_{\mathcal{O}_X}^T(\tilde{E}, \mathcal{F})$ and $\mathrm{Hom}_S^{\mathcal{X}(G)}(E, \widehat{\mathcal{F}}) \to \mathrm{Hom}_{\mathcal{O}_X}^T(\tilde{E}, \mathcal{F})$, respectively, which are induced by sheafification. By (ii), the first map is surjective, hence the second must be surjective, too. □

We can show our main results now, which in particular implies that lifting is left-exact.

THEOREM 3.10. *The lifting functor from the category of T-equivariant quasi-coherent sheaves on X to the category of $\mathcal{X}(G)$-graded S-modules is right adjoint to the sheafification functor.*

PROOF. We first consider the affine situation and assume that $\hat{X} = \mathrm{Spec}(S)$ and $X = \mathrm{Spec}(R) = \mathrm{Spec}(S_{(0)})$. Denote E an $\mathcal{X}(G)$-graded S-module and F an $\mathcal{X}(T)$-graded (and therefore $\mathcal{X}(G)$-graded, see 2.6) R-module. For simplicity, we write $\widehat{F}$ for the lifting of F. Then we have the isomorphisms of $\mathcal{X}(G)$-graded R-modules

$$\begin{aligned}\mathrm{HOM}_S^{\mathcal{X}(G)}(E, \widehat{F}) &= \mathrm{HOM}_S^{\mathcal{X}(G)}\left(E, \mathrm{HOM}_R^{\mathcal{X}(G)}(S, F)\right)\\ &\cong \mathrm{HOM}_R^{\mathcal{X}(G)}(E \otimes_S S, F) \cong \mathrm{HOM}_R^{\mathcal{X}(G)}(E, F)\,.\end{aligned}$$

Taking invariants with respect to the $\mathcal{X}(H)$-grading, we get

$$\mathrm{HOM}_S^{\mathcal{X}(G)}(E, \widehat{F})_{(0)} = \mathrm{HOM}_R^{\mathcal{X}(G)}(E, F)_{(0)} = \mathrm{Hom}_R^{\mathcal{X}(G)}(E_{(0)}, F) = \mathrm{Hom}_R^{\mathcal{X}(T)}(E_{(0)}, F)\,,$$

where the second equality follows form the fact that F is concentrated in $\mathcal{X}(H)$-degree zero.

For the general case, consider a T-equivariant sheaf $\mathcal{F}$ on X and an $\mathcal{X}(G)$-graded S-module E whose restriction to $\hat{X}$ corresponds to a G-equivariant quasi-coherent sheaf $\mathcal{E}$. As above, denote $\{U_i\}_{i\in I}$, $\{\hat{U}_i\}_{i\in I}$ a T-invariant (resp. G-invariant) affine cover of X (resp. $\hat{X}$). The affine case considered before corresponds to isomorphisms

$$\Gamma\big(\hat{U}_i, \mathcal{H}om^G_{\mathcal{O}_{\hat{U}_i}}(\mathcal{E}|_{\hat{U}_i}, \mathcal{F}_G|_{\hat{U}_i})\big) \to \Gamma\big(U_i, \mathcal{H}om^T_{\mathcal{O}_{U_i}}(\widetilde{E}|_{U_i}, \mathcal{F}|_{U_i})\big)$$

for every $i \in I$. These isomorphisms commute naturally with the restrictions

$$\Gamma(\hat{U}_i, \mathcal{H}om^G_{\mathcal{O}_{\hat{U}_i}}(\mathcal{E}|_{\hat{U}_i}, \mathcal{F}_G|_{\hat{U}_i})) \to \Gamma(\hat{U}_i \cap \hat{U}_j, \mathcal{H}om^G_{\mathcal{O}_{\hat{U}_i\cap\hat{U}_j}}(\mathcal{E}|_{\hat{U}_i\cap\hat{U}_j}, \mathcal{F}_G|_{\hat{U}_i\cap\hat{U}_j}))$$

and

$$\Gamma\big(U_i, \mathcal{H}om^T_{\mathcal{O}_{U_i}}(\widetilde{E}|_{U_i}, \mathcal{F}|_{U_i})\big) \to \Gamma\big(U_i \cap U_j, \mathcal{H}om^T_{\mathcal{O}_{U_i\cap U_j}}(\widetilde{E}|_{U_i\cap U_j}, \mathcal{F}|_{U_i\cap U_j})\big)\,,$$

respectively for $i, j \in I$. Therefore we obtain an induced homomorphism

$$\operatorname{Hom}^G_{\mathcal{O}_{\hat{X}}}(\mathcal{E}, \mathcal{F}_G) = \Gamma\big(\hat{X}, \mathcal{H}om^G_{\mathcal{O}_{\hat{X}}}(\mathcal{E}, \mathcal{F}_G)\big) \to \Gamma\big(X, \mathcal{H}om^T_{\mathcal{O}_X}(\widetilde{E}, \mathcal{F})\big) = \operatorname{Hom}^T_{\mathcal{O}_X}(\widetilde{E}, \mathcal{F})\,.$$

By the naturality of the local isomorphisms and the property that $\mathcal{H}om^G_{\mathcal{O}_{\hat{X}}}(\mathcal{E}, \mathcal{F}_G)$ is a sheaf, it follows that this homomorphism is an isomorphism. It remains to show that $\operatorname{Hom}^{\mathcal{X}(G)}_S(E, \widehat{\mathcal{F}}) = \operatorname{Hom}^G_{\mathcal{O}_W}(E, \widehat{\mathcal{F}})$ equals $\operatorname{Hom}^G_{\mathcal{O}_{\hat{X}}}(\mathcal{E}, \mathcal{F}_G)$. For this, consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}^G_{\mathcal{O}_W}(E, \widehat{\mathcal{F}}) \arrow[d, "\phi"'] \arrow[dr, "\psi"] & \\
\operatorname{Hom}^G_{\mathcal{O}_{\hat{X}}}(\mathcal{E}, \mathcal{F}_G) \arrow[r, "\cong"'] & \operatorname{Hom}^T_{\mathcal{O}_X}(\widetilde{E}, \mathcal{F})\,,
\end{tikzcd}$$

where ϕ is the restriction map and ψ the map induced by the sheafification functor. ϕ is injective because $\widehat{\mathcal{F}}$ is an extension of $\mathcal{F}_G$ from $\hat{X}$ to W and therefore does not have torsion with support on Z. Now, ψ is surjective by Lemma 3.9 (iii), hence both ϕ and ψ are isomorphisms. □

REMARK 3.11. From the proofs of Proposition 3.8 and Theorem 3.10, it follows that the counit of the adjunction is for every T-equivariant quasicoherent sheaf $\mathcal{E}$ given by the natural map $(\widehat{\mathcal{E}})^\sim \to \mathcal{E}$ which, using notation from the proof of Proposition 3.8, is locally given by the natural isomorphisms $\operatorname{Hom}^{\mathcal{X}(T)}_{R_i}(R_i, E_i) \xrightarrow{\equiv} E_i$. This is an interesting observation, as it implies that the category of T-equivariant sheaves on X is a reflective localization of the category of $\mathcal{X}(G)$-graded S-modules by the kernel of the sheafification functor. This was previously only known for the case where X is smooth. As was pointed out to me by Barakat and Lange-Hegermann, this is relevant for recent work [BL12] related to computational toric geometry.

4. Coherence.

We have seen in Proposition 3.5 that a T-equivariant torsion-free coherent sheaf $\mathcal{E}$ on X lifts to a torsion-free coherent sheaf $\mathcal{E}_G$ on $\hat{X}$. In this section we want to give similar and refined criteria for the lifting $\widehat{\mathcal{E}}$.

REMARK 4.1. Recall that by Proposition 3.5 and Remark 3.7 there is no loss of generality if we only consider the $X(H)$-graded structure of $\widehat{\mathcal{E}}$ in order to prove properties such as coherence and torsion-freeness. This has the additional technical advantage that in the proof of Theorem 4.4 we will need to find suitable open subsets of X, which might not necessarily exist if they had to be T-invariant.

PROPOSITION 4.2. *Let D be a Weil divisor on X and denote $\alpha \in \mathcal{X}(H) \cong A_{d-1}(X)$ the corresponding class. Then $\widehat{\mathcal{O}(D)} \cong S(\alpha)$. In particular, $\widehat{\mathcal{O}}_X \cong S$.*

PROOF. By the isomorphism $\mathcal{O}(D) \cong \mathcal{O}(\alpha)$, we have a decomposition as observed in Remark 3.7:

$$\widehat{\mathcal{O}(D)} \cong \bigoplus_{\beta \in \mathcal{X}(H)} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}(\beta), \mathcal{O}(\alpha)) \cong \bigoplus_{\beta \in \mathcal{X}(H)} \Gamma(\hat{X}, \mathcal{O}(\alpha - \beta)) \cong S(\alpha)\,.$$

□

4.3. By the general properties of good quotients, any open subset U of X can be represented as a good quotient $\hat{U} /\!/ H$, where $\hat{U}$ is the preimage of U in $\hat{X}$ under the quotient map. If $U = \mathrm{Spec}(R)$, then from the proof of Propositions 3.5 (ii) and 4.2, we can conclude that $\hat{U} = \mathrm{Spec}(\widehat{R})$ and $R = \widehat{R}_{(0)}$ with respect to the natural $\mathcal{X}(H)$-grading of $\widehat{R}$.

THEOREM 4.4. *Let $\mathcal{E}$ be a T-equivariant coherent sheaf on X.*
(i) *If $\mathcal{E}$ is torsion-free then $\widehat{\mathcal{E}}$ is torsion-free and finitely generated.*
(ii) *If $\mathcal{E}$ is reflexive then $\widehat{\mathcal{E}}$ is reflexive and finitely generated.*

PROOF. First we note that by the fact that Z has codimension 2 in W, coherence (as well as torsion-freeness and reflexivity, respectively, see [Har80, §1]) of $\mathcal{E}_H$ implies that the S-module $\widehat{\mathcal{E}}$ is finitely generated (and torsion-free, respectively reflexive). So, assertion (i) follows from Proposition 3.5 (iv).

Now we prove (ii). If $\mathcal{E}$ is reflexive, then by [Har80, Proposition 1.1], we can choose for every point in X a neighbourhood $U = \mathrm{Spec}(R)$ such that there exists a short exact sequence

$$0 \longrightarrow \Gamma(U, \mathcal{E}) \longrightarrow R^n \longrightarrow F \longrightarrow 0\,,$$

where F is a finitely generated, torsion-free R-module. By 4.3, we have $U \cong \hat{U} /\!/ H$ with $\hat{U} = \mathrm{Spec}(\widehat{R})$ and we can lift this sequence to

$$0 \longrightarrow \Gamma(\hat{U}, \mathcal{E}_H) \longrightarrow \widehat{R}^n \longrightarrow P \longrightarrow 0\,,$$

where P is the homomorphic image of S^n in $\widehat{F}$ and therefore torsion-free by Proposition 3.5 (iv). Applying again [Har80, Proposition 1.1], we conclude that $\mathcal{E}_G$ locally reflexive and therefore reflexive. Hence, as the complement of $\hat{X}$ in W has codimension at least two, the module $\widehat{\mathcal{E}}$ is reflexive by [Har80, Proposition 1.6]. □

We will see in Example 5.6 that, in general, coherence is not preserved for sheaves with torsion.

5. The case of toric sheaves. We now assume that X is a d-dimensional toric variety with associated fan Δ and $\hat{X} \subseteq \mathbf{A}_K^{\Delta(1)} = W$ its standard quotient presentation. As a general reference to toric geometry we refer to [CLS11]; for specifics of our setting see also [Per11].

5.1. It is customary to denote $M := \mathcal{X}(T) \cong \boldsymbol{Z}^d$ and $\hat{T} := G$, such that $\mathcal{X}(G) \cong \boldsymbol{Z}^{\Delta(1)}$. Moreover, we denote $N = M^*$ and assume that Δ consists of strictly convex polyhedral cones in $N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. We denote $\{l_\rho\}_{\rho \in \Delta(1)}$ the set of primitive vectors of the rays in Δ, which we interpret as linear forms on M. Elements $m \in M$ can be considered as regular functions on T and therefore as rational functions on X. In this case, we write $\chi(m)$, where

$\chi(m+m') = \chi(m)\chi(m')$ for any $m, m' \in M$. We have $\mathcal{X}(H) \cong A_{d-1}(X)$ and the inclusion of M in to $\boldsymbol{Z}^{\Delta(1)}$ yields the short exact sequence

$$0 \longrightarrow M \xrightarrow{L} \boldsymbol{Z}^{\Delta(1)} \longrightarrow A_{d-1}(X) \longrightarrow 0\,,$$

where L can be represented as a matrix whose rows are formed by the l_ρ. For any strictly convex rational polyhedral cone $\sigma \in \Delta$, we get an affine toric variety U_σ whose M-graded coordinate ring is given by $K[\sigma_M]$ with $\sigma_M = \check{\sigma} \cap M$ and $\check{\sigma}$ denotes the dual cone of σ in $M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Similarly, we get an exact sequence

$$M \xrightarrow{L_\sigma} \boldsymbol{Z}^{\sigma(1)} \longrightarrow A_{d-1}(U_\sigma) \longrightarrow 0\,,$$

where L_σ is the submatrix of L consisting of the rows which correspond to rays in $\sigma(1)$.

We start by recalling some facts about toric sheaves on affine toric varieties and poset representations from [Per04] and [Per11]. Assume that σ is a cone and $S = K[\boldsymbol{N}^{\sigma(1)}]$ the homogeneous coordinate ring. For any $m, m' \in M$ we write $m \le_\sigma m'$ if and only if $m' - m \in \sigma_M$. This way we get a preordered set $(M, \le_\sigma)$, which is partially ordered if $\dim \sigma = d$. Equivalently, M becomes a small category, where the morphisms are given by pairs (m, m') whenever $m \le_\sigma m'$. By the preorder $\le_\sigma$, M also becomes a topological space. Its topology is generated by open sets $U(m) = \{m' \in M; m \le_\sigma m'\}$ for every $m \in M$.

PROPOSITION 5.2 ([Per04, Proposition 5.5], [Per11, Proposition 2.5]). *The following categories are equivalent:*

(i) *Toric sheaves on U_σ.*
(ii) *M-graded $K[\sigma_M]$-modules.*
(iii) *Functors from $(M, \le_\sigma)$ to the category of K-vector spaces.*
(iv) *Sheaves of K-vector spaces on M.*

Note that, given a representation E of $(M, \le_\sigma)$, the associated sheaf assigns to any open subset U of M the limit $\varprojlim E_m$ for $m \in U$ (see [Per11, Proposition 2.5]).

Similarly, $\boldsymbol{N}^{\sigma(1)}$ induces a partial order "$\le$" on $\boldsymbol{Z}^{\sigma(1)}$, which is compatible with $\le_\sigma$ in the following way.

LEMMA 5.3. *$L_\sigma(m) \le L_\sigma(m')$ if and only if $m \le_\sigma m'$.*

PROOF. We observe

$$\begin{aligned} L_\sigma(m) \le L_\sigma(m') &\Leftrightarrow L_\sigma(m') - L_\sigma(m) \in \boldsymbol{N}^{\sigma(1)} \\ &\Leftrightarrow l_\rho(m'-m) \ge 0 \text{ for every } \rho \in \sigma(1) \\ &\Leftrightarrow m' - m \in \sigma_M\,. \end{aligned}$$

□

So, with respect to a fixed cone σ, it is natural to write $m \le m'$ instead of $L_\sigma(m) \le L_\sigma(m')$, i.e., $m \le m'$ if and only if $m \le_\sigma m'$. Moreover, for every $\underline{c} \in \boldsymbol{Z}^n$ there exists some $m \in M$ such that $\underline{c} \le m$. To see this, we observe that we always can choose some $m \in \sigma_M$ with $l_\rho(m) > 0$ for every $\rho \in \sigma(1)$ and some integer $r > 0$ such that $\underline{c} \le r \cdot m$. So, for every

$\underline{c} \in \boldsymbol{Z}^{\sigma(1)}$ we obtain a nonempty open subset $U_{\underline{c}}$ of M which is given as

$$U_{\underline{c}} = \bigcup_{\underline{c} \leq m} U(m)\,.$$

By Proposition 5.2, every M-graded module E is equivalent to a sheaf of K-vector spaces on M which assigns to every open subset U of M the vector space $E(U) = \varprojlim E_m$, where the limit is taken over $m \in U$. We use this to define a representation $\overline{E}$ of $(\boldsymbol{Z}^{\sigma(1)}, \leq)$ by setting

$$\overline{E}_{\underline{c}} := E(U_{\underline{c}})\,.$$

By the functoriality of sheaves we have restriction maps $\overline{E}_{\underline{c}} \to \overline{E}_{\underline{c}'}$ whenever $\underline{c} \leq \underline{c}'$. Hence we obtain a functor from $(\boldsymbol{Z}^{\sigma(1)}, \leq)$ to the category of K-vector spaces and thus a $\boldsymbol{Z}^{\sigma(1)}$-graded S-module $\overline{E} := \bigoplus_{\underline{c} \in \boldsymbol{Z}^{\sigma(1)}} \overline{E}_{\underline{c}}$ by Proposition 5.2. Clearly this construction is functorial.

PROPOSITION 5.4. *Denote $\widehat{E} \cong \bigoplus_{\underline{c} \in \boldsymbol{Z}^{\sigma(1)}} \widehat{E}_{\underline{c}}$ the $\boldsymbol{Z}^{\sigma(1)}$-graded lifting of the sheaf over U_σ associated to E in the sense of Definition* 3.6. *Then the modules $\widehat{E}$ and $\overline{E}$ are naturally isomorphic. In particular, $\widehat{E}_{\underline{c}} \cong \mathrm{Hom}^M_{K[\sigma_M]}(S_{(\underline{c})}, E)$ is naturally isomorphic to $\overline{E}_{\underline{c}}$ for every $\underline{c} \in \boldsymbol{Z}^{\sigma(1)}$.*

PROOF. We write $\underline{c} = \big(c_\rho;\, \rho \in \sigma(1)\big)$. We can consider $S_{(\underline{c})}$ as an M-graded $K[\sigma_M]$-submodule of the group ring $K[M]$ with $S_{(\underline{c})} \cong \bigoplus_m K\chi(m)$, where the sum is taken over all $m \in M$ with $l_\rho(m) \geq -c_\rho$. Choose a minimal set of generators $s_1, \ldots, s_t$ of $S_{(\underline{c})}$ with degrees $m_1, \ldots, m_t$. Then any M-homogeneous homomorphism is determined by the images of the s_i in the homogeneous components E_{m_i}. Hence, we can identify $\mathrm{Hom}^M_{K[\sigma_M]}(S_{(\underline{c})}, E)$ in a natural way with a subvector space of $\bigoplus_{i=1}^t E_{m_i}$ consisting of tuples $(e_1, \ldots, e_t)$ such that $\chi(m - m_i)e_i = \chi(m - m_j)e_j$ whenever $m_i, m_j \leq_\sigma m$. But this vector space has the universal properties of the limit $\varprojlim E_m$ and thus we can naturally identify it with $\widehat{E}_{\underline{c}} = \varprojlim E_m$. The isomorphism of the modules $\widehat{E}$ and $\overline{E}$ then follows from the naturality of this identification.

□

REMARK 5.5. By Theorem 3.10 the lifting functor is left-exact and to any toric sheaf $\mathcal{E}$ we can consider its right derived modules

$$\widehat{\mathcal{E}} = \widehat{\mathcal{E}}^{(0)}, \widehat{\mathcal{E}}^{(1)}, \ldots\,.$$

By Proposition 5.4 we have now a very nice interpretation of these modules, as we can identify them degree-wise with the right derived functors of the limit functor $\varprojlim$. Right derived limit functors $\varprojlim^i$ have been pioneered by Roos [Roo61] and have since been studied extensively. Roos also gives a combinatorial analog of the Cech complex which allows in simple cases the explicit computation of the derived functors. We can now understand the left-exactness of the lifting functor combinatorially by the fact that the posets $\{m \in M;\, l_\rho(m) \geq -c_\rho$ for all $\rho \in \sigma(1)\}$ are not filtered, i.e., for any $m, m' \in U_{\underline{c}}$ there may not exist any $m'' \in U_{\underline{c}}$ with $m'' \leq_\sigma m$ and $m'' \leq_\sigma m'$, which otherwise would imply the exactness of the limit functor (see [Jen72, Corollary 7.2]).

The following example shows both that lifting in general does not preserve exactness, and the existence of nontrivial right derived modules $\widehat{E}^{(i)}$.

EXAMPLE 5.6. Let $\sigma \subset N_{\boldsymbol{R}} \cong \boldsymbol{R}^3$ be the cone generated by the primitive vectors $l_1 = (1, 0, 0), l_2 = (0, 1, 0), l_3 = (-1, 1, 1), l_4 = (0, 0, 1)$. Denote $\mathfrak{m} \subset K[\sigma_M]$ the maximal homogeneous ideal and consider $K = K[\sigma_M]/\mathfrak{m}$ as a simple module in degree 0. Now for a given $\underline{c} = (c_1, c_2, c_3, c_4) \in \boldsymbol{Z}^4$, it is straightforward to see that $0 \in M$ is a minimal element in $\{m \in M; l_i(m) \geq -c_i\}$ if and only if

$$c_1, c_3 \leq 0\,, \quad c_2 = c_4 = 0 \quad \text{or} \quad c_1 = c_3 = 0\,, \quad c_2, c_4 \leq 0\,.$$

If $\underline{c}$ satisfies one of these conditions, then $\widehat{K}_{\underline{c}} \cong K$, and $\widehat{K}_{\underline{c}} = 0$ otherwise. So there is no lower bound for the c_i such that $\widehat{K}_{\underline{c}}$ vanishes and so $\widehat{K}$ cannot be finitely generated. We observe that $\widehat{K}$ is Artinian and is supported precisely on those torus orbits which get contracted to the fixed point under the quotient map $\mathbf{A}^4_K \to U_\sigma$.

Moreover, by Lemma 4.2, we have $\widehat{K[\sigma_M]} \cong S$, and the long exact derived sequence of $0 \to \mathfrak{m} \to K[\sigma_M] \to K \to 0$ starts by

$$0 \longrightarrow \widehat{\mathfrak{m}} \longrightarrow S \longrightarrow \widehat{K} \longrightarrow \widehat{\mathfrak{m}}^{(1)}\,.$$

By degree-wise inspection one can see that $\widehat{\mathfrak{m}} = (x_1, x_2, x_3, x_4)$, and therefore $\widehat{\mathfrak{m}}^{(1)}$ cannot be finitely generated as well.

By adjointness, the lifting functor transports injective M-graded $K[\sigma_M]$-modules to injective $\boldsymbol{Z}^{\sigma(1)}$-graded S-modules. In [Per11], codivisorial modules have been considered. For given $\underline{c} \in \boldsymbol{Z}^{\sigma(1)}$, such a module can be defined as $K[-M^{\underline{c},I}] = \bigoplus_{m \in -M^{\underline{c},I}} K\chi(m)$, where I is any subset of $\sigma(1)$ and $M^{\underline{c},I} = \{m \in M; l_\rho(m) \geq c_\rho \text{ for } \rho \in I\}$. If $\underline{c} = L_\sigma(m)$ for some $m \in M$, then $K[-M^{\underline{c},I}]$ is an injective object in M-$K[\sigma_M]$-Mod. However, if $K[-M^{\underline{c},I}]$ is not injective, the following example shows that lifting can exhibit a more bizarre behavior than in the previous example.

EXAMPLE 5.7. Let $K[\sigma_M]$ be as in Example 5.6 and consider the module $K[-M^{\underline{c},I}]$ with $\underline{c} = 0$ and $I = \{1, 3\}$. A similar computation as in Example 5.6 shows that $\widehat{K[-M^{\underline{c},I}]}_{(c_1,0,c_3,0)} \cong K^{1-c_1-c_3}$ whenever $c_1 + c_3 \leq 0$. So this module exhibits an infinite family of graded components of any finite dimension. This shows that the lifting functor does not respect combinatorial finiteness in the sense of [Per11].

5.8. Rather than limits, we can also consider colimits associated to representations of $(M, \leq_\sigma)$. That is, for any M-graded $K[\sigma_M]$-module E, there is its colimit $\varinjlim E_m$. As the preordered set $(M, \leq_\sigma)$ is filtered, forming the colimit is exact. Given an M-graded $K[\sigma_M]$-module $E \cong \bigoplus_{m \in M} E_m$, we can associate to it the colimit $\mathbf{E} := \varinjlim E_m$. Similarly, for the lifted S-module $\widehat{E}$ we have the colimit $\widehat{\mathbf{E}} := \varinjlim \widehat{E}_{\underline{c}}$, which is formed over the poset $(\boldsymbol{Z}^{\sigma(1)}, \leq)$.

PROPOSITION 5.9. *In the above situation we have* $\mathbf{E} = \widehat{\mathbf{E}}$.

PROOF. It is easy to see that for every $\underline{c} \in \mathbf{Z}^{\sigma(1)}$ we can find some $m \in M$ such that $\underline{c} \leq m$. Conversely, for every $m \in M$ we can find some $\underline{c} \in \mathbf{Z}^{\sigma(1)}$ such that $m \leq \underline{c}$. It follows that $\varinjlim E_m$ and $\varinjlim \widehat{E}_{\underline{c}}$ are cofinal. □

If $\dim \sigma < d$, then we have $m \leq_\sigma m'$ and $m' \leq_\sigma m$ whenever $m' - m \in \sigma_M^\perp$. In particular, such a pair (m, m') is an isomorphism in the *category* M. The following proposition states that, up to natural equivalence, we do not loose anything essential if we pass from the preordered set $(M, \leq_\sigma)$ to $M/\sigma_M^\perp$ with the induced partial order:

PROPOSITION 5.10 ([Per11, Proposition 2.8]). *Let $\Lambda \subseteq \sigma_M$ be a subgroup. Then there is an equivalence of categories between the category of M-graded $K[\sigma_M]$-modules and the category of M/Λ-graded $K[\sigma_M/\Lambda]$-modules.*

Note that we state Proposition 5.10 in slightly greater generality than [Per11].

5.11. Now, we are ready to consider the non-affine case. Denote $\{U_\sigma\}_{\sigma\in\Delta}$ the standard covering of X and $\{\hat{U}_\sigma = \operatorname{Spec}(S_\sigma)\}_{\sigma\in\Delta}$ the corresponding cover of $\hat{X}$ given by the preimages of the U_σ. If we take a T-equivariant, i.e., toric sheaf $\mathcal{E}$ on X, we see by Proposition 5.10 that the S_σ-modules $\Gamma(\hat{U}_\sigma, \mathcal{E}^{\hat{T}})$ are naturally equivalent to the lifts of $\Gamma(U_\sigma, \mathcal{E})$ to $K[\mathbf{N}^{\sigma(1)}]$. In particular, it is straightforward to check that coherence, torsion-freeness, and reflexivity are preserved by passing back and forth between $K[\mathbf{N}^{\sigma(1)}]$ and S_σ.

5.12. Given a quasi-coherent sheaf $\mathcal{E}$ on X, we obtain a family of colimits $\mathbf{E}^\sigma := \varinjlim \Gamma(U_\sigma, \mathcal{E})_m$ for $\sigma \in \Delta$. For every pair of cones τ, σ such that τ is a face of σ, the restriction $\Gamma(U_\sigma, \mathcal{E}) \to \Gamma(U_\tau, \mathcal{E})$ induces a map of directed families over $(M, \leq_\sigma)$ and $(m, \leq_\tau)$, respectively, and by the universal property of colimits we obtain an induced K-linear isomorphism $\mathbf{E}^\sigma \to \mathbf{E}^\tau$ (see [Per04, §5.4]). Since the face poset of Δ has the zero cone 0 as the unique minimal element, we can use the isomorphisms $\mathbf{E}^\sigma \to \mathbf{E}^0$ to identify the $\mathbf{E}^\sigma$ with $\mathbf{E}^0 =: \mathbf{E}$. For the case that $\mathcal{E}$ is coherent, it has been shown in [Per04, §5.4] that $\dim \mathbf{E}$ equals the rank of $\mathcal{E}$. We can do the same construction for $\widehat{\mathcal{E}}$ and obtain a colimit $\widehat{\mathbf{E}}$, which, using Proposition 5.9, we can in a natural way identify with $\mathbf{E}$.

5.13. This construction becomes most interesting for the case that $\mathcal{E}$ (and thus $\widehat{\mathcal{E}}$ by Theorem 4.4) is finitely generated and torsion-free. Then the maps $\Gamma(U_\sigma, \mathcal{E})_M \xrightarrow{\cdot\chi(m')} \Gamma(U_\sigma, \mathcal{E})_{m+m'}$ are injective for every $\sigma \in \Delta$, $m \in M$, and $m' \in \sigma_M$. It follows that the induced maps $\Gamma(U_\sigma, \mathcal{E})_m \to \mathbf{E}$ are injective as well for every $\sigma \in \Delta$ and $m \in M$, and analogously so for the induced maps $\widehat{\mathcal{E}}_{\underline{c}} \to \mathbf{E}$ for $\underline{c} \in \mathbf{Z}^{\Delta(1)}$. This allows a greatly condensed representation of torsion-free toric sheaves in terms of families of subvector spaces of a fixed vector space $\mathbf{E}$ which are parameterized by the family of posets $\{(M, \leq_\sigma)\}_{\sigma\in\Delta}$ (see [Per04, Theorem 5.18]).

For the case of reflexive sheaves, we have the following structural theorem due to Klyachko.

THEOREM 5.14 ([Kly90], [Kly91], see also [Per04]). *The category of coherent reflexive toric sheaves on a toric variety X is equivalent to the category of vector spaces $\mathbf{E}$ endowed with filtrations $0 \subseteq \cdots \subseteq E^\rho(i) \subseteq E^\rho(i+1) \subseteq \cdots \subseteq \mathbf{E}$ for $\rho \in \Delta(1)$ which are full in the sense that $E^\rho(i) = 0$ for $i \ll 0$ and $E^\rho(i) = \mathbf{E}$ for $i \gg 0$.*

5.15. Over U_σ, we observe that for a torsion-free $K[\sigma_M]$-module E we have $\varprojlim E_m$ equals the intersection $\bigcap_{m \le_\sigma m'} E_{m'}$ in $\mathbf{E}$. Therefore, given $\mathbf{E}$ and $E^\rho(i)$ for $\rho \in \sigma(1)$ as in Theorem 5.14, one constructs a reflexive module E from this data by setting $E = \bigoplus_{m \in M} E_m$ and $E_m = \bigcap_{\rho \in \sigma(1)} E^\rho\big(l_\rho(m)\big) \subseteq \mathbf{E}$.

By Theorem 4.4 we know that for a reflexive toric sheaf $\mathcal{E}$ on X, its lifting $\widehat{\mathcal{E}}$ is reflexive as well. The fan $\widehat{\Delta}$ associated to $\hat{X}$ in general contains more cones than Δ, but we have a one-to-one correspondence between $\Delta(1)$ and $\hat{\Delta}(1)$ given by, say, $\rho \mapsto \hat{\rho}$. So we know a priori that $\mathcal{E}$ and $\widehat{\mathcal{E}}$ are described by the same number of filtrations. The following result shows that these filtrations (in an almost tautological sense) indeed coincide and, moreover, that lifting is indeed "the" correct functor to translate reflexive toric sheaves into $\mathbf{Z}^{\Delta(1)}$-graded S-modules.

THEOREM 5.16. *A toric sheaf $\mathcal{E}$ is coherent and reflexive if and only if $\widehat{\mathcal{E}}$ is coherent and reflexive. Moreover, if $\mathcal{E}$ and $\widehat{\mathcal{E}}$ are coherent and reflexive, then they are canonically described by the same data, i.e., $\widehat{\mathbf{E}} = \mathbf{E}$ and $\widehat{E}^{\hat{\rho}}(i) = E^\rho(i)$ for any $\rho \in \Delta(1)$. In particular, lifting induces equivalences of categories between the category of reflexive toric sheaves on X, the category of reflexive toric sheaves on $\hat{X}$, and the category of reflexive $\mathbf{Z}^{\Delta(1)}$-graded S-modules.*

PROOF. The statements on coherence and reflexivity follow from Theorem 4.4. It suffices to consider the case that X is affine, i.e., $X = U_\sigma$. So, assume that E is a reflexive M-graded $K[\sigma_M]$-module, given by filtrations $E^\rho(i)$ of the vector space $\mathbf{E}$. From this data we can construct a reflexive $\mathbf{Z}^{\sigma(1)}$-graded S-module F by setting $\mathbf{F} = \mathbf{E}$ and $F^{\hat{\rho}}(i) = E^\rho(i)$. Similarly, if we start with the reflexive S-module F, we get a reflexive $K[\sigma_M]$-module E' by simply identifying the filtrations. We show that $F \cong \widehat{E}$ and $E' = F_{(0)} = E$.

The equality $E' = F_{(0)} = E$ follows from the fact that $E_m = \bigcap_{\rho \in \sigma(1)} E^\rho\big(l_\rho(m)\big) = \bigcap_{\rho \in \sigma(1)} F^{\hat{\rho}}\big(l_\rho(m)\big) = F_m$ (see 5.15), where in the latter equation we identify m with its image $L_\sigma(m) \in \mathbf{Z}^{\sigma(1)}$. Now consider $\widehat{E}_{\underline{c}}$ for some $\underline{c} \in \mathbf{Z}^n$. By 5.15 we have $\widehat{E}_{\underline{c}} = \varprojlim E_m = \bigcap_{\underline{c} \le m} E_m = \bigcap_{\underline{c} \le m} \bigcap_{\rho \in \Delta(1)} E^\rho\big(l_\rho(m)\big) \subseteq \mathbf{E}$. Now by the fact that the l_ρ are primitive elements in N, we can always choose for any $\rho \in \Delta(1)$ some $m \in M$ such that $l_\rho(m) = c_\rho$. It follows that $\widehat{E}_{\underline{c}} = \bigcap_{\rho \in \Delta(1)} E^\rho(c_\rho) = F_{\underline{c}}$.

For the equivalence of categories, it suffices to remark that for any two reflexive toric sheaves $\mathcal{E}, \mathcal{F}$, there is a natural bijection $\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \to \operatorname{Hom}(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$, as any homomorphism of vector spaces $\mathbf{E} \to \mathbf{F}$ which respects the filtrations also respects any of their intersections. $\square$

REMARK 5.17. For $\mathcal{E}$ reflexive, one can easily show that the S-module $\widehat{\mathcal{E}}$ is isomorphic to $(\Gamma_* \mathcal{E})^{\vee\vee}$, the reflexive hull of $\Gamma_* \mathcal{E}$. Note that more generally, if $\mathcal{E}$ is torsion-free, then $\widehat{\mathcal{E}}$ does not necessarily coincide with $\Gamma_* \mathcal{E}$ modulo torsion.

REMARK 5.18. In [Per11], reflexive M-graded $K[\sigma_M]$-modules have been investigated in terms of the vector space arrangements underlying the associated filtrations. Given such a module E, it is not difficult to see that in general not all possible intersections are realized as the graded components $\Gamma(U_\sigma, \mathcal{E})_m = \bigcap_{\rho \in \sigma(1)} E^\rho\big(l_\rho(m)\big)$. However, for the vector space arrangement underlying the filtrations associated to $\widehat{\mathcal{E}}$, all possible intersections indeed are realized this way. In this sense, one can consider vector space arrangement in $\mathbf{E}$ underlying the filtrations associated to $\widehat{E}$ as the intersection completion of the vector space arrangement underlying the filtrations associated to E.

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