

## EINSTEIN METRICS AND YAMABE INVARIANTS OF WEIGHTED PROJECTIVE SPACES

This article is dedicated to the memory of Friedrich Hirzebruch

JEFF A. VIACLOVSKY

(Received June 21, 2012, revised October 9, 2012)

**Abstract.** An orbifold version of the Hitchin-Thorpe inequality is used to prove that certain weighted projective spaces do not admit orbifold Einstein metrics. Also, several estimates for the orbifold Yamabe invariants of weighted projective spaces are proved.

**1. Introduction.** This article is concerned with certain orbifolds in dimension four with isolated singularities modeled on  $\mathbf{R}^4/\Gamma$ , where  $\Gamma$  is a finite subgroup of  $\mathrm{SO}(4)$  acting freely on  $\mathbf{R}^4 \setminus \{0\}$ . The examples considered are weighted projective spaces:

**DEFINITION 1.1.** For relatively prime integers  $1 \leq r \leq q \leq p$ , the *weighted projective space*  $\mathbf{CP}_{(r,q,p)}^2$  is  $S^5/S^1$ , where  $S^1$  acts by

$$(1.1) \quad (z_0, z_1, z_2) \mapsto (e^{ir\theta}z_0, e^{iq\theta}z_1, e^{ip\theta}z_2),$$

for  $0 \leq \theta < 2\pi$ .

The weighted projective space  $\mathbf{CP}_{(r,q,p)}^2$  has no singular points if and only if  $(r, q, p) = (1, 1, 1)$ . In general, the orbifold group at each singular point is a cyclic group, with action described below in Subsection 2.2.

A *Riemannian metric* on an orbifold is a smooth Riemannian metric away from the singular set, such that near any singular point the metric locally lifts to a smooth  $\Gamma$ -invariant metric on  $B^4$ .

**1.1. Einstein metrics.** The first result is the following non-existence theorem.

**THEOREM 1.2.** *If  $p > 1$ , then the weighted projective space  $\mathbf{CP}_{(r,q,p)}^2$  does not admit any Kähler-Einstein metric with respect to any complex structure. Furthermore, if*

$$(1.2) \quad p \geq (\sqrt{q} + \sqrt{r})^2,$$

*then the weighted projective space  $\mathbf{CP}_{(r,q,p)}^2$  does not admit any Einstein metric.*

---

2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 58J20.

*Key words and phrases.* Weighted projective spaces, Einstein metrics, Kähler-Einstein metrics, orbifold Yamabe invariants.

Research partially supported by NSF Grants DMS-0804042 and DMS-1105187.

REMARK 1.3. Assuming the complex structure is standard, non-existence of a Kähler-Einstein metric on weighted projective spaces for  $p > 1$  was shown in previous works [Mab87, GMSY07, RT11]. It is emphasized that the non-existence proof given in this paper does not make any assumptions about the complex structure.

Robert Bryant proved that every weighted projective space admits a Bochner-Kähler metric [Bry01], and subsequently, David and Gauduchon gave an alternative construction and showed that this metric is the unique Bochner-Kähler metric on a given weighted projective space [DG06, Appendix D]. Consequently, this metric will be called the *canonical* Bochner-Kähler metric. It is noted that this metric is the quotient of a Sasakian structure on  $S^5$  under the  $S^1$ -action, which implies that it is an orbifold Riemannian metric in the above sense.

Note that in real dimension four, Bochner-Kähler metrics are the same as self-dual Kähler metrics. Derdzinski [Der83] proved that for self-dual Kähler metric  $g$ , the conformal metric  $\tilde{g} = R_g^{-2}g$  is a self-dual Hermitian *Einstein* metric, away from the zero set of the scalar curvature  $R_g$ . This conformal metric is not Kähler unless  $R_g$  is a constant.

For a weighted projective space  $\mathbf{CP}_{(r,q,p)}^2$  with Bochner-Kähler metric  $g$ , the zero set of the scalar curvature is easily identified using [DG06, (2.32)], which yields the following 3 cases:

- If  $p < r + q$ , then  $R_g > 0$  everywhere, and  $\tilde{g}$  is a positive Einstein metric.
- If  $p = r + q$ , then  $R_g > 0$  except at one point, and  $\tilde{g}$  is Ricci-flat away from this point.
- If  $p > r + q$ ,  $R_g$  vanishes along a hypersurface and the complement consists of two open sets on which  $\tilde{g}$  has negative Einstein constant.

REMARK 1.4. In relation to Theorem 1.2,  $\tilde{g}$  is a global Einstein metric in the case  $p < r + q$ , but the author does not know if there exists an Einstein metric on  $\mathbf{CP}_{(r,q,p)}^2$  in the range  $r + q \leq p < (\sqrt{q} + \sqrt{r})^2$ ; this is a very interesting problem.

The main tools used in proving Theorem 1.2 are an orbifold version of the Hitchin-Thorpe inequality [Hit74, Tho69] and the triple reciprocity law for Dedekind sums of Rademacher [Rad54]. Similar computations for the signature were previously done by Hirzebruch and Zagier [HZ74, Zag72]. For another recent application of this reciprocity law, see [LV12].

The weighted projective space  $\mathbf{CP}_{(1,1,p)}^2$  is the one-point compactification of  $\mathcal{O}(-p)$ , the complex line bundle over  $\mathbf{CP}^1$ , which will be denoted by  $\widehat{\mathcal{O}(-p)}$  (noting that  $\mathcal{O}(-p)$  is diffeomorphic to  $\mathcal{O}(p)$ ). The above theorem in this special case is then simply as follows.

THEOREM 1.5. *If  $p \geq 4$  then  $\widehat{\mathcal{O}(-p)}$  does not admit any Einstein metric.*

The case  $p = 1$  is just  $\mathbf{CP}^2$  which of course admits an Einstein metric, the Fubini-Study metric. The author does not know if either  $\widehat{\mathcal{O}(-2)}$  or  $\widehat{\mathcal{O}(-3)}$  admits an Einstein metric. Exactly as above,  $\mathcal{O}(-2)$  does admit a complete Ricci-flat Einstein metric, the well-known

Eguchi-Hanson metric [EH79], but this does not yield an Einstein metric on the compactification  $\widehat{\mathcal{O}(-2)}$ .

**1.2. Orbifold Yamabe invariants.** The next results deal with orbifold Yamabe invariants (see [AB04] for background on the orbifold Yamabe problem). The conformal orbifold Yamabe invariant is defined by

$$(1.3) \quad Y_{\text{orb}}(M, [g]) = \inf_{\tilde{g} \in [g]} \text{Vol}(\tilde{g})^{-1/2} \int_M R_{\tilde{g}} dV_{\tilde{g}},$$

where  $[g]$  denotes the conformal class of  $g$ . The orbifold Yamabe invariant is then defined as

$$(1.4) \quad Y_{\text{orb}}(M) = \sup_{[g]} Y_{\text{orb}}(M, [g]),$$

where the supremum is taken over all conformal classes.

If  $M$  is a weighted projective space satisfying  $1 \leq r \leq q \leq p$ , then since  $p$  is the size of the largest orbifold group, any conformal class satisfies the estimate

$$(1.5) \quad Y_{\text{orb}}(M, [g]) \leq \frac{8\pi\sqrt{6}}{\sqrt{p}}.$$

This follows from [AB04, Corollary 2.10], and will be called the *elementary estimate* of Akutagawa-Botvinnik.

The main estimate for the orbifold Yamabe invariants of weighted projective spaces is the following:

**THEOREM 1.6.** *If  $M = \mathbf{CP}^2_{(r,q,p)}$ , then*

$$(1.6) \quad Y_{\text{orb}}(M) \leq 4\pi\sqrt{2} \frac{(r + q + p)}{\sqrt{rqp}},$$

and if

$$(1.7) \quad p < (\sqrt{r} + \sqrt{q})^2,$$

then the lower estimate

$$(1.8) \quad Y_{\text{orb}}(M) \geq 4\pi\sqrt{6} \sqrt{\frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr}}$$

is satisfied. Furthermore, if  $r + q \leq p < (\sqrt{r} + \sqrt{q})^2$  then strict inequality holds in (1.8).

The upper and lower estimates on the Yamabe invariant in Theorem 1.6 coincide only for  $(p, q, r) = (1, 1, 1)$ . In this case, the Fubini-Study metric is a *supreme* Einstein metric, using terminology of LeBrun [Leb99]. In the case  $p < q + r$ , the lower bound in (1.8) is in fact the Yamabe energy of the Einstein metric  $\tilde{g}$ . Interestingly, the upper bound in (1.6) turns out to be the Yamabe energy of the canonical Bochner-Kähler metric. However, for  $p > 1$ , this is *not* a Yamabe minimizer in its conformal class; it does not even have constant scalar curvature. The upper estimate in (1.6) is likely not sharp; except for the Fubini-Study metric, the upper bound in (1.6) is not attained by any conformal class:

THEOREM 1.7. *If  $M = \mathbf{CP}^2_{(r,q,p)}$  and  $p > 1$ , then any conformal class  $[g]$  satisfies*

$$(1.9) \quad Y_{\text{orb}}(M, [g]) < 4\pi\sqrt{2}\frac{(r+q+p)}{\sqrt{rqp}}.$$

Note that in case

$$(1.10) \quad 4\pi\sqrt{2}\frac{(r+q+p)}{\sqrt{rqp}} > \frac{8\pi\sqrt{6}}{\sqrt{p}},$$

Theorem 1.7 is trivial and follows from the elementary estimate (1.5). However, there are many cases when the upper bound in (1.9) is strictly smaller than the elementary estimate (see Theorem 1.8 below).

The proof of (1.8) follows more or less immediately from the Hitchin-Thorpe inequality on orbifolds used to prove Theorem 1.2. However, the proof of (1.6) is more subtle, and follows the idea of Gursky-LeBrun [GL98] adapted to orbifolds by Akutagawa-Botvinnik [AB04]. For convenience, a slightly simplified proof of this result is given in Section 3, which is also used to prove Theorem 1.7. In [AB04], the estimate (1.6) was applied to the example of  $\mathcal{O}(-p)$  (the case of  $\mathbf{CP}^2_{(1,1,p)}$ ), but the upper estimate (1.6) is not “effective” for  $p > 1$  since (1.6) is larger than the elementary estimate (1.5) in that case. So it is only interesting when the upper estimate given in (1.6) is strictly smaller than the elementary estimate (1.5). This turns out to hold for a large class of weighted projective spaces:

THEOREM 1.8. *Let  $M = \mathbf{CP}^2_{(r,q,p)}$ , with  $1 \leq r \leq q \leq p$ . If*

$$(1.11) \quad p < (2\sqrt{3} - 3)q + r \sim 0.464q + r,$$

then

$$(1.12) \quad 0 < 4\pi\sqrt{6}\sqrt{\frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr}} \\ \leq Y_{\text{orb}}(M) \leq 4\pi\sqrt{2}\frac{(r+q+p)}{\sqrt{rqp}} < \frac{8\pi\sqrt{6}}{\sqrt{p}}.$$

To conclude, it is remarked that only a few orbifold Yamabe invariants are known exactly. For example, in [Via10] it was shown that the orbifold conformal compactification of a hyperkähler ALE metric in dimension four has maximal orbifold Yamabe invariant. That argument also gives an exact determination of the orbifold Yamabe invariant in the “critical” case  $p = q + r$ :

THEOREM 1.9. *Let  $M = \mathbf{CP}^2_{(r,q,p)}$ , and let  $g$  be the canonical Bochner-Kähler metric. If  $p = q + r$ , then there is no constant scalar curvature metric in the conformal class of  $g$ , and*

$$(1.13) \quad Y_{\text{orb}}(M, [g]) = \frac{8\pi\sqrt{6}}{\sqrt{p}}.$$

Consequently,

$$(1.14) \quad Y_{\text{orb}}(M) = \frac{8\pi\sqrt{6}}{\sqrt{p}}.$$

The proof of this result is based on the Obata argument [Oba72], and is more or less the same as [Via10, Theorem 1.3], with a few minor modifications.

REMARK 1.10. The author does not know if the orbifold Yamabe problem has a solution if  $p > r + q$  on  $\mathbf{CP}^2_{(r,q,p)}$  in the conformal class of the canonical Bochner-Kähler metric  $g$ . However, *symmetric* solutions were ruled out in the case  $(1, 1, p)$  in [Via10, Theorem 1.4].

*Acknowledgments.* The author would like to thank Michael T. Lock for very useful discussions, and Xiaodong Wang for assistance with the argument in Theorem 3.1. The author would also like to thank the anonymous referee who made numerous helpful suggestions to improve the exposition of the paper.

**2. Einstein metrics.** Let  $(M, g)$  be a Riemannian orbifold with singular points  $x_i, i = 1, \dots, N$ . The Euler characteristic is given by

$$(2.1) \quad \chi(M) = \frac{1}{8\pi^2} \int_M \left( |W|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) dV_g + \sum_{i=1}^N \frac{|\Gamma_i| - 1}{|\Gamma_i|},$$

where  $E$  denotes the traceless Ricci tensor  $E = \text{Ric} - (R/4)g$ , and the signature is given by

$$(2.2) \quad \tau(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dV_g - \sum_{i=1}^N \eta(S^3/\Gamma_i),$$

where  $\Gamma_i \subset \text{SO}(4)$  is the orbifold group around the point  $p_i$ , and  $\eta(S^3/\Gamma_i)$  is the eta-invariant. See [Hit97, Nak90] for a discussion of the formulas (2.1) and (2.2).

**2.1. Cyclic group actions.** For  $1 \leq q < p$  relatively prime integers, denote by  $\Gamma_{(q,p)}$  the cyclic action

$$(2.3) \quad \begin{pmatrix} \exp^{2\pi ik/p} & 0 \\ 0 & \exp^{2\pi ikq/p} \end{pmatrix}, \quad 0 \leq k < p,$$

acting on  $\mathbf{R}^4 \simeq \mathbf{C}^2$ . The action  $\Gamma_{(q,p)}$  will be referred to as a type  $(q, p)$ -action. If  $\Gamma_i$  is conjugate to a  $\Gamma_{(q,p)}$  action in  $\text{SO}(4)$ , then

$$(2.4) \quad \eta(S^3/\Gamma_i) = 4s(q, p),$$

where

$$(2.5) \quad s(q, p) = \frac{1}{4p} \sum_{j=1}^{p-1} \left[ \cot\left(\frac{\pi}{p}j\right) \cot\left(\frac{\pi}{p}qj\right) \right]$$

is the well-known Dedekind sum [APS75].

**2.2. Weighted projective spaces.** For relatively prime integers  $a < b$ , let  $a^{-1:b}$  denote the inverse of  $a$  modulo  $b$ . On  $\mathbf{CP}^2_{(r,q,p)}$  there are three possible orbifold points:

- (1)  $[1, 0, 0]$  with a type  $(q^{-1:r}p, r)$ -action.
- (2)  $[0, 1, 0]$  with a type  $(p^{-1;q}r, q)$ -action.
- (3)  $[0, 0, 1]$  with a type  $(r^{-1;p}q, p)$ -action.

Consequently, on a weighted projective space, the Chern-Gauss-Bonnet formula is

$$(2.6) \quad \chi(M) = \frac{1}{8\pi^2} \int_M \left( |W|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) dV_g + \left[ \frac{|r|-1}{|r|} + \frac{|q|-1}{|q|} + \frac{|p|-1}{|p|} \right].$$

Since  $\chi(M) = 3$  (see [Kaw73]), this may be rewritten as

$$(2.7) \quad \frac{1}{8\pi^2} \int_M \left( |W|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) dV_g = \frac{1}{r} + \frac{1}{q} + \frac{1}{p}.$$

Next, on a weighted projective space, the Hirzebruch signature formula is

$$(2.8) \quad \tau(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dV_g - 4[s(q^{-1:r}p, r) + s(p^{-1;q}r, q) + s(r^{-1;p}q, p)].$$

Rademacher’s triple reciprocity for Dedekind sums [Rad54]

$$(2.9) \quad s(q^{-1:r}p, r) + s(p^{-1;q}r, q) + s(r^{-1;p}q, p) = -\frac{1}{4} + \frac{1}{12} \left( \frac{r}{pq} + \frac{q}{pr} + \frac{p}{qr} \right),$$

implies that

$$(2.10) \quad \tau(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dV_g + 1 - \frac{1}{3} \left( \frac{r}{pq} + \frac{q}{pr} + \frac{p}{qr} \right).$$

Since  $\tau(M) = 1$  (see [Kaw73]), this may be rewritten as

$$(2.11) \quad \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dV_g = \frac{1}{3} \left( \frac{r}{pq} + \frac{q}{pr} + \frac{p}{qr} \right).$$

The following argument to rule out Kähler-Einstein metrics for  $p > 1$  is an adaptation of the argument of [Der83, Lemma 3] to weighted projective spaces:

**THEOREM 2.1.** *Let  $M = \mathbf{CP}^2_{(r,q,p)}$ . Then  $M$  admits a Kähler-Einstein metric if and only if  $(r, q, p) = (1, 1, 1)$ .*

**PROOF.** Any Kähler metric satisfies

$$(2.12) \quad |W^+|^2 = \frac{R^2}{24}.$$

Consequently, the Gauss-Bonnet formula for any Kähler metric on  $M$  is

$$(2.13) \quad \frac{1}{8\pi^2} \int_M \left( 2|W^+|^2 + |W^-|^2 - \frac{1}{2}|E|^2 \right) dV_g = \frac{1}{r} + \frac{1}{q} + \frac{1}{p}.$$

Subtracting (2.13) from 3 times (2.11) yields

$$\begin{aligned}
 & -\frac{3}{8\pi^2} \int_M |W^-|^2 dV_g + \frac{1}{16\pi^2} \int_M |E|^2 dV_g \\
 (2.14) \quad & = \frac{r}{pq} + \frac{q}{pr} + \frac{p}{qr} - \frac{1}{r} - \frac{1}{q} - \frac{1}{p} \\
 & = \frac{1}{rqp} (r^2 + q^2 + p^2 - pq - pr - qr) \\
 & = \frac{1}{2rqp} ((p-r)^2 + (p-q)^2 + (q-r)^2).
 \end{aligned}$$

Consequently, if  $g$  is Kähler-Einstein, this gives a contradiction since the left-hand side is nonpositive and the right-hand side is strictly positive unless  $(p, q, r) = (1, 1, 1)$  in which case the Fubini-Study metric is a Kähler-Einstein metric.  $\square$

The following theorem is a generalization of the Hitchin-Thorpe inequality [Hit74, Tho69] to weighted projective spaces:

**THEOREM 2.2.** *If*

$$(2.15) \quad p \geq (\sqrt{q} + \sqrt{r})^2,$$

*then the weighted projective space  $\mathbf{CP}_{(r,q,p)}^2$  does not admit any Einstein metric.*

**PROOF.** Subtracting 3 times (2.11) from 2 times (2.7) yields

$$(2.16) \quad \frac{1}{4\pi^2} \int_M \left( 2|W^-|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) dV_g = \frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr},$$

for any metric  $g$ . Next, assume that  $g$  is an Einstein metric on  $M = \mathbf{CP}_{(r,q,p)}^2$ . Then (2.16) yields the inequality

$$(2.17) \quad \frac{r}{pq} + \frac{q}{pr} + \frac{p}{qr} \leq \frac{2}{r} + \frac{2}{q} + \frac{2}{p},$$

whereupon multiplication by  $pqr$  results in the inequality

$$(2.18) \quad r^2 + q^2 + p^2 \leq 2(pq + pr + qr),$$

which is rewritten as

$$(2.19) \quad p^2 - 2(q+r)p + (q-r)^2 \leq 0.$$

For fixed  $q$  and  $r$ , consider the left-hand side of the above equation as a quadratic polynomial in  $p$ . By the quadratic formula, the roots are

$$(2.20) \quad p_{\pm} = q + r \pm 2\sqrt{qr},$$

Clearly then, the inequality in (2.19) is satisfied if

$$(2.21) \quad p_- = (\sqrt{q} - \sqrt{r})^2 \leq p \leq (\sqrt{q} + \sqrt{r})^2 = p_+.$$

Since  $1 \leq r \leq q \leq p$ , it follows that

$$(2.22) \quad p_- = q + r - 2\sqrt{qr} \leq q + r - 2r = q - r < q,$$

so the lower inequality is already satisfied. Consequently, the only requirement is that

$$(2.23) \quad p \leq (\sqrt{q} + \sqrt{r})^2 = p_+.$$

In the case of equality  $p = p_+$ , from (2.16), the metric must be Ricci-flat and self-dual, so the bundle  $\Lambda_-^2$  is flat. Since  $\mathbf{CP}_{(r,q,p)}^2$  is simply connected,  $\Lambda_-^2$  must be trivial, and the holonomy reduces to  $SU(2)$ . The metric is therefore Kähler with zero Ricci tensor, which contradicts Theorem 2.1.  $\square$

Theorem 1.2 immediately follows from Theorems 2.1 and 2.2.

**3. Orbifold Yamabe invariants.** The following Proposition is a restatement of Theorem 1.7, and immediately implies the upper estimate on the orbifold Yamabe invariant in Theorem 1.6. The proof is based on the idea of Gursky-LeBrun [GL98] adapted to orbifolds by Akutagawa-Botvinnik [AB04].

PROPOSITION 3.1. *If  $g$  is any Riemannian metric on  $M = \mathbf{CP}_{(r,q,p)}^2$ , then*

$$(3.1) \quad Y_{\text{orb}}(M, [g]) \leq 4\pi\sqrt{2}\frac{(r+q+p)}{\sqrt{rqp}}.$$

Furthermore, if  $p > 1$ , then strict inequality holds in (3.1).

PROOF. First, one may assume that  $g$  has positive scalar curvature. Let  $L$  be the  $\text{Spin}^c$  structure associated to the almost complex structure  $J$  on  $M$ , and let  $D$  denote the Dirac operator:

$$(3.2) \quad D : \Gamma(S^+) \rightarrow \Gamma(S^+).$$

From [Fuk05, Theorem 2], it follows that  $\text{Ind}(D) = 1$ . Therefore, there exists a positive harmonic spinor  $\psi \neq 0$ . By the Lichnerowicz-Bochner formula,

$$(3.3) \quad \nabla^*\nabla\psi + \frac{R}{4}\psi + \frac{1}{2}F^+ \cdot \psi = 0,$$

where  $F$  is the curvature form of the line bundle and one chooses the connection such that  $F$  is a harmonic 2-form. Pairing this with  $\psi$  and using the Kato inequality

$$(3.4) \quad |\nabla\psi|^2 \geq \frac{4}{3}|\nabla|\psi||^2,$$

yield

$$(3.5) \quad \frac{1}{2}\Delta|\psi|^2 \geq \frac{4}{3}|\nabla|\psi||^2 + \frac{R}{4}|\psi|^2 + \frac{1}{2}\langle F^+ \cdot \psi, \psi \rangle.$$

It follows from the Cauchy-Schwarz inequality  $|\langle F^+ \cdot \psi, \psi \rangle| \leq \sqrt{2}|F^+||\psi|^2$  that

$$(3.6) \quad |\psi|\Delta|\psi| \geq \frac{1}{3}|\nabla|\psi||^2 + \frac{R}{4}|\psi|^2 - \frac{\sqrt{2}}{2}|F^+||\psi|^2.$$

Letting  $u = |\psi|^{2/3}$ , it follows that

$$(3.7) \quad -\Delta u + \frac{R}{6}u \leq \frac{\sqrt{2}}{3}|F^+|u.$$



Multiplying by  $6 \cdot u$  and integrating by parts,

$$(3.8) \quad \frac{\int (6|\nabla u|^2 + Ru^2)dV}{(\int u^4 dV)^{1/2}} \leq 2\sqrt{2} \left( \int |F^+|^2 dV \right)^{1/2}.$$

Since  $b_-^2 = 0$ ,  $c_1(L) = \frac{\sqrt{-1}}{2\pi} F$ , which yields

$$(3.9) \quad Y_{\text{orb}}(M, [g]) \leq 4\pi\sqrt{2} \left( \int_M c_1(L)^2 \right)^{1/2}.$$

Since  $L$  is the anti-canonical bundle, the first Chern class satisfies  $c_1(L) = c_1(M)$ , and from elementary complex geometry,  $p_1(M) = c_1(M)^2 - 2c_2(M)$ . By Chern-Weil theory and (2.7) and (2.11) above,

$$\begin{aligned} \int_M c_1(L)^2 &= \frac{3}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dV_g + \frac{2}{8\pi^2} \int_M \left( |W|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) dV_g \\ &= \frac{r}{pq} + \frac{q}{pr} + \frac{p}{qr} + \frac{2}{r} + \frac{2}{q} + \frac{2}{p} = \frac{(r+p+q)^2}{rpq}, \end{aligned}$$

and (3.1) follows.

If equality held in (3.1), then the function  $u$  in the above argument must be a minimizer of the Yamabe energy, so it satisfies the elliptic PDE  $-6\Delta u + Ru = cu^3$  where  $c > 0$  is a positive constant. By elliptic regularity and the Harnack inequality,  $u$  is a smooth positive function. The metric  $g' = u^2g$  has constant scalar curvature and  $\psi' = u^{-3/2}\psi = \psi/|\psi|$  is a  $g'$ -harmonic spinor [LM89, Theorem 5.24]. Replacing  $g$  and  $\psi$  by  $g'$  and  $\psi'$  in the above proof, one may then assume that  $\psi$  is a unit spinor and  $g$  has constant scalar curvature. In the above argument, all the inequalities used must be equalities. In particular  $|F^+| = (\sqrt{2}/4)R$  and  $\langle F^+ \cdot \psi, \psi \rangle = -\sqrt{2}|F^+|$ . Therefore  $F^+ \cdot \psi = -\sqrt{2}|F^+|\psi$ . The equation (3.3) then implies that  $\psi$  is parallel, which implies that  $g$  is Kähler [Mor97, Theorem 1.1].

Adding 2 times (2.7) with 3 times (2.11) yields

$$(3.10) \quad \frac{1}{4\pi^2} \int_M \left( 2|W^+|^2 - \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) dV_g = \frac{(r+q+p)^2}{rqp}.$$

Since  $g$  is Kähler, using (2.12), it follows that

$$(3.11) \quad \frac{1}{32\pi^2} \int_M R^2 dV_g = \frac{(r+q+p)^2}{rqp} + \frac{1}{8\pi^2} \int_M |E|^2 dV_g.$$

This implies that  $g$  is also Einstein, since  $g$  attains the maximal value of the Yamabe energy in (3.1). Thus  $g$  is Kähler-Einstein, and this contradicts Theorem 2.1, unless  $p = 1$ .  $\square$

The next lemma will be used in both the proofs of Theorems 1.6 and 1.9.

LEMMA 3.2. *Let  $g$  be the canonical Bochner-Kähler metric on  $\mathbf{CP}_{(r,q,p)}^2$ . If  $p \geq r+q$  then there is no Einstein metric in the conformal class of  $g$ .*

PROOF. To begin, it is shown in [DG06, (2.32)] that with the scaling so that

$$(3.12) \quad \text{Vol}(g) = \frac{\pi^2}{2} \frac{1}{pqr},$$

the scalar curvature of  $g$  is given by

$$(3.13) \quad R_g = 24(r(-r + q + p)|u_1|^2 + q(r - q + p)|u_2|^2 + p(r + q - p)|u_3|^2),$$

where  $(u_1, u_2, u_3)$  are coordinates on the Sasakian sphere  $S^5 \subset \mathbb{C}^3$ . Consequently, in the case  $p = q + r$ ,

$$(3.14) \quad R_g = 48rq(|u_1|^2 + |u_2|^2),$$

which is positive except at the single point  $[0, 0, 1]$  (the orbifold point with group of order  $p$ ). The metric  $\tilde{g} = R_g^{-2}g$  is Ricci-flat.

Since there are two Einstein metrics in the conformal class, the complete manifold  $(M \setminus [0, 0, 1], \tilde{g})$  admits a nonconstant solution of the equation

$$(3.15) \quad \nabla^2 \phi = \frac{\Delta \phi}{m} \tilde{g},$$

which is called a concircular scalar field, and complete manifolds which admit a non-zero solution were classified by Tashiro [Tas65] (see also [Küh88]), who showed that  $(X, g)$  must be conformal to one of the following:

- (A) A direct product  $V \times J$ , where  $V$  is an  $(m - 1)$ -dimensional complete Riemannian manifold and  $J$  is an interval,
- (B) Hyperbolic space  $\mathcal{H}^m$ ,
- (C) the round sphere  $S^m$ .

If  $M \setminus [0, 0, 1]$  were diffeomorphic to a product, then any element in  $H_2(M)$  would have zero self-intersection. However, from the determination of the cohomology ring of weighted projective spaces in [Kaw73], this cannot happen, so case (A) is ruled out. Cases (B) and (C) cannot happen since  $g$  is obviously not locally conformally flat. This is a contradiction, and the nonexistence is proved.

In the case  $p > q + r$ , from (3.13), the scalar curvature vanishes along a hypersurface which divides  $M$  into two components  $U_+$  and  $U_-$ , with  $U_-$  containing the orbifold point  $[0, 0, 1]$  and  $U_+$  containing the other two orbifold points  $[1, 0, 0]$  and  $[0, 1, 0]$ . On  $U_{\pm}$ , the metric  $\tilde{g} = R_g^{-2}g$  is complete Einstein with negative Einstein constant. If there were an Einstein metric in the conformal class of  $g$ , then  $U_{\pm}$  would admit a concircular scalar field, and the same argument above rules out this possibility.  $\square$

Next, the lower estimate in Theorem 1.6 is given by the following.

PROPOSITION 3.3. *Let  $g$  be the canonical Bochner-Kähler metric on  $\mathbb{C}P_{(r,q,p)}^2$ . If*

$$(3.16) \quad p < (\sqrt{r} + \sqrt{q})^2,$$

then

$$(3.17) \quad Y_{\text{orb}}(\mathbf{CP}^2_{(r,q,p)}, [g]) \geq 4\pi\sqrt{6}\sqrt{\frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr}}.$$

Furthermore, if  $r + q \leq p < (\sqrt{r} + \sqrt{q})^2$  then strict inequality holds in (3.17).

If  $p < r + q$  then

$$(3.18) \quad Y_{\text{orb}}(\mathbf{CP}^2_{(r,q,p)}, [g]) = 4\pi\sqrt{6}\sqrt{\frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr}}.$$

PROOF. Since  $W^-(g) = 0$ , any metric  $\hat{g}$  conformal to  $g$  also satisfies  $W^-(\hat{g}) = 0$ . Formula (2.16) above becomes

$$(3.19) \quad \frac{1}{4\pi^2} \int_M \left( -\frac{1}{2}|E|^2 + \frac{1}{24}R^2 \right) dV_{\hat{g}} = \frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr},$$

for any metric  $\hat{g}$  in the conformal class of  $g$ . If  $p < r + q$ , the Bochner-Kähler metric  $g$  on  $\mathbf{CP}^2_{(r,q,p)}$  is conformal to a positive self-dual Einstein metric. Using the fact that an Einstein metric achieves the Yamabe invariant in its conformal class [Oba72], the equality in (3.18) follows.

Next, consider the case

$$(3.20) \quad r + q \leq p < (\sqrt{r} + \sqrt{q})^2.$$

Rewriting (3.19),

$$(3.21) \quad \begin{aligned} \frac{1}{4 \cdot 24 \cdot \pi^2} \int_M R^2 dV_{\hat{g}} &= \frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr} + \frac{1}{8\pi^2} \int_M |E|^2 dV_{\hat{g}} \\ &\geq \frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr}. \end{aligned}$$

Note the important fact that

$$(3.22) \quad \frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr} > 0,$$

precisely when  $p < (\sqrt{r} + \sqrt{q})^2$ , this was the inequality above in the proof of Theorem 1.2. Furthermore, the orbifold conformal Yamabe invariant of  $[g]$  is positive; this follows from [DG06, equation (2.37)] which implies that

$$(3.23) \quad \text{Vol}(g)^{-1/2} \int_M R_g dV_g = 4\pi\sqrt{2} \frac{r+q+p}{\sqrt{rqp}} > 0,$$

together with [AB04, Lemma 3.4]. In contrast to the case of smooth manifolds, one is not assured that there is a solution to the orbifold Yamabe problem. So to proceed, assume by contradiction that

$$(3.24) \quad Y_{\text{orb}}(M, [g]) < 4\pi\sqrt{6}\sqrt{\frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr}}.$$

If  $p < (\sqrt{r} + \sqrt{q})^2$ , then the inequality

$$(3.25) \quad 4\pi\sqrt{6}\sqrt{\frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr}} < \frac{8\pi\sqrt{6}}{\sqrt{p}}$$

is satisfied. To see this, squaring both sides of (3.25) results in

$$(3.26) \quad \frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr} < \frac{4}{p}.$$

Multiplying by  $pqr$ , and rearranging, this inequality is equivalent to

$$(3.27) \quad r^2 + q^2 + p^2 - 2pq - 2pr + 2qr > 0.$$

But the left-hand side is a perfect square,

$$(3.28) \quad r^2 + q^2 + p^2 - 2pq - 2pr + 2qr = (p - (r + q))^2$$

which is strictly positive since  $p < r + q$ .

Therefore, by [AB03, Theorem 5.2] or [Aku12, Theorem 3.1], there exists a solution to the orbifold Yamabe problem which has constant scalar curvature. Choosing  $\hat{g}$  to be this Yamabe minimizer, the inequality (3.21) is then

$$(3.29) \quad \frac{1}{4 \cdot 24 \cdot \pi^2} (Y_{\text{orb}}(M, [g]))^2 \geq \frac{2}{r} + \frac{2}{q} + \frac{2}{p} - \frac{r}{pq} - \frac{q}{pr} - \frac{p}{qr},$$

which contradicts (3.24) and therefore (3.17) holds.

Finally, if equality holds in the inequality (3.21), then  $\hat{g}$  is Einstein. But Lemma 3.2 says there is no global Einstein metric in the conformal class of  $g$  for  $p \geq r + q$ , so strict inequality must hold in (3.17) when  $p \geq r + q$ . □

PROOF OF THEOREM 1.6. This clearly follows from Propositions 3.1 and 3.3. □

PROOF OF THEOREM 1.8. The inequality

$$(3.30) \quad 4\pi\sqrt{2}\frac{(r + q + p)}{\sqrt{rqp}} < \frac{8\pi\sqrt{6}}{\sqrt{p}}$$

is equivalent to

$$(3.31) \quad (r + q + p)^2 < 12rq.$$

Rewrite this

$$(3.32) \quad (x + y + 1)^2 < 12xy,$$

where  $x = r/p$  and  $y = q/p$ . Since  $1 \leq r \leq q \leq p$ , one must determine the region where the inequality (3.32) is satisfied in the triangle  $V = ([0, 1] \times [0, 1]) \cap \{y \geq x\}$ . The level set  $(x + y + 1)^2 = 12xy$  is a convex curve in this region, so lies below the line connecting its endpoints on the boundary. It is easy to verify that this line is given by

$$(3.33) \quad y = \left(1 + \frac{2}{\sqrt{3}}\right)(1 - x).$$

The inequality (3.32) is then satisfied for points above this line. Converting back to the original variables, this is

$$(3.34) \quad \frac{q}{p} > \left(1 + \frac{2}{\sqrt{3}}\right) \left(1 - \frac{r}{p}\right)$$

which is equivalent to

$$(3.35) \quad p < (2\sqrt{3} - 3)q + r \sim 0.464q + r.$$

Finally, if  $p < (2\sqrt{3} - 3)q + r$ , then  $p < q + r$ , so (1.7) is satisfied, and the lower estimate (1.12) holds also.  $\square$

PROOF OF THEOREM 1.9. As noted above in the proof of Lemma 3.2, in the case  $p = q + r$ ,

$$(3.36) \quad R_g = 48rq(|u_1|^2 + |u_2|^2),$$

which is positive except at the single point  $[0, 0, 1]$  (the orbifold point with group of order  $p$ ).

Assume by contradiction that  $\hat{g}$  is a constant scalar curvature metric on  $M = \mathbf{CP}^2_{(r,q,p)}$  in the conformal class of the Bochner-Kähler metric  $g$ . Letting  $E$  denote the traceless Ricci tensor, since  $\tilde{g} = R_g^{-2}g$  is Ricci-flat, it follows that

$$(3.37) \quad E_{\hat{g}} = \phi^{-1}(-2\nabla^2\phi + (\Delta\phi/2)\tilde{g}),$$

where  $\tilde{g} = \phi^{-2}\hat{g}$ , and the covariant derivatives are taken with respect to  $\hat{g}$ . Next, using the argument of Obata [Oba72] by integrating on  $M$  it follows that

$$(3.38) \quad \begin{aligned} \int_M \phi |E_{\hat{g}}|^2 d\hat{V} &= \int_M \phi E_{\hat{g}}^{ij} \{ \phi^{-1}(-2\nabla^2\phi + (\Delta\phi/2)\tilde{g})_{ij} \} d\hat{V} \\ &= -2 \int_M E_{\hat{g}}^{ij} \nabla_i \nabla_j \phi d\hat{V} = -2 \lim_{\varepsilon \rightarrow 0} \int_{M \setminus B([0,0,1], \varepsilon)} E_{\hat{g}}^{ij} \nabla_i \nabla_j \phi d\hat{V}. \end{aligned}$$

Since  $\tilde{g} = R_g^{-2}g = \phi^{-2}\hat{g}$ , and  $\hat{g}$  and  $g$  are related by a strictly positive conformal factor, it follows from (3.36) that  $\phi \sim R_g \sim \rho^2$  as  $\rho \rightarrow 0$ , where  $\rho$  is the distance to  $[0, 0, 1]$  with respect to the metric  $\hat{g}$ . Integration by parts yields

$$(3.39) \quad \begin{aligned} \int_M \phi |E_{\hat{g}}|^2 d\hat{V} \\ = -2 \lim_{\varepsilon \rightarrow 0} \left( \int_{\partial B([0,0,1], \varepsilon)} E_{\hat{g}}^{ij} \nabla_i \phi \nu_j d\sigma - \int_{M \setminus B([0,0,1], \varepsilon)} (\nabla_j E_{\hat{g}}^{ij} \cdot \nabla_i \phi) d\hat{V} \right). \end{aligned}$$

By the Bianchi identity, the second term on the right-hand side is zero since the scalar curvature of  $\hat{g}$  is constant. By [TV05, Theorem 6.4],  $\hat{g}$  is a smooth Riemannian orbifold, which implies that the curvature is bounded near  $[0, 0, 1]$ . Since  $|\nabla\phi| \sim \rho$  near  $[0, 0, 1]$ , the first term on the right-hand side of (3.39) therefore limits to zero as  $\varepsilon \rightarrow 0$ . Consequently,  $E_{\hat{g}} \equiv 0$ , and  $\hat{g}$  is Einstein. This is ruled out by Lemma 3.2.  $\square$

REMARK 3.4. In the case  $p > r + q$ , there is a complete conformal Einstein metric away from the zero set of the scalar curvature, which is a hypersurface. The above Obata argument does not work in this case to prove that a possible Yamabe minimizer must be

Einstein. Indeed, there are many known examples of Bach-flat extremal Kähler metrics which are conformal to complete Einstein metrics away from a hypersurface on smooth manifolds (see for example [TF02]). There is a Yamabe minimizer in any such conformal class by the solution of the Yamabe problem on smooth manifolds [Sch84], which in these examples is easily seen to be a non-Einstein metric.

## REFERENCES

- [AB03] K. AKUTAGAWA AND B. BOTVINNIK, Yamabe metrics on cylindrical manifolds, *Geom. Funct. Anal.* 13 (2003), 259–333.
- [AB04] K. AKUTAGAWA AND B. BOTVINNIK, The Yamabe invariants of orbifolds and cylindrical manifolds, and  $L^2$ -harmonic spinors, *J. Reine Angew. Math.* 574 (2004), 121–146.
- [Aku12] K. AKUTAGAWA, Computations of the orbifold Yamabe invariant, *Math. Z.* 271 (2012), 611–625.
- [APS75] M. F. ATIYAH, V. K. PATODI AND I. M. SINGER, Spectral asymmetry and Riemannian geometry. I, *Math. Proc. Cambridge Philos. Soc.* 77 (1975), 43–69.
- [Bry01] R. L. BRYANT, Bochner-Kähler metrics, *J. Amer. Math. Soc.* 14 (2001), 623–715.
- [Der83] A. DERDZIŃSKI, Self-dual Kähler manifolds and Einstein manifolds of dimension four, *Compositio Math.* 49 (1983), 405–483.
- [DG06] L. DAVID AND P. GAUDUCHON, The Bochner-flat geometry of weighted projective spaces, *Perspectives in Riemannian geometry*, 109–156, CRM Proc. Lecture Notes 40, Amer. Math. Soc., Providence, RI, 2006.
- [EH79] T. EGUCHI AND A. J. HANSON, Self-dual solutions to Euclidean gravity, *Ann. Physics* 120 (1979), 82–106.
- [Fuk05] Y. FUKUMOTO, The index of the  $\text{Spin}^c$  Dirac operator on the weighted projective space and the reciprocity law of the Fourier-Dedekind sum, *J. Math. Anal. Appl.* 309 (2005), 674–685.
- [GL98] M. J. GURSKY AND C. LEBRUN, Yamabe invariants and  $\text{Spin}^c$  structures, *Geom. Funct. Anal.* 8 (1998), 965–977.
- [GMSY07] J. P. GAUNTLETT, D. MARTELLI, J. SPARKS, AND S.-T. YAU, Obstructions to the existence of Sasaki-Einstein metrics, *Comm. Math. Phys.* 273 (2007), 803–827.
- [Hit74] N. J. HITCHIN, Compact four-dimensional Einstein manifolds, *J. Differential Geom.* 9 (1974), 435–441.
- [Hit97] N. J. HITCHIN, Einstein metrics and the eta-invariant, *Boll. Un. Mat. Ital. B* (7) 11 (1997), no. 2, suppl. 95–105.
- [HZ74] F. HIRZEBRUCH AND D. ZAGIER, The Atiyah-Singer theorem and elementary number theory, *Mathematics Lecture Series*, No. 3, Publish or Perish Inc., Boston, Mass. 1974.
- [Kaw73] T. KAWASAKI, Cohomology of twisted projective spaces and lens complexes, *Math. Ann.* 206 (1973), 243–248.
- [Küh88] W. KÜHNEL, Conformal transformations between Einstein spaces, *Conformal geometry* (Bonn, 1985/1986), 105–146, *Aspects Math.* E12, Vieweg, Braunschweig, 1988.
- [Leb99] C. LEBRUN, Einstein metrics and the Yamabe problem, *Trends in mathematical physics* (Knoxville, TN, 1998), 353–376, *AMS/IP Stud. Adv. Math.* vol. 13, Amer. Math. Soc. Providence, RI, 1999.
- [LM89] H. B. LAWSON, JR. AND M.-L. MICHELSON, *Spin geometry*, Princeton Math. Ser. 38, Princeton University Press, Princeton, NJ, 1989.
- [LV12] M. T. LOCK AND J. A. VIACLOVSKY, Anti-self-dual orbifolds with cyclic quotient singularities, *arXiv:1205.4059*, 2012.
- [Mab87] T. MABUCHI, Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties, *Osaka J. Math.* 24 (1987), 705–737.

- [Mor97] A. MOROIANU, Parallel and Killing spinors on  $\text{Spin}^c$  manifolds, *Comm. Math. Phys.* 187 (1997), 417–427.
- [Nak90] H. NAKAJIMA, Self-duality of ALE Ricci-flat 4-manifolds and positive mass theorem, *Recent topics in differential and analytic geometry*, 385–396, Academic Press, Boston, MA, 1990.
- [Oba72] M. OBATA, The conjectures on conformal transformations of Riemannian manifolds, *J. Differential Geom.* 6 (1971/72), 247–258.
- [Rad54] H. RADEMACHER, Generalization of the reciprocity formula for Dedekind sums, *Duke Math. J.* 21 (1954), 391–397.
- [RT11] J. ROSS AND R. THOMAS, Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics, *J. Differential Geom.* 88 (2011), 109–159.
- [Sch84] R. SCHOEN, Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Differential Geom.* 20 (1984), 479–495.
- [Tas65] Y. TASHIRO, Complete Riemannian manifolds and some vector fields, *Trans. Amer. Math. Soc.* 117 (1965), 251–275.
- [TF02] C. W. TØNNESEN-FRIEDMAN, Extremal Kähler metrics and Hamiltonian functions. II, *Glasg. Math. J.* 44 (2002), 241–253.
- [Tho69] J. A. THORPE, Some remarks on the Gauss-Bonnet integral, *J. Math. Mech.* 18 (1969), 779–786.
- [TV05] G. TIAN AND J. VIACLOVSKY, Moduli spaces of critical Riemannian metrics in dimension four, *Adv. Math.* 196 (2005), 346–372.
- [Via10] J. VIACLOVSKY, Monopole metrics and the orbifold Yamabe problem, *Annales de L'Institut Fourier* 60 (2010), 2503–2543.
- [Zag72] D. B. ZAGIER, Equivariant Pontrjagin classes and applications to orbit spaces. Applications of the  $G$ -signature theorem to transformation groups, symmetric products and number theory, *Lecture Notes in Mathematics*, Vol. 290, Springer-Verlag, Berlin, 1972.

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF WISCONSIN  
MADISON, WI, 53706  
U.S.A.

*E-mail address:* jeffv@math.wisc.edu