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A NOTE ON THE SHADOWING LEMMA OF LIAO: A GENERALIZED AND IMPROVED VERSION

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Abstract. The shadowing lemma of Shantao Liao asserts that a "recurrent" quasi-hyperbolic string of a C^1-class diffeomorphism of a closed manifold might be closed up by a periodic orbit. In this note, we further show that the string can be closed exponentially up by the periodic orbit. Moreover, this statement also holds for a C^1-class map which is only locally diffeomorphic.

Introduction. Consider a discrete-time dynamical system $f\colon \mathbf{Z}\times M \to M$ on a compact metric space (M, d). We say a motion $f(t, w)$ can be *ε-closed up* if to any $\varepsilon > 0$ there exists some $\delta > 0$ such that, when $d(w, f^n w) < \delta$ for an integer $n \geq 1$, one can find $z \in M$ satisfying $f^n z = z$ and $d(f^t z, f^t w) < \varepsilon$ for $0 \leq t \leq n$. For the modern differentiable dynamical systems, it is an important problem that whether or not a recurrent orbit of f can be closed up by a periodic motion of f. For example, in [8, 9], this kind of closing property plays an essential role.

However, in smooth ergodic theory, frequently the ε-closing property is not enough for us to obtain a "good" periodic orbit near a recurrent motion. We often need a stronger shadowing property, named (ε, ρ)-exponential closing property in [3, 4], first established in the context of non-uniformly hyperbolic dynamical systems by A. Katok in [6], where he uses it to prove the existence of hyperbolic periodic orbits near a Pesin set of a $C^{1+\alpha}$-class diffeomorphism, where $0 < \alpha \leq 1$.

Precisely speaking, the motion $f(t, w)$ is said to obey the *(ε, ρ)-exponential closing property*, provided that, for any $\varepsilon > 0$ and small $\rho > 0$, there exists $\delta_* > 0$ such that, if $d(w, f^n w) < \delta_*$ for an integer $n \geq 2$, then there can be found a periodic point z of period n satisfying

$$d(f^t z, f^t w) < \begin{cases} \varepsilon e^{-\rho t} & \text{for } 1 \leq t \leq [n/2]\,, \\ \varepsilon e^{-\rho(n-t)} & \text{for } [n/2] < t \leq n\,. \end{cases}$$

We notice that the $(\varepsilon, 0)$-exponential closing property is the classical ε-closing property. If $f(t, w)$ is hyperbolic, the (ε, ρ)-exponential shadowing just means that the periodic motion $f(t, z)$ moves more close to the stable foliation of $f(t, w)$ during the period $[0, n/2]$, and more close to the unstable foliation during $[n/2 + 1, n]$.

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In the classical Pesin theory, the (ε, ρ)-exponential shadowing just panders to the Hölder condition via the so-called Lyapunov neighborhoods. In [6], Katok proves that almost every recurrent motions of a non-uniformly hyperbolic $C^{1+\alpha}$-class diffeomorphism f can be closed (ε, ρ)-exponentially up. Using the (ε, ρ)-exponential closing property, in [12], Wang and Sun further show that every Lyapunov exponents of a hyperbolic ergodic measure of a $C^{1+\alpha}$-class diffeomorphism f can be approximated by ones of periodic measures. In [3], it is proved that every Lyapunov exponents of an ergodic measure, not necessarily hyperbolic, of a Hölder-continuous linear cocycle, can also be approximated arbitrarily by that of periodic measures, whenever the driving dynamical system obeys the (ε, ρ)-exponential closing property. Moreover, [3] shows that every hyperbolic C^1-class diffeomorphism f, not necessarily $C^{1+\alpha}$, has also the (ε, ρ)-exponential closing property. Thus, to study the Lyapunov characteristic spectra of a Hölder-continuous linear cocycle driven by a hyperbolic differentiable dynamical system, we need to consider only the Lyapunov spectra of the periodic points.

In [7], Liao presents an important shadowing lemma, which asserts that every "recurrent" quasi-hyperbolic orbit string of a C^1-class diffeomorphism can be ε-closed up by a periodic motion. The aim of the present paper is to generalize and improve Liao's lemma by the (ε, ρ)-exponential closing property that is important for the study of Lyapunov exponents in the smooth ergodic theory (for example, see [4]).

By M, in the sequel, we denote a closed manifold of finite dimension d. Let $\mathrm{Diff}^1_{\mathrm{loc}}(M)$ be the set of all C^1-class local diffeomorphisms of M. Assume that we have chosen a smooth Riemannian structure $\langle\cdot,\cdot\rangle$ on M, and then it induces a norm $\|\cdot\|_x$ in every tangent space T_xM to $x \in M$. Let $\mathrm{dist}(x, y)$ be the local geodetic distance connecting the points x and y in M induced naturally by $\langle\cdot,\cdot\rangle$. By SM we denote the unit tangent sphere bundle of M endowed with the natural topological metric $\widetilde{\mathrm{dist}}$. For any $K \subset SM$ and $\delta > 0$, $B_{\widetilde{\mathrm{dist}}}(K, \delta)$ denotes the δ-neighborhood of K in SM with respect to $\widetilde{\mathrm{dist}}$.

Next, we introduce the basic concept of quasi-hyperbolic orbit string and state our main result (Theorem 2 below).

For any $f \in \mathrm{Diff}^1_{\mathrm{loc}}(M)$, any $x \in M$ and any integer $n > 0$, we denote by $(x, f^n(x))$ the f-forward orbit string $\{x, f^1(x), \dots, f^n(x)\}$.

DEFINITION 1 ([7]). Let Λ be a forwardly invariant set of f, and $T_\Lambda M = E \oplus F$ a forwardly Df-invariant splitting. Let $\lambda > 0$ be arbitrarily given. A string $(x, f^n(x))$ in Λ is said to be *λ-quasi-hyperbolic of index* i respecting $E \oplus F$, provided that $\dim E(x) = \mathrm{i}$, $1 \le \mathrm{i} \le d-1$ and there hold the following three conditions:

(1) $\prod_{j=0}^{k-1} \|(D_{f^jx}f)|_{E(f^jx)}\| \le \exp(-\lambda k)$ for $k = 1, \dots, n$;

(2) $\prod_{j=1}^{k} \|(D_{f^{n-j}x}f)|_{F(f^{(n-j)}x)}\|_{\mathrm{co}} \ge \exp(\lambda k)$ for $k = 1, \dots, n$;

(3) $\log\|(D_{f^jx}f)|_{F(f^jx)}\|_{\mathrm{co}} - \log\|(D_{f^jx}f)|_{E(f^jx)}\| \ge 2\lambda$ for $j = 0, \dots, n-1$.

Here and in the future $\|\cdot\|_{\mathrm{co}}$ means the minimum norm (also called co-norm) defined as

$$\|A\|_{\mathrm{co}} = \|A^{-1}\|^{-1} = \min_{\|x\|=1}\|Ax\|\,.$$

Notice here that, although condition (3) is just a domination inequality, we need not have a uniform dominated splitting on Λ.

In this paper, we mainly generalize and improve Liao's shadowing lemma [7] as follows:

THEOREM 2. *Given any $f \in \mathrm{Diff}^1_{\mathrm{loc}}(M)$, let Λ be a forwardly invariant set of f with a forwardly Df-invariant splitting $T_\Lambda M = E \oplus F$ and $\lambda > 0$. Then, to any $\varepsilon > 0$ and $0 < \rho < \lambda/2$, there exist $\delta_* > 0$ and $N_* > 1$ such that, for any λ-quasi-hyperbolic string $(x, f^n(x))$ respecting $E \oplus F$, where $1 \leq \dim E(x) < d$, in Λ with*

$$(*) \qquad n \geq N_* \quad \text{and} \quad E(f^n x) \cap S_{f^n x}M \subset B_{\widetilde{\mathrm{dist}}}(E(x) \cap S_x M, \delta_*)\,,$$

there can be found a forward periodic point b of period n, which closes (ε, ρ)-exponentially up $(x, f^n(x))$, i.e.,

$$\mathrm{dist}(f^i x, f^i b) < \begin{cases} \varepsilon e^{-\rho i} & \text{for } 0 \leq i \leq [n/2]\,, \\ \varepsilon e^{-\rho(n-i)} & \text{for } [n/2] < i \leq n\,. \end{cases}$$

Moreover, if Λ is a closed neighborhood of the motion $f(t, x)$ and $E(w) \oplus F(w)$ is continuous with respect to w on it, then we can further require b to be hyperbolic of index $\dim E(x)$ *and each of the Lyapunov exponents of b to lie off the interval* $[-\lambda/2, \lambda/2]$.

We note here that condition $(*)$ is the so-called "recurrence" of a quasi-hyperbolic string $(x, f^n(x))$ mentioned before.

REMARK 3. Compared to the original shadowing lemma of Liao, in the statement of which b only ε-shadows $(x, f^n(x))$ in the sense of $\mathrm{dist}(f^i x, f^i b) < \varepsilon$ for $0 \leq i \leq n$, the (ε, ρ)-exponential closing property and the hyperbolicity of b in our statement are right the new ingredients of the improved version above. In addition, we do not impose here the assumption that $T_\Lambda M = E \oplus F$ is a continuous and uniformly dominated splitting as supposed in [5].

REMARK 4. Let there be given a forwardly invariant set Λ of a C^1-class local diffeomorphism f. Recall that f is said to be hyperbolic on Λ, provided that there exists $\lambda > 0$ and a forwardly Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ such that

$$\|(Df)v\| \leq e^{-\lambda}\|v\| \quad \text{for all } v \in E^s \quad \text{and} \quad \|(Df)u\| \geq e^{\lambda}\|u\| \ \text{ for all } u \in E^u.$$

Obviously, from Theorem 2 above, the following theorem, which generalizes [3, Theorem 1.2], could be obtained as a corollary.

THEOREM 5. *If $f \in \mathrm{Diff}^1_{\mathrm{loc}}(M)$ is hyperbolic on a forwardly invariant subset $\Lambda \subset M$, then f obeys the (ε, ρ)-exponential closing property on Λ.*

According to [3], the above Theorem 5 is possibly useful for the approximation of Lyapunov exponents of Hölder-continuous linear cocycles by that of periodic orbits.

This paper is simply organized as follows. In Section 1, we introduce the notion of Liao standard systems of difference equations for a C^1-class local diffeomorphism. It is very expedient for the author himself. In Section 2, we consider an abstract hyperbolic difference equations. There is proved Theorem 2.4, which is an organic combination of Katok's (ε, ρ)-exponential shadowing idea, Newhouse's semi-invariant disc technique, and Liao's standard system theory of differential systems. Finally, in Section 3, we complete the proof of Theorem 2 in the framework of Liao theory by applying Theorem 2.4 to a Liao standard difference system associated with a forwardly recurrent quasi-hyperbolic orbit string.

1. Standard systems of local diffeomorphisms. We will prove our generalized and improved version of Liao's shadowing lemma in the framework of Liao theory, so we need to introduce the notion of Liao standard systems of difference equations for a local diffeomorphism from [1, 2]. In this section, we consider a C^1-class local diffeomorphism

$$f\colon M \to M\,,$$

and by

$$Df\colon TM \to TM$$

we denote its tangent map on the tangent bundle TM. Since M is compact, there exist two universal constants $\varepsilon_1 < \varepsilon_2$ such that, for any $x \in M$, the restriction of f to the ε_2-ball $B_{\mathrm{dist}}(x, \varepsilon_2)$ around x is diffeomorphic into M and $B_{\mathrm{dist}}(f(x), \varepsilon_1) \subset f(B_{\mathrm{dist}}(x, \varepsilon_2))$.

1.1. A natural triangularization of Df. Let $d = \dim M$. For any orthonormal d-frame $\gamma = \{\vec{u}_1, \dots, \vec{u}_d\}$ of T_xM, by the Gram-Schmidt orthonormalization procedure, for any $t \in \boldsymbol{Z}_+$, there exists a unique orthonormal d-frame γ_t of $T_{f^tx}M$ at the base point f^tx, defined from the collection of the tangent vectors $\{(D_xf^t)\vec{u}_1, \dots, (D_xf^t)\vec{u}_d\}$. See Figure 1 below for an illustration for the case $d = 2$.

Associated to (x, γ), define a non-autonomous orthogonal coordinate transformation

$$\{T_t\}_{t\in\boldsymbol{Z}_+}\colon \boldsymbol{R}^d \to T_{f^tx}M; \quad y \mapsto \gamma_t y := y_1\mathrm{col}_1\gamma_t + \cdots + y_d\mathrm{col}_d\gamma_t\,,$$

where for any $t > 0$,

$$\mathrm{col}_i\colon \gamma_t = (\vec{u}_1(t), \dots, \vec{u}_d(t)) \mapsto \vec{u}_i(t) \in T_{f^tx}M \quad (1 \le i \le d)\,.$$

Then, we obtain a linear difference system

$$y_{t+1} = R_{x,\gamma}(t)y_t\,, \quad t \in \boldsymbol{Z}_+,\ y_t \in \boldsymbol{R}^d$$

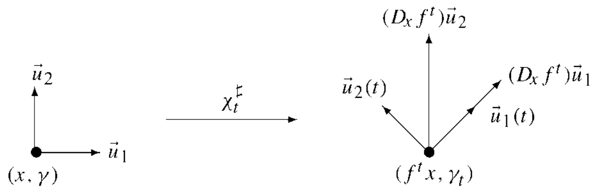

FIGURE 1. Skew-product semi-dynamical system on the d-frame bundle $\mathscr{F}_d^\sharp$.

with the commutative diagrams

$$\begin{array}{ccc} \boldsymbol{R}^d & \xrightarrow{R_{x,\gamma}(t)} & \boldsymbol{R}^d \\ {\scriptstyle T_t}\big\downarrow & & \big\downarrow{\scriptstyle T_{t+1}} \\ T_{f^t x}M & \xrightarrow{D_{f^t x} f} & T_{f^{t+1}x}M \end{array} \quad \text{for all } t \in \boldsymbol{Z}_+ .$$

According to the Liao theory [1, 2], $R_{x,\gamma}(t)$ is upper triangular and both $\|R_{x,\gamma}(t)\|$ and $\|R_{x,\gamma}(t)\|_{\mathrm{co}}$ are uniformly bounded away from 0 and $+\infty$ for t and (x, γ), since f is of class C^1 and M is compact.

Therefore, we obtain the following lemma.

LEMMA 1.1. *There exists a constant $\mu_4 > 0$ such that*

$$\exp(-\mu_4) \leq \inf_{t,(x,\gamma)} \|R_{x,\gamma}(t)\|_{\mathrm{co}} \leq \sup_{t,(x,\gamma)} \|R_{x,\gamma}(t)\| \leq \exp(\mu_4) ,$$

where $t \in \boldsymbol{Z}_+$ and (x, γ) runs over the orthonormal d-frame bundle $\mathscr{F}_d^\sharp$ of M. Hence

$$-\mu_4 \leq \inf_{t\in \boldsymbol{N},\vec{u}\in SM} \frac{1}{t} \log \|(Df^t)\vec{u}\| \leq \sup_{t\in \boldsymbol{N},\vec{u}\in SM} \frac{1}{t} \log \|(Df^t)\vec{u}\| \leq \mu_4 .$$

Particularly, if Λ is a forwardly f-invariant compact set and if $T_\Lambda M = E \oplus F$ is a continuous forwardly Df-invariant splitting, then we could define an adapted Lyapunov metric $\langle\cdot,\cdot\rangle'$ on $T_\Lambda M$ which is equivalent to $\langle\cdot,\cdot\rangle$, such that $\langle E(x), F(x)\rangle'_x = 0$ for all $x \in \Lambda$. So, if in such case we choose an orthonormal d-frame γ at the base point $x \in \Lambda$ with $\{\mathrm{col}_1\gamma, \dots, \mathrm{col}_{\mathrm{i}}\gamma\} \subset E(x)$ and $\{\mathrm{col}_{\mathrm{i}+1}\gamma, \dots, \mathrm{col}_d\gamma\} \subset F(x)$ where $\mathrm{i} = \dim E(x)$, then $R_{x,\gamma}(t)$ has the blockwise diagonal form

$$R_{x,\gamma}(t) = \begin{bmatrix} R_{x,\gamma|E}(t) & 0_{\mathrm{i}\times(d-\mathrm{i})} \\ 0_{(d-\mathrm{i})\times\mathrm{i}} & R_{x,\gamma|F}(t) \end{bmatrix}$$

under the adapted metric $\langle\cdot,\cdot\rangle'$.

1.2. Standard systems. From here on, we fix a constant $\theta > 0$ such that for any $x \in M$, the exponential projection

$$\exp_x \colon \boldsymbol{B}_{\mathbf{0}_x}(\theta) \to M , \quad \text{where } \boldsymbol{B}_{\mathbf{0}_x}(\theta) := \{\vec{v} \in T_x M \,;\, \|\vec{v}\| < \theta\} ,$$

is smoothly diffeomorphic from $\boldsymbol{B}_{\mathbf{0}_x}(\theta)$ into M, where $\mathbf{0}_x$ denotes the zero tangent vector to x. For any $x \in M$, define

$$\tilde{f}_x \colon T_x M \to T_{fx} M$$

locally in the following way:

$$\tilde{f}_x(\vec{v}) = (\exp_{fx}^{-1} \circ f \circ \exp_x)\vec{v} \quad \text{for all } \vec{v} \in \boldsymbol{B}_{\mathbf{0}_x}(\theta) .$$

Let

$$\mathfrak{S}_{x,\gamma}(t) \colon B(\theta) \to M \,; \quad y \mapsto \exp_{f^t x}(\gamma_t y) ,$$

where $B(\theta) = \{y \in \boldsymbol{R}^d \,;\, \|y\| < \theta\}$.

DEFINITION 1.2 ([1, 2]). For any given orthonormal d-frame (x, γ) of TM, we locally define

$$f_{x,\gamma}(t, y) = \mathfrak{S}_{x,\gamma}(t+1)^{-1} \circ f \circ \mathfrak{S}_{x,\gamma}(t)y \quad \text{for all } (t, y) \in \mathbf{Z}_+ \times B(\theta)\,.$$

The difference system

$$y_{t+1} = f_{x,\gamma}(t, y_t)\,, \quad (t, y_t) \in \mathbf{Z}_+ \times B(\theta)\,,$$

equivalently written as

$$(f_{x,\gamma}): \quad y_{t+1} = R_{x,\gamma}(t)y_t + f_{\mathrm{rem}(x,\gamma)}(t, y_t)\,, \quad (t, y_t) \in \mathbf{Z}_+ \times B(\theta)\,,$$

is called the *standard system of f under the d-frame* (x, γ).

Thus, $f_{\mathrm{rem}(x,\gamma)}(t, \mathbf{0}) = \mathbf{0}$ for all $t \in \mathbf{Z}_+$, where $\mathbf{0}$ is the origin of $\boldsymbol{R}^d$, and there the commutativity

$$\begin{array}{ccc}
\boldsymbol{R}^d & \xrightarrow{f_{x,\gamma}(t,\cdot)} & \boldsymbol{R}^d \\
{\scriptstyle T_t}\big\downarrow & & \big\downarrow{\scriptstyle T_{t+1}} \\
T_{f^t x}M & \xrightarrow{\tilde{f}_{f^t x}} & T_{f^{t+1}x}M \\
{\scriptstyle \exp_{f^t x}}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{f^{t+1}x}} \\
M & \xrightarrow{f} & M
\end{array}$$

holds locally for every $t \in \mathbf{Z}_+$ and

$$\mathfrak{S}_{x,\gamma}(t)y = \exp_{f^t x}(\gamma_t y) = \mathfrak{S}_{f^t x,\gamma_t}(0)y \quad \text{for all } t \in \mathbf{Z}_+\,.$$

The following result asserts a uniform Lipschitz property of the remainder $f_{\mathrm{rem}(x,\gamma)}(t, y)$ for $y \in B(\theta)$.

LEMMA 1.3 ([1]). *The standard systems $(f_{x,\gamma})$ satisfy that to any $\mu > 0$, there exists a constant $\nu > 0$ such that uniformly for $t \in \mathbf{Z}_+$ and for d-frame (x, γ)*

$$\| f_{\mathrm{rem}(x,\gamma)}(t, y) - f_{\mathrm{rem}(x,\gamma)}(t, y')\| \le \mu \|y - y'\|$$

for any $y, y' \in B(\nu)$.

Notice here that the constant ν is independent of the choice of the d-frame (x, γ). This point is very important for us to prove Theorem 2 later.

2. Hyperbolic difference equations. In this section, we shall prove an abstract (ε, ρ)-exponential shadowing property and a certain "semi-invariant disc" property for hyperbolic difference equations. The isomorphism of homology groups in Theorem 2.4 (iii) below is a key manner for us to finally get a good periodic motion which exponentially shadows a recurrent quasi-hyperbolic orbit string. This idea is inspired by Liao [7].

We first consider a linear system of difference equations

$$(A): \quad x_{t+1} = A(t)x_t\,, \quad t \in \mathbf{Z},\ x_t \in \boldsymbol{R}^d\,,$$

where $A(t) = (A_{ij}(t))$ is a real upper triangular d-by-d matrix such that

(a) $A_{ii}(t) \neq 0$ for all $t \in \boldsymbol{Z}$ and $1 \le i \le d$;

(b) $\|A^{\pm}\| := \sup_{t\in\boldsymbol{Z}}\{\|A(t)\|, \|A(t)^{-1}\|\} \le \hat{\eta}$ for some constant $\hat{\eta} > 0$.

It is very helpful for us to keep in mind that $A(t) = R_{x,\gamma}(t)$ for all $t \in \boldsymbol{Z}_+$.

DEFINITION 2.1. Let i, $1 \le \mathrm{i} < d$, be an integer and $\eta_1 > 0$. The system (A) is said to be (η_1, i)-*hyperbolic*, if there is a decomposition of $\boldsymbol{R}^d$ into subspaces

$$\boldsymbol{R}^d = E^s \oplus E^u\,, \quad \dim E^s = \mathrm{i}$$

such that, for all $i \in \boldsymbol{Z}$ and $t \in \boldsymbol{N}$,

(1) $\|x_A(i+t, v_s)\| \le \|x_A(i, v_s)\| \exp(-\eta_1 t)$ for any $v_s \in E^s$;

(2) $\|x_A(i+t, v_u)\| \ge \|x_A(i, v_u)\| \exp(\eta_1 t)$ for any $v_u \in E^u$.

Here $\{x_A(t, v)\}_{t\in\boldsymbol{Z}}$ denotes the solution of (A) satisfying $x_A(0, v) = v$ for $v \in \boldsymbol{R}^d$.

As (A) is uniformly hyperbolic, it follows from [1, Lemma 2.3] that the angle between $x_A(t, E^s)$ and $x_A(t, E^u)$ is uniformly bounded away from 0 for $t \in \boldsymbol{Z}$. So, without loss of generality, we might further assume

(c) for all $t \in \boldsymbol{Z}$

$$A(t) = \begin{bmatrix} A^s(t) & 0_{\mathrm{i}\times(d-\mathrm{i})} \\ 0_{(d-\mathrm{i})\times\mathrm{i}} & A^u(t) \end{bmatrix} \quad \text{and} \quad E^s = \boldsymbol{R}^{\mathrm{i}} \times \{\boldsymbol{0}\}\,, \quad E^u = \{\boldsymbol{0}\} \times \boldsymbol{R}^{d-\mathrm{i}}\,,$$

where $A^s(t)$ and $A^u(t)$ are i-by-i and $(d-\mathrm{i})$-by-$(d-\mathrm{i})$ upper-triangular subblocks of $A(t)$, respectively.

In light of condition (c), we define an equivalent box-norm $|\cdot|$ on $\boldsymbol{R}^d$ by

$$|v| = \max\{\|v_s\|, \|v_u\|\} \quad \text{for all } \ v = v_s + v_u \in E^s \oplus E^u\,.$$

Write

$$O(\mathrm{i}) = \{G \in \mathrm{GL}(d, \boldsymbol{R})\,;\ GG^{\mathrm{T}} = \mathrm{Id}_{\boldsymbol{R}^d},\ G(E^s) = E^s\}\,,$$

where the linear isomorphism $G\colon \boldsymbol{R}^d \to \boldsymbol{R}^d$ is regarded as a d-by-d matrix and G^{T} denotes the transpose operator of G.

Next, we consider a perturbation of (A)

$$(A, g)\colon \quad z_{t+1} = G_t(z_t) = A(t)z_t + g(t, z_t)\,, \quad t \in \boldsymbol{Z},\ \ z_t \in \boldsymbol{R}^d\,,$$

where $g(t, z)$ is Lipschitz continuous with respect to z such that $g(t, \boldsymbol{0}) = \boldsymbol{0}$ and

$$\|g(t, z) - g(t, z')\| \le L_g\|z - z'\| \quad \text{for all } \ z, z' \in \boldsymbol{R}^d$$

uniformly for $t \in \boldsymbol{Z}$ for some constant $L_g > 0$. Write

$$\|g\| = \sup_{t\in\boldsymbol{Z}, z\in\boldsymbol{R}^d}\|g(t, z)\|$$

and we denote by $z_{A,g}(t, v)$ the solution of (A, g) with $z_{A,g}(0, v) = v$ for all $v \in \boldsymbol{R}^d$.

Since we always keep in mind that (A, g) is equal to $(f_{x,\gamma})$, there is no loss of generality in assuming that $G_t\colon \boldsymbol{R}^d \to \boldsymbol{R}^d$ is invertible.

DEFINITION 2.2. Given any $0 < \rho < \eta_1$, let $\rho_k = \exp(-\rho k)$ for $k \in \mathbf{Z}_+$. For any $\varepsilon > 0$, the *Lyapunov neighborhood* $L[\varepsilon\rho_k]$ around $\mathbf{0}$ with size $\varepsilon\rho_k$ is defined as the closed rectangle $B^s[\varepsilon\rho_k] \oplus B^u[\varepsilon\rho_k]$ in $\boldsymbol{R}^d$, where

$$B^s[\varepsilon\rho_k] = \{v_s \in E^s\ ;\ \|v_s\| \leq \varepsilon\rho_k\} \quad \text{and} \quad B^u[\varepsilon\rho_k] = \{v_u \in E^u\ ;\ \|v_u\| \leq \varepsilon\rho_k\}.$$

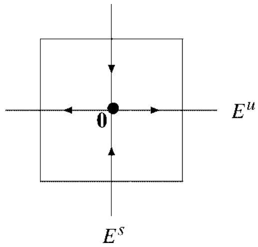

FIGURE 2. Neighborhood $L[\varepsilon\rho_k]$ of $\mathbf{0}$ in $\boldsymbol{R}^d$.

As in the classical Pesin theory, we need to use the hyperbolicity of (A) to describe how $L[\varepsilon\rho_k]$ transforms under $(G_t)_{t\in\mathbf{Z}}$.

LEMMA 2.3. *Let $0 < \rho < \eta_1$. If the Lipschitz constant L_g of $g(t, \cdot)$ satisfies*

$$L_g \leq e^{-\rho} - e^{-\eta_1} \quad \textit{and} \quad L_g \leq e^{\eta_1} - e^{\rho},$$

then, for any $\varepsilon > 0$ and any integer $k \geq 0$, the image $G_t(L[\varepsilon\rho_k])$ meets $L[\varepsilon\rho_{k+1}]$ transversely for $t \in \boldsymbol{N}$ in the sense that their configuration is homeomorphic to the diagram in Figure 3.

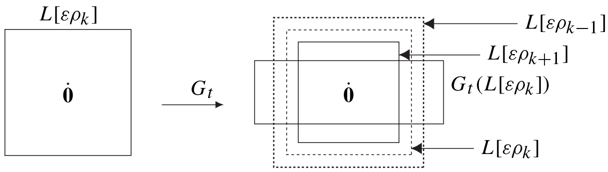

FIGURE 3. A transverse intersection of rectangles.

Moreover, the result remains true if we replace $L[\varepsilon\rho_{k+1}]$ by either $L[\varepsilon\rho_k]$ or $L[\varepsilon\rho_{k-1}]$ for all $k \geq 1$.

PROOF. It is easy to see that, for any $v = v_s + v_u \in E^s \oplus E^u$,

$$|G_t(v)| \leq \|v_s\|e^{-\rho} \quad \text{if } |v| = \|v_s\|$$

and

$$|G_t(v)| \geq \|v_u\|e^{\rho} \qquad \text{if } |v| = \|v_u\|.$$

For the rest discussion, its similar idea has been used in the proof of [3, Lemma 2.2], and so we omit the details here. □

In the following theorem, the topological ball plays the role of "semi-invariant disc" as in Newhouse [10].

THEOREM 2.4. *Let (A, g) be (η_1, i)-hyperbolic and $0 < \rho < \eta_1$. Then, there exist constants*

$$\lambda_* = \lambda_*(\hat{\eta}, \rho) > 1\,, \quad N_* = N_*(\hat{\eta}, \rho) \geq 2\,, \quad \textit{and} \quad \hat{\hat{\xi}} = \hat{\hat{\xi}}(\hat{\eta}, \eta_1) \geq 1$$

which satisfy the following. For any constant $\varepsilon > 0$ and any integer $n \geq N_*$, if

$$L_g \leq \min\left\{ e^{-\rho} - e^{-\eta_1},\ e^{\eta_1} - e^{\rho},\ \frac{1}{\hat{\hat{\xi}}(1 + 2\hat{\eta}\hat{\hat{\xi}})^d} \right\} \quad \textit{and} \quad \|g\| \leq \frac{\varepsilon}{4\lambda_*\hat{\hat{\xi}}(1 + 2\hat{\eta}\hat{\hat{\xi}})^d}\,,$$

then there can be found a d-dimensional topological closed ball $B[n, \varepsilon]$ in $\boldsymbol{R}^d$, containing $\mathbf{0}$ as an interior point, with the following properties (i), (ii) *and* (iii).

(i) *For all $v \in B[n, \varepsilon]$*

$$|z_{A,g}(i, v)| < \begin{cases} \varepsilon e^{-\rho i} & \textit{for } 0 \leq i \leq [n/2]\,, \\ \varepsilon e^{-\rho(n-i)} & \textit{for } [n/2] < i \leq n\,. \end{cases}$$

(ii) *For all $v \in \partial B[n, \varepsilon]$ and for all $G \in O(\mathrm{i})$,*

$$|G(z_{A,g}(n, v)) - v| \geq \frac{\varepsilon}{\lambda_*}\,.$$

(iii) *For any given $G \in O(\mathrm{i})$, the map*

$$h\colon \partial B[n, \varepsilon] \to \boldsymbol{R}^d \setminus \{\mathbf{0}\}\,; \quad v \mapsto G(z_{A,g}(n, v)) - v$$

induces an isomorphism

$$h_*\colon H_{d-1}(\partial B[n, \varepsilon], \boldsymbol{Z}) \xrightarrow{\approx} H_{d-1}(\boldsymbol{R}^d \setminus \{\mathbf{0}\}, \boldsymbol{Z})$$

of the homology groups $H_{d-1}(\partial B[n, \varepsilon], \boldsymbol{Z})$ onto $H_{d-1}(\boldsymbol{R}^d \setminus \{\mathbf{0}\}, \boldsymbol{Z})$.

PROOF. From the L_g condition and Lemma 2.3, we see that, for any $\varepsilon > 0$ and any $n \geq 2$, there is a narrow "rectangle" $B[n, \varepsilon] \subset L[\varepsilon\rho_0]$ such that for all $v \in B[n, \varepsilon]$

$$|z_{A,g}(i, v)| < \begin{cases} \varepsilon e^{-\rho i} & \text{for } 0 \leq i \leq l \\ \varepsilon e^{-\rho(n-i)} & \text{for } l < i \leq n \end{cases}$$

for $l = [n/2]$. In fact, from the sequence of morphisms

$$L[\varepsilon\rho_0] \xrightarrow{G_0} L[\varepsilon\rho_1] \xrightarrow{G_1} \cdots \xrightarrow{G_{l-2}} L[\varepsilon\rho_{l-1}] \xrightarrow{G_{l-1}} L[\varepsilon\rho_l] \xrightarrow{G_l} L[\varepsilon\rho_{n-(l+1)}] \xrightarrow{G_{l+1}}$$
$$L[\varepsilon\rho_{n-(l+2)}] \xrightarrow{G_{l+2}} \cdots \xrightarrow{G_{n-2}} L[\varepsilon\rho_1] \xrightarrow{G_{n-1}} L[\varepsilon\rho_0]\,,$$

we obtain $B[n, \varepsilon] = L[\varepsilon\rho_0] \cap (\bigcap_{i=1}^{l} G_{i-1}^{-i}(L[\varepsilon\rho_i])) \cap (\bigcap_{i=l+1}^{n} G_{i-1}^{-i}(L[\varepsilon\rho_{n-i}]))$.

Next, we are going to choose the desired constants λ_* and N_*. For that, we need to solve the inequality equation

$$\lambda_* \geq \frac{1}{1 - \exp(-\rho N_*)}\,.$$

It is easily seen that as N_* tends to 0, $(1-\exp(-\rho N_*))^{-1}$ goes to $+\infty$ and as N_* tends to $+\infty$, $(1-\exp(-\rho N_*))^{-1}$ goes to 1, so we can choose two constants $\lambda_* = \lambda_*(\rho) > 1$ and $N_* = N_*(\rho) \geq 2$ such that

$$\lambda_* \geq \varphi(n) := \frac{1}{1-\exp(-\rho n)} \quad \text{for all } \ n \geq N_*$$

from Figure 4.

Hence, for any $n \geq N_*$ and $\varepsilon > 0$, from Lemma 2.3 it follows that, for any $v \in \partial B[n, \varepsilon]$ and all $G \in O(\mathrm{i})$,

$$|G(z_{A,g}(n, v)) - v| \geq \varepsilon - \varepsilon e^{-\rho n} \geq \frac{\varepsilon}{\lambda_*}$$

as desired.

This proves the assertions (i) and (ii) of the theorem.

We now proceed to the proof of the assertion (iii) of the theorem.

According to the Liao theory [1], there is a constant

$$\hat{\xi} = \hat{\xi}(\hat{\eta}, \eta_1) \geq 1$$

such that if the Lipschitz constant $L_g \leq 1/\hat{\xi}(1+2\hat{\eta}\hat{\xi})^d$, then there is a homeomorphism $\Delta\colon \boldsymbol{R}^d \to \boldsymbol{R}^d$ which satisfies that $\Delta(\mathbf{0}) = \mathbf{0}$ and, for any $v \in \boldsymbol{R}^d$ and $t \in \boldsymbol{Z}$,

$$\|z_{A,g}(t, v) - x_A(t, \Delta(v))\| \leq \|g\|\hat{\xi}(1+2\hat{\eta}\hat{\xi})^d\,,$$

where $x_A(t, \Delta(v))$ is the solution of (A) with the initial value $\Delta(v)$.

Next, let $\widetilde{B} = \Delta(B[n, \varepsilon])$, which is also a topological closed ball containing $\mathbf{0}$ as an interior point in $\boldsymbol{R}^d$ since Δ is homeomorphic. Therefore, if

$$\|g\| \leq \frac{\varepsilon}{4\lambda_*\hat{\xi}(1+2\hat{\eta}\hat{\xi})^d}\,,$$

then

$$\|z_{A,g}(t, u) - x_A(t, \Delta(u))\| \leq \frac{\varepsilon}{4\lambda_*} \quad \text{for all } \ u \in \boldsymbol{R}^d \text{ and } t \in \boldsymbol{Z}\,,$$

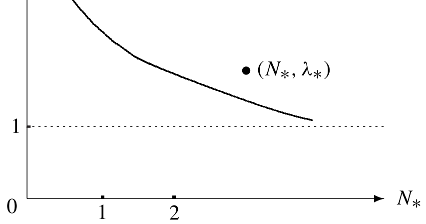

FIGURE 4. Choices of λ_* and N_*.

and hence from (ii) we get

$$\|G(x_A(n,v)) - v\| \geq \frac{\varepsilon}{2\lambda_*} \quad \text{for all } v \in \partial\widetilde{B} \text{ and } G \in O(\mathrm{i})\,.$$

Given any $G \in O(\mathrm{i})$, the above inequality implies that $L\colon v \mapsto G(x_A(n,v)) - v$ for $v \in \boldsymbol{R}^d$ is an isomorphism from $\boldsymbol{R}^d$ onto itself. Thus,

$$(L \circ \Delta)_*\colon H_{d-1}(\partial B[n,\varepsilon], \boldsymbol{Z}) \xrightarrow{\approx} H_{d-1}(\boldsymbol{R}^d \setminus \{\boldsymbol{0}\}, \boldsymbol{Z})\,.$$

Let $h\colon v \mapsto G(z_{A,g}(n,v)) - v$ for all $v \in \partial B[n,\varepsilon]$. To show that

$$h_*\colon H_{d-1}(\partial B[n,\varepsilon], \boldsymbol{Z}) \to H_{d-1}(\boldsymbol{R}^d \setminus \{\boldsymbol{0}\}, \boldsymbol{Z})$$

is isomorphic, it is sufficient to show that h is homotopic to $L \circ \Delta$ from $\partial B[n,\varepsilon]$ into $\boldsymbol{R}^d \setminus \{\boldsymbol{0}\}$. Indeed, since for all $v \in \partial B[n,\varepsilon]$

$$\begin{aligned}\|h(v) - L \circ \Delta(v)\| &= \|G(z_{A,g}(n,v) - x_A(n,\Delta(v))) - (v - \Delta(v))\| \\ &\leq \frac{\varepsilon}{\lambda_*}\,,\end{aligned}$$

we could easily obtain $h \simeq L \circ \Delta\colon \partial B[n,\varepsilon] \to \boldsymbol{R}^d \setminus \{\boldsymbol{0}\}$ by property (ii).

This proves the statement (iii) and then the proof of Theorem 2.4 is complete. □

REMARK 2.5. The choices of the constants λ_*, N_* and $\hat{\xi}$ rely only on the numbers $\hat{\eta}, \rho, \eta_1$, and do not depend upon the explicit form of $A(t)$, where $\|A^{\pm}\| \leq \hat{\eta}$ as in condition (b) before.

3. The generalized and improved shadowing lemma. This section is devoted to the proof of Theorem 2 stated in the Introduction using Theorem 2.4 and Lemma 1.3. There let $f \in \mathrm{Diff}^1_{\mathrm{loc}}(M)$, Λ be a forwardly invariant set of f, and let $T_\Lambda M = E \oplus F$ be a forwardly Df-invariant splitting. From now on, let $\varepsilon > 0$ and $0 < \rho < \lambda/2$, be arbitrarily given constants as in the statement of Theorem 2.

3.1. One technical lemma. The following technical lemma is a discrete form of Liao [7, Lemma 3.6]. For it, there can be read two different and simple proofs in [5, 13].

LEMMA 3.1. *Let $n > 1$ be an integer and $0 < \aleph \leq \mathfrak{b}$ two constants. Let*

$$h_-, h_+\colon \{0, 1, \dots, n\} \to \boldsymbol{R}$$

be two sequences such that for $k = 1, \dots, n$,

(1) $h_-(k) - h_-(0) \leq -\aleph k$;
(2) $h_+(n) - h_+(k-1) \geq \aleph(n-k+1)$;
(3) $[h_+(k) - h_+(k-1)] - [h_-(k) - h_-(k-1)] \geq 2\aleph$;
(4) $\max\{|h_-(k) - h_-(k-1)|, |h_+(k) - h_+(k-1)|\} \leq \mathfrak{b}$.

Then, there exists a sequence $h\colon \{0, 1, \dots, n+1\} \to \boldsymbol{R}$ such that:

(a) $0 = h(0) = h(n) = h(n+1) \geq h(t)$ *for* $t = 1, \dots, n-1$;
(b) $\sup_{0 \leq t \leq n}\{|h(t) - h(t+1)|\} \leq \mathfrak{b} - \aleph$;

(c) *for* $0 \le i < i + t \le n$,

$$[h_-(i+t) - h_-(i)] - [h(i+t) - h(i)] \le -t\aleph/2$$
$$[h_+(i+t) - h_+(i)] - [h(i+t) - h(i)] \ge t\aleph/2\,.$$

3.2. Proof of Theorem 2. Now, we are ready to prove our main result Theorem 2. We will divide our arguments into subsections.

3.2.1. At first, for some $x \in \Lambda$, assume that with respect to $T_xM = E(x) \oplus F(x)$, $1 \le \mathfrak{i} = \dim E(x) < d$, the forward orbit string $(x, f^n(x))$ is λ-quasi-hyperbolic of index $\mathfrak{i}$; that is to say,

$$\sum_{j=1}^{k} \log \|Df|_{E(f^{j-1}x)}\| \le -\lambda k$$
$$\sum_{j=k}^{n} \log \|Df|_{F(f^{j-1}x)}\|_{\mathrm{co}} \ge \lambda(n-k+1)$$
$$\log \|Df|_{F(f^{k-1}x)}\|_{\mathrm{co}} - \log \|Df|_{E(f^{k-1}x)}\| \ge 2\lambda$$

for each $k = 1, \dots, n$.

3.2.2. Put

$$h_-(k) = \begin{cases} 0 & \text{if } k = 0\,, \\ \sum_{j=1}^{k} \log \|Df|_{E(f^{j-1}x)}\| & \text{if } k = 1, \dots, n\,; \end{cases}$$
$$h_+(k) = \begin{cases} 0 & \text{if } k = 0, \\ \sum_{j=1}^{k} \log \|Df|_{F(f^{j-1}x)}\|_{\mathrm{co}} & \text{if } k = 1, \dots, n\,. \end{cases}$$

By Lemma 1.1 we can choose a constant $\mu_4 \ge \lambda$ such that

$$\max\{|h_-(k) - h_-(k-1)|,\ |h_+(k) - h_+(k-1)|\} \le \mu_4$$

for $k = 1, \dots, n$. Then by Lemma 3.1 instead of $\aleph$ and $\mathfrak{b}$ by λ and μ_4, respectively, we could obtain the sequence

$$h\colon \{0, 1, \dots, n+1\} \to \boldsymbol{R} \tag{3.1}$$

such that

$$0 = h(0) = h(n) = h(n+1) \ge h(t) \quad \text{for all } t = 1, \dots, n-1\,, \tag{3.2a}$$
$$\sup_{0 \le t \le n} |h(t) - h(t+1)| \le \mu_4 - \lambda\,, \tag{3.2b}$$

and, for all $0 \le i < i + t \le n$,

$$[h_-(i+t) - h(i+t)] - [h_-(i) - h(i)] \le -t\lambda/2\,, \tag{3.2c}$$
$$[h_+(i+t) - h(i+t)] - [h_+(i) - h(i)] \ge t\lambda/2\,. \tag{3.2d}$$

3.2.3. Now, under any given orthonormal d-frame γ at the base point x with $\mathrm{col}_k\gamma \in E(x)$ for every $1 \le k \le \mathrm{i}$, we consider the natural difference systems

$$y_{t+1} = R_{x,\gamma}(t)y_t\,, \quad (t, y_t) \in \boldsymbol{Z}_+ \times \boldsymbol{R}^d \tag{3.3a}$$

and

$$y_{t+1} = R_{x,\gamma}(t)y_t + f_{\mathrm{rem}(x,\gamma)}(t, y_t)\,, \quad (t, y_t) \in \boldsymbol{Z}_+ \times B(\theta) \tag{3.3b}$$

defined as in Section 1. After the change of variables

$$z_t = y_t \exp(-h(t)) \quad \text{for all } t \in \{0, 1, \dots, n+1\}, \tag{3.4}$$

we could obtain two corresponding systems of difference equations

$$z_{t+1} = A_{x,\gamma}(t)z_t\,, \quad t \in \{0, 1, \dots, n\}, z_t \in \boldsymbol{R}^d \tag{3.5a}$$

and

$$z_{t+1} = A_{x,\gamma}(t)z_t + g_{x,\gamma}(t, z_t)\,, \quad t \in \{0, 1, \dots, n\}, z_t \in e^{-h(t)}B(\theta)\,, \tag{3.5b}$$

where, for $t \in \{0, 1, \dots, n\}$,

$$A_{x,\gamma}(t) = e^{h(t)-h(t+1)}R_{x,\gamma}(t) \tag{3.5c}$$

and

$$g_{x,\gamma}(t, z_t) = e^{-h(t+1)}f_{\mathrm{rem}(x,\gamma)}(t, z_t e^{h(t)})\,. \tag{3.5d}$$

It is obvious that if $\{y(t, v)\}_{t\in\boldsymbol{Z}}$ is a solution of (3.3a), then $z(t, v) = e^{-h(t)}y(t, v)$ for all $0 \le t \le n+1$ is a local solution of (3.5a).

3.2.4. Put

$$\hat{\eta} = \exp(2\mu_4)\,. \tag{3.6}$$

Evidently, from Lemma 1.1 and Lemma 3.1(b), it follows that

$$\max_{0\le t\le n} \|A_{x,\gamma}(t)^{\pm 1}\| \le \hat{\eta} \quad \text{and} \quad \|g_{x,\gamma}(t, z_t)\| \le \hat{\eta}L_{f_{\mathrm{rem}(x,\gamma)}}\|z_t\|\,.$$

3.2.5. By the subadditivity and the definition of h_- and h_+, we easily see that, for any $\vec{u} \in E(x) \setminus \{\mathbf{0}_x\}$ and $0 \le i < i+t \le n$, one has

$$\log\|(Df^{i+t})\vec{u}\| - \log\|(Df^i)\vec{u}\| \le h_-(i+t) - h_-(i)\,,$$

and, for any $\vec{u} \in F(x) \setminus \{\mathbf{0}_x\}$ and $0 \le i < i+t \le n$, one has

$$\log\|(Df^{i+t})\vec{u}\| - \log\|(Df^i)\vec{u}\| \ge h_+(i+t) - h_+(i)\,.$$

Thus, by (3.2c) and (3.2d), we have

$$\log\|(Df^{i+t})\vec{u}\| - \log\|(Df^i)\vec{u}\| \le h(i+t) - h(i) - t\lambda/2 \tag{3.7a}$$

for $\vec{u} \in E(x) \setminus \{\mathbf{0}_x\}$ and $0 \le i < i+t \le n$, and

$$\log\|(Df^{i+t})\vec{u}\| - \log\|(Df^i)\vec{u}\| \ge h(i+t) - h(i) + t\lambda/2 \tag{3.7b}$$

for any $\vec{u} \in F(x) \setminus \{\mathbf{0}_x\}$ and any $0 \leq i < i + t \leq n$.

3.2.6. Put

$$E^s = \{z \in \boldsymbol{R}^d\,;\ \gamma z \in E(x)\} \quad \text{and} \quad E^u = \{z \in \boldsymbol{R}^d\,;\ \gamma z \in F(x)\}, \tag{3.8}$$

and

$$\eta_1 = \lambda/2 > 0\,. \tag{3.9}$$

Obviously, $\boldsymbol{R}^d = E^s \oplus E^u$ and $\boldsymbol{R}^{\mathfrak{i}} \times \{\mathbf{0}\} = E^s$ where $\mathbf{0}$ is the origin of $\boldsymbol{R}^{d-\mathfrak{i}}$. By (3.7), we can easily check that (3.5a) has the following property:

$$\|z_{A_{x,\gamma}}(i+t, v_-)\| \leq \|z_{A_{x,\gamma}}(i, v_-)\| \exp(-\eta_1 t) \tag{3.10a}$$

for any $v_- \in E^s$, $0 \leq i < i + t \leq n$, and

$$\|z_{A_{x,\gamma}}(i+t, v_+)\| \geq \|z_{A_{x,\gamma}}(i, v_+)\| \exp(\eta_1 t) \tag{3.10b}$$

for any $v_+ \in E^u$, $0 \leq i < i + t \leq n$.

3.2.7. For the constants $\hat{\eta}$ and η_1 defined by (3.6) and (3.9), respectively, from Theorem 2.4 we could choose constants

$$\lambda_* = \lambda_*(\hat{\eta}, \rho) > 1\,, \quad N_* = N_*(\hat{\eta}, \rho) \geq 2\,, \quad \hat{\xi} = \hat{\xi}(\hat{\eta}, \eta_1) \geq 1\,. \tag{3.11}$$

We note here that these constants λ_*, N_*, and $\hat{\xi}$ all are independent of the explicit form of (A, g) in Section 2, but depend only upon the numbers $\hat{\eta}$, ρ and η_1 with $\|A^{\pm}\| \leq \hat{\eta}$ in Theorem 2.4 (see Remark 2.5).

3.2.8. Now, define

$$A(t) = \begin{cases} A_{x,\gamma}(t) & \text{if } t \in \{0, 1, \dots, n\}\,, \\ \operatorname{diag}(\overbrace{e^{-\eta_1}, \dots, e^{-\eta_1}}^{\mathfrak{i}}, \overbrace{e^{\eta_1}, \dots, e^{\eta_1}}^{d-\mathfrak{i}}) & \text{if } t \notin \{0, 1, \dots, n\}\,. \end{cases}$$

Then

$$z_{t+1} = A(t) z_t\,, \quad (t, z_t) \in \boldsymbol{Z} \times \boldsymbol{R}^d\,, \tag{3.12}$$

is $(\eta_1, \mathfrak{i})$-hyperbolic, as in Definition 2.1.

3.2.9. Next, take a C^∞ bump function $\vartheta\colon \boldsymbol{R} \to [0, 1]$ such that

$$\vartheta|_{(-\infty, 1/4]} \equiv 1\,, \quad \vartheta|_{[1/2, \infty)} \equiv 0 \quad \text{and} \quad 4 \leq m_\vartheta := \sup_{t \in \boldsymbol{R}} |d\vartheta(t)/dt| < \infty\,.$$

Take

$$\mu = \min\left\{\frac{1}{\hat{\xi}(1 + 2\hat{\eta}\hat{\xi})^d}, e^{\eta_1} - e^{\rho}, e^{-\rho} - e^{-\eta_1}\right\} \min\left\{\frac{1}{16\lambda_*}, \frac{1}{2(1 + m_\vartheta)}\right\}, \tag{3.13}$$

then from Lemma 1.3, there exists a constant $\nu > 0$ with $\nu < \theta$, which is independent of the choice of (x, γ) but relies only on the value μ, such that

$$\sup_{t \in \boldsymbol{Z}_+, \|y\| \leq \nu} \|f_{\mathrm{rem}(x,\gamma)}(t, y)\| < \mu\nu\,, \tag{3.14}$$

and, uniformly for $t \in \boldsymbol{Z}_+$,

$$\|f_{\mathrm{rem}(x,\gamma)}(t, y) - f_{\mathrm{rem}(x,\gamma)}(t, y')\| \leq \mu\|y - y'\| \tag{3.15}$$

for all $y, y' \in B(\nu)$.

3.2.10. Put

$$g(t,z) \begin{cases} \vartheta(\|ze^{h(t)}\|/\nu) f_{\mathrm{rem}(x,\gamma)}(t, ze^{h(t)}) & \text{if } \|ze^{h(t)}\| < \nu \text{ and } t \geq 0\,, \\ \mathbf{0} & \text{if } \|ze^{h(t)}\| \geq \nu \text{ or } t < 0\,. \end{cases}$$

It is easily seen from (3.13), (3.14) and (3.15) that

$$\|g\| \leq \mu\nu \leq \frac{1}{4\lambda_* \hat{\xi}(1 + 2\hat{\eta}\hat{\xi})^d} \frac{\nu}{4}\,, \tag{3.16}$$

and

$$L_g \leq \min\left\{\frac{1}{\hat{\xi}(1 + 2\hat{\eta}\hat{\xi})^d},\ e^{\eta_1} - e^{\rho},\ e^{-\rho} - e^{-\eta_1}\right\}. \tag{3.17}$$

Next, we will consider the difference system

$$z_{t+1} = A(t)z_t + g(t, z_t)\,, \quad (t, z_t) \in \boldsymbol{Z} \times \boldsymbol{R}^d\,. \tag{3.18}$$

For any $v \in \boldsymbol{R}^d$, let $z_{x,\gamma}(t, v)$ be the solution of (3.18) with $z_{x,\gamma}(0, v) = v$.

Applying Theorem 2.4 to (3.18), for the λ-quasi-hyperbolic orbit string $(x, f^n(x))$ we could obtain the following assertion.

ASSERTION 3.2. *If $n \geq N_*$ and $0 < \varepsilon \leq \nu/8$, then for any orthonormal d-frame γ at the base point x with* $\mathrm{col}_k\gamma \in E(x)$ *for* $1 \leq k \leq \imath$, *there exists a d-dimensional topological closed ball $D_{x,\gamma}$ in $\boldsymbol{R}^d$ such that, for any $v \in D_{x,\gamma}$,* (3.18) *has a solution $z_{x,\gamma}(t, v)$ satisfying the following conditions.*

(i*) *If $l = [n/2]$ then*

$$|z_{x,\gamma}(t, v)| \leq \begin{cases} \varepsilon e^{-\rho t} & \textit{for } 0 \leq t \leq l\,, \\ \varepsilon e^{-\rho(n-t)} & \textit{for } l < t \leq n\,. \end{cases}$$

(ii*) *For all $v \in \partial D_{x,\gamma}$ and all $G \in O(\imath)$,*

$$|G(z_{x,\gamma}(n, v)) - v| \geq \frac{\varepsilon}{\lambda_*}.$$

(iii*) *For any $G \in O(\imath)$, the map*

$$h\colon \partial D_{x,\gamma} \to \boldsymbol{R}^d \setminus \{\mathbf{0}\}\,; \quad v \mapsto G(z_{x,\gamma}(n, v)) - v\,,$$

induces an isomorphism

$$h_*\colon H_{d-1}(\partial D_{x,\gamma}, \boldsymbol{Z}) \xrightarrow{\approx} H_{d-1}(\boldsymbol{R}^d \setminus \{\mathbf{0}\}, \boldsymbol{Z})\,.$$

3.2.11. By (3.4) and (3.2a), from Assertion 3.2 we can easily obtain the following assertion.

ASSERTION 3.3. *If $n \geq N_*$ and $0 < \varepsilon \leq \nu/8$, then for any orthonormal d-frame γ at the base point x with $\mathrm{col}_k\gamma \in E(x)$ for all $1 \leq k \leq \mathrm{i}$, there exists a d-dimensional topological closed ball $B_{x,\gamma}$ in $\boldsymbol{R}^d$ such that, for any $v \in B_{x,\gamma}$,* (3.3b) *has a solution $y_{x,\gamma}(t, v)$, $0 \leq t \leq n+1$ with $y_{x,\gamma}(0, v) = v$ satisfying the following conditions.*

(I*) *If $l = [n/2]$ then*

$$|y_{x,\gamma}(t,v)| \leq \begin{cases} \varepsilon e^{-\rho t} & \text{for } 0 \leq t \leq l\,, \\ \varepsilon e^{-\rho(n-t)} & \text{for } l < t \leq n\,. \end{cases}$$

(II*) *For all $v \in \partial B_{x,\gamma}$ and all $G \in O(\mathrm{i})$,*

$$|G^{\mathrm{T}}(y_{x,\gamma}(n,v)) - v| \geq \frac{\varepsilon}{\lambda_*}\,.$$

(III*) *For any $G \in O(\mathrm{i})$, the map*

$$\mathcal{H}\colon \partial B_{x,\gamma} \to \boldsymbol{R}^d \setminus \{\boldsymbol{0}\}; \quad v \mapsto G^{\mathrm{T}}(y_{x,\gamma}(n,v)) - v\,,$$

induces an isomorphism

$$\mathcal{H}_*\colon H_{d-1}(\partial B_{x,\gamma}, \boldsymbol{Z}) \xrightarrow{\approx} H_{d-1}(\boldsymbol{R}^d \setminus \{\boldsymbol{0}\}, \boldsymbol{Z})\,.$$

(IV*) *For any $(t, v) \in \{0, 1, \ldots, n\} \times B_{x,\gamma}$,*

$$\mathfrak{S}_{x,\gamma}(t)y_{x,\gamma}(t,v) = f^t(\mathfrak{S}_{x,\gamma}(0)v) \quad \text{or} \quad \exp_{f^t x}(\gamma_t y_{x,\gamma}(t,v)) = \exp_x(\gamma v)\,.$$

3.2.12. Since M is compact and $\exp\colon TM \to M$ is smooth, there holds the following assertion.

ASSERTION 3.4. *For the given x, there is a constant δ_* with $0 < \delta_* < \nu/8$ which satisfies the following. If $E(f^n x) \cap S_{f^n x}M \subset B_{\widetilde{\mathrm{dist}}}(E(x) \cap S_x M, \delta_*)$ then, for any orthonormal d-frame β at $f^n x$ with $\mathrm{col}_k\beta \in E(f^n x)$ for $1 \leq k \leq \dim E(f^n x)$, there can be found an orthonormal d-frame α at the base point x with $\mathrm{col}_k\alpha \in E(x)$ for $1 \leq k \leq \dim E(x)$, such that $\|v_1 - v_2\| \leq \varepsilon/(2\lambda_*)$ whenever $v_1, v_2 \in B(\theta)$, $\exp_x(\alpha v_1) = \exp_{f^n x}(\beta v_2)$.*

In what follows, assume that $(x, f^n(x))$ in Λ, $n \geq N_*$, is λ-quasi-hyperbolic respecting the splitting $T_x M = E(x) \oplus F(x)$ and let $E(f^n x) \cap S_{f^n x}M \subset B_{\widetilde{\mathrm{dist}}}(E(x) \cap S_x M, \delta_*)$.

3.2.13. For the d-frame γ at the base point x given as in Assertion 3.3, let γ_n be defined as in Figure 1 of Section 1, which is an orthonormal d-frame at $f^n x$ such that $\mathrm{col}_k\gamma_n \in E(f^n x)$ for all $1 \leq k \leq \mathrm{i}$. Then from Assertion 3.4, we could take some orthonormal d-frame γ' at x such that

$$\text{(3.19)} \qquad \|v_1 - v_2\| \leq \frac{\varepsilon}{2\lambda_*} \quad \text{whenever } v_1, v_2 \in B(\theta) \text{ and } \exp_x(\gamma' v_1) = \exp_{f^n x}(\gamma_n v_2)\,.$$

Put $G = \big(\langle \mathrm{col}_i\gamma, \mathrm{col}_j\gamma'\rangle_x\big)_{d\times d}$. Clearly, $G \in O(\mathrm{i})$, $\gamma' = \gamma G$ where we think of γ' and γ as row vectors, and $\exp_x(\gamma' v) = \exp_x(\gamma G v)$ for all $v \in B(\theta)$.

3.2.14. Now, we take a topological closed ball $B_{x,\gamma}$ as in Assertion 3.3. It is clear by (I*) that $B_{x,\gamma} \subset B(\nu/8)$ and $y_{x,\gamma}(n, v) \in B(\nu/8)$ for all $v \in B_{x,\gamma}$, since there we have $\varepsilon < \nu/8$.

Next, we shall verify another assertion.

ASSERTION 3.5. *There exists a vector $v_x \in B_{x,\gamma}$ such that*

$$\exp_x(\gamma v_x) = \exp_{f^n x}(\gamma_n y_{x,\gamma}(n, v_x)) .$$

PROOF. Indeed, for any given $v \in B_{x,\gamma}$, we may exactly take one corresponding point $\hat{v} \in B(\nu/4)$ such that $\exp_x(\gamma\hat{v}) = \exp_{f^n x}(\gamma_n y_{x,\gamma}(n, v))$. Since the exponential projection exp is smooth, $\hat{v}$ is continuous with respect to $v \in B_{x,\gamma}$ such that

$$\|G\hat{v} - y_{x,\gamma}(n, v)\| \leq \frac{\varepsilon}{2\lambda_*}$$

by Assertion 3.4. Thus,

$$\|G^{\mathrm{T}}(y_{x,\gamma}(n, v)) - \hat{v}\| \leq \frac{\varepsilon}{2\lambda_*} \quad \text{for all } v \in B_{x,\gamma} . \tag{3.20}$$

Define $\widetilde{\mathcal{H}}\colon B_{x,\gamma} \to \boldsymbol{R}^d$ by $v \mapsto \hat{v} - v$. By (II*) of Assertion 3.3 and (3.20), we get

$$\widetilde{\mathcal{H}}|_{\partial B_{x,\gamma}} \simeq \mathcal{H}\colon \partial B_{x,\gamma} \to \boldsymbol{R}^d \setminus \{\boldsymbol{0}\} .$$

Then from (III*) of Assertion 3.3, it follows that

$$\big(\widetilde{\mathcal{H}}|_{\partial B_{x,\gamma}}\big)_*\colon H_{d-1}(\partial B_{x,\gamma}, \boldsymbol{Z}) \xrightarrow{\approx} H_{d-1}(\boldsymbol{R}^d \setminus \{\boldsymbol{0}\} , \boldsymbol{Z}),$$

which implies that there must exist a point $v_x \in B_{x,\gamma}$ such that $\widetilde{\mathcal{H}}(v_x) = \boldsymbol{0} \in \boldsymbol{R}^d$, the origin. Otherwise, $\widetilde{\mathcal{H}}|_{\partial B_{x,\gamma}}$ is homotopic to the constant map $\boldsymbol{0}$ by the fact that $\widetilde{\mathcal{H}}$ is defined on $B_{x,\gamma}$. That is to say, $\hat{v}_x = v_x$. □

3.2.15. Let $b = \exp_x(\gamma v_x)$. Then $b = f^n b$ from (IV*), and

$$\mathrm{dist}(f^i x, f^i b) = \|y_{x,\gamma}(t, v_x)\|$$

for all $0 \leq i \leq n$.

3.2.16. If Λ is a closed neighborhood of the motion $f(t, x)$ and the splitting $E \oplus F$ is continuous on it, then from the Alexseev theorem [3, Lemma 3.6], we can further require b to be hyperbolic of index $\dim E(x)$ and each of the Lyapunov exponents of b to lie off the interval $[-\lambda/2, \lambda/2]$.

This proves Theorem 2.

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