

OPTIMAL NORM ESTIMATE OF OPERATORS RELATED TO THE HARMONIC BERGMAN PROJECTION ON THE BALL

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Abstract. We first obtain an optimal norm estimate for one-parameter family of operators associated with the weighted harmonic Bergman projections on the ball. We then use this result and derive an optimal norm estimate for the weighted harmonic Bergman projections.

Introduction. For a fixed positive integer $n \geq 2$, let B denote the open unit ball in \mathbf{R}^n . For $\alpha > -1$, we denote by dV_α the weighted measure defined by $dV_\alpha(x) = \lambda_\alpha(1 - |x|^2)^\alpha dx$ where dx is the Lebesgue volume measure on B . The constant λ_α is chosen so that $V_\alpha(B) = 1$, i.e.,

$$\lambda_\alpha = \frac{2}{n|B|} \cdot \frac{\Gamma(n/2 + \alpha + 1)}{\Gamma(n/2)\Gamma(\alpha + 1)}$$

where $|B|$ denotes the volume of B .

Given $\alpha > -1$ and $1 \leq p < \infty$, the weighted harmonic Bergman space $b_\alpha^p = b_\alpha^p(B)$ is the space of all complex-valued harmonic functions u on B such that

$$\|u\|_{L_\alpha^p} = \left(\int_B |u|^p dV_\alpha \right)^{1/p} < \infty.$$

As is well known, each b_α^p is a closed subspace of the Lebesgue space $L_\alpha^p = L_\alpha^p(B, dV_\alpha)$ and thus is a Banach space. In particular, b_α^2 is a Hilbert space for each α . By mean value property of harmonic functions, it is easily seen that point evaluations are continuous on b_α^2 . Thus, to each $x \in B$, there corresponds a unique $R_\alpha(x, \cdot) \in b_\alpha^2$ which has the following reproducing property:

$$(1) \quad u(x) = \int_B u(y) R_\alpha(x, y) dV_\alpha(y), \quad x \in B$$

for all $u \in b_\alpha^2$. The kernel $R_\alpha(x, y)$, called the reproducing kernel for b_α^2 , is real and symmetric; see (6) below.

Note that the Hilbert space orthogonal projection from L_α^2 onto b_α^2 , denoted by Π_α and called the α -weighted harmonic Bergman projection, can be realized as an integral operator

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by (1):

$$\Pi_\alpha \psi(x) = \int_B \psi(y) R_\alpha(x, y) dV_\alpha(y), \quad x \in B$$

for functions $\psi \in L^2_\alpha$. For each $\alpha > -1$, it is well known that Π_α is bounded on L^p_α for $1 < p < \infty$, but not for $p = 1$.

The growth rate of the kernel $R_\alpha(x, y)$ is well known (see [4]):

$$(2) \quad |R_\alpha(x, y)| \leq \frac{C_\alpha}{[x, y]^{n+\alpha}}$$

for all $x, y \in B$. This motivates us to consider an operator with the kernel $[x, y]^{-(n+\alpha)}$. More explicitly, we define an operator Λ_α by

$$\Lambda_\alpha \psi(x) = \int_B \frac{\psi(y)}{[x, y]^{n+\alpha}} dV_\alpha(y), \quad x \in B$$

for functions ψ that make the integral well defined. For each $\alpha > -1$, the operator Λ_α is also bounded on L^p_α if and only if $1 < p < \infty$. This must have been known to experts, even though we could not find an explicit reference in the literature.

Our first result is the following optimal norm estimates of Λ_α acting on L^p_α when p varies throughout the full range and when α is fixed:

$$(3) \quad \|\Lambda_\alpha\|_p \approx \frac{p^2}{p-1} \quad (\alpha \text{ fixed})$$

for all $1 < p < \infty$. Here and in elsewhere, $\|T\|_p$ denotes the operator norm of a bounded linear operator $T : L^p_\alpha \rightarrow L^p_\alpha$. Also, for positive quantities X and Y , the notation $X \approx Y$ means that X/Y is bounded below and above by some positive constant that depends only on allowed parameters.

Note $|\Pi_\alpha \psi| \leq C_\alpha \Lambda_\alpha |\psi|$. Thus (3) gives an upper estimate of the norm $\|\Pi_\alpha\|_p$ of Π_α acting on L^p_α . When α is fixed, we show that such an estimate for $\|\Pi_\alpha\|_p$ is optimal, which is our second result:

$$(4) \quad \|\Pi_\alpha\|_p \approx \frac{p^2}{p-1} \quad (\alpha \text{ fixed})$$

for all $1 < p < \infty$. Earlier, Zhu [9] obtained the same estimate in the context of holomorphic Bergman spaces over the unit ball of \mathbf{C}^n . Zhu's result was then extended by the last two authors [5] to the setting of harmonic Bergman spaces over the upper half-space.

In Section 1 we express the weighted harmonic Bergman kernel by means of fractional derivatives. Such an integral representation allows us to estimate the weighted harmonic Bergman kernels. Section 2 is devoted to the proof of (3) with some additional information on behavior as $\alpha \rightarrow -1$; see Theorem 2.2. In fact Theorem 2.2 states that the dependency of the upper estimate on parameter α as $\alpha \rightarrow -1$ is at most like $(\alpha + 1)^{-1}$. However, we do not know whether such behavior with parameter α is sharp. In Section 3 we establish an estimate, with the help of (3), that implies the upper estimate of (4). Most part of the section is devoted to a careful analysis of the behavior of C_α in (2) as $\alpha \rightarrow -1$. Our estimate shows that C_α stays bounded as $\alpha \rightarrow -1$. So, the growth rate of $\|\Pi_\alpha\|_p$ as $\alpha \rightarrow -1$ is at most like

$(\alpha + 1)^{-1}$. However, as in the case of $\|A_\alpha\|_p$, we do not know whether such behavior with parameter α is sharp, either. In Section 4 we establish the lower estimate of (4).

CONSTANTS. Throughout the paper we use the same letter C , always depending on the dimension n , to denote various positive constants which may change at each occurrence. Those constants will often depend on other parameters such as α and p as well. Such additional dependency will be explicitly indicated in subscripts. For example, C_α will stand for constants depending only on n and α .

1. Weighted harmonic Bergman kernel. The series expansion of the unweighted kernel function $R_0(x, y)$ is well known:

$$(5) \quad R_0(x, y) = \frac{2}{n} \sum_{k=0}^{\infty} \left(k + \frac{n}{2}\right) Z_k(x, y), \quad x, y \in B$$

where $Z_k(x, y)$ denotes extended zonal harmonics of order k ; see [1, Theorem 8.9] where some extra constant factor, caused by non-normalization, appears. We refer to [1, Chapter 5] for definition of zonal harmonics and related facts. However, we would like to mention that each $Z_k(x, y)$ has the k -th order homogeneity in each variable separately, which we will use later.

The series expansion (5) can be easily modified to the weighted cases. More explicitly, we have

$$(6) \quad R_\alpha(x, y) = \omega_\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k + n/2 + \alpha + 1)}{\Gamma(k + n/2)} Z_k(x, y)$$

for general $\alpha > -1$ where $\omega_\alpha = \Gamma(n/2)/\Gamma(n/2 + \alpha + 1)$; see [4, Section 2] or [7, Proposition 3]. This series converges absolutely and uniformly on $K \times B$ for every compact set $K \subset B$. When $n = 2$, note that (6) can be written (in complex notation) in a closed form

$$(7) \quad R_\alpha(x, y) = \frac{1}{(1 - x\bar{y})^{\alpha+2}} + \frac{1}{(1 - \bar{x}y)^{\alpha+2}} - 1.$$

The series expansion (5) is also well known to be closely related to the extended Poisson kernel $P(x, y)$ for B defined by

$$P(x, y) = \frac{1 - |x|^2|y|^2}{[x, y]^n}, \quad x, y \in B.$$

Here and elsewhere, we let

$$[x, y] = \sqrt{1 - 2x \cdot y + |x|^2|y|^2}$$

to simplify the notation. Using the series expansion

$$(8) \quad P(x, y) = \sum_{k=0}^{\infty} Z_k(x, y)$$

and utilizing homogeneity of extended zonal harmonics, one can verify by a straightforward calculation

$$(9) \quad R_m(x, y) = \omega_m [\partial_r^{m+1} (r^{n/2+m} P(x, ry))]_{r=1}$$

for integers $m \geq 0$ where $\partial_r = \partial/\partial r$. We remark in passing that (9) (with $m = 0$) also leads to the explicit formula for $R_0(x, y)$ as in [1, Theorem 8.13].

Here, we need a versions of (9) for general $\alpha > -1$, which is not necessarily an integer. To this end we first introduce a notion of fractional derivatives. Given s real, denote by $[s]$ the smallest integer bigger than s . For $s > 0$ and an $[s]$ -times continuously differentiable function f on $[0, 1)$, we define the fractional derivative $\mathcal{D}^s f$ of order s by

$$\mathcal{D}^s f(r) = \frac{r^{-s}}{\Gamma([s] - s)} \int_0^1 (1 - t)^{[s]-s-1} \partial_t^{[s]} [f(rt)] dt .$$

Now, we have the following representation of weighted harmonic Bergman kernels for $n \geq 3$ in terms of fractional derivatives. Recall that for $n = 2$ we have the explicit formula (7).

PROPOSITION 1.1 ($n \geq 3$). *Let $\alpha > -1$. Then*

$$\begin{aligned} R_\alpha(x, y) &= \omega_\alpha [\mathcal{D}_r^{\alpha+1} (r^{n/2+\alpha} P(x, ry))]_{r=1} \\ &= \frac{\omega_\alpha}{\Gamma([\alpha] - \alpha)} \int_0^1 (1 - t)^{[\alpha]-\alpha-1} \partial_t^{[\alpha]+1} [t^{n/2+\alpha} P(x, ty)] dt \end{aligned}$$

for $x, y \in B$.

PROOF. Fix $x, y \in B$ and put

$$f(r) = r^{n/2+\alpha} P(x, ry), \quad 0 \leq r < 1 .$$

In case α is an integer, note that $f^{(\alpha+1)}(0) = 0$, because $n \geq 3$. So, the claimed integral representation reduces to

$$\omega_\alpha \int_0^1 f^{(\alpha+2)}(t) dt = \omega_\alpha f^{(\alpha+1)}(1),$$

which is the same as (9).

Now, assume that α is not an integer. So, let $\alpha = m + \varepsilon$ where m is an integer and $0 < \varepsilon < 1$. Note $[\alpha] + 1 = m + 2$ and $[\alpha] - \alpha = 1 - \varepsilon$. Using (8) and homogeneity of $Z_k(x, \cdot)$, we have

$$f(r) = \sum_{k=0}^{\infty} r^{k+n/2+\alpha} Z_k(x, y)$$

so that

$$\partial_t^{m+2} [f(rt)] = r^{n/2+\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(k + n/2 + \alpha + 1)}{\Gamma(k + n/2 + \alpha - m - 1)} t^{k+n/2+\alpha-m-2} Z_k(x, ry)$$

for $r, t \in [0, 1)$. Note $k + n/2 + \alpha - m - 1 > 0$ for all $k \geq 0$, because $\alpha > m$. Thus, multiplying both sides of the above by $(1 - t)^{-\varepsilon}$ and then integrating term by term against the

measure dt , we have

$$\begin{aligned} \mathcal{D}^{\alpha+1} f(r) &= \frac{r^{-(\alpha+1)}}{\Gamma(1-\varepsilon)} \int_0^1 (1-t)^{-\varepsilon} \partial_t^{m+2} [f(rt)] dt \\ &= r^{n/2-1} \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+n/2+\alpha+1)}{\Gamma(k+n/2+\alpha-m-1)} Z_k(x, ry) \end{aligned}$$

where

$$a_k = \frac{1}{\Gamma(1-\varepsilon)} \int_0^1 (1-t)^{-\varepsilon} t^{k+n/2+\alpha-m-2} dt.$$

Also, recall $\alpha = m + \varepsilon$. Thus we have

$$a_k = \frac{\Gamma(k+n/2+\alpha-m-1)}{\Gamma(k+n/2)}$$

for each $k \geq 0$. Combining these observations, we obtain

$$\begin{aligned} \mathcal{D}^{\alpha+1} f(r) &= r^{n/2-1} \sum_{k=0}^{\infty} \frac{\Gamma(k+n/2+\alpha+1)}{\Gamma(k+n/2)} Z_k(x, ry) \\ &= \omega_{\alpha}^{-1} r^{n/2-1} R_{\alpha}(x, ry) \end{aligned}$$

where the second equality comes from (6). Now, evaluating both sides of the above at $r = 1$, we have the first equality. For the second equality, note that $f(rt) = (rt)^{n/2+\alpha} P(x, rty)$ and thus

$$\begin{aligned} \mathcal{D}^{\alpha+1} f(r) &= \frac{r^{-(\alpha+1)}}{\Gamma(1-\varepsilon)} \int_0^1 (1-t)^{-\varepsilon} \partial_t^{m+2} [f(rt)] dt \\ &= \frac{r^{n/2-1}}{\Gamma(1-\varepsilon)} \int_0^1 (1-t)^{-\varepsilon} \partial_t^{m+2} [t^{n/2+\alpha} P(x, rty)] dt. \end{aligned}$$

So, evaluating both sides of the above at $r = 1$, we have the second equality. The proof is complete. \square

2. Estimate of $\|A_{\alpha}\|_p$. In this section we prove the optimal norm estimate asserted in (3). Given $\alpha > -1$ and $s > 0$, consider the function $J_{\alpha,s}$ defined by

$$J_{\alpha,s}(x) = \int_B \frac{(1-|y|^2)^{\alpha} dy}{[x, y]^{n+\alpha+s}}$$

for $x \in B$. It is known that $J_{\alpha,s}(x)$ grows like $(1-|x|^2)^{-s}$ as $|x| \rightarrow 1$; see, for example, [2, Lemma 2.5]. However, such boundedness is not enough for our purpose. In order to obtain our optimal upper estimate, we need to keep track of how the growth rate of $J_{\alpha,s}(x)$ depends on parameters α and s .

Given a, b, c real with $c \neq 0, -1, -2, \dots$, let $F(a, b, c; t)$ be the hypergeometric function

$$F(a, b, c; t) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{t^j}{j!}$$

for $0 \leq t < 1$. Here, $(a)_0 = 1$ and $(a)_k = a(a+1) \cdots (a+k-1)$ for $k \geq 1$ as usual. The role of this hypergeometric function in our argument lies in the following equality (see [6, Lemma 2.1]) for c real:

$$(10) \quad \int_{\partial B} \frac{d\sigma(\xi)}{|x - \xi|^c} = F\left(\frac{c}{2}, \frac{c-n}{2} + 1, \frac{n}{2}; |x|^2\right), \quad x \in B$$

where $d\sigma$ denotes the normalized surface area measure on ∂B .

LEMMA 2.1. *Given $v > 0$, there is a constant $C_v > 0$ such that*

$$C_v^{-1} \Gamma(s) \Gamma(\alpha + 1) \leq \frac{J_{\alpha,s}(x) - J_{\alpha,s}(0)}{(1 - |x|^2)^{-s} - 1} \leq C_v \Gamma(s) \Gamma(\alpha + 1), \quad x \in B, \quad x \neq 0$$

for $0 < \alpha + 1 < v$ and $0 < s < v$.

PROOF. Let $x \in B$. Integrating in polar coordinates and then using (10), we have

$$(11) \quad \begin{aligned} J_{\alpha,s}(x) &= n|B| \int_0^1 (1 - t^2)^\alpha t^{n-1} F\left(\frac{n + \alpha + s}{2}, \frac{\alpha + s}{2} + 1, \frac{n}{2}; t^2|x|^2\right) dt \\ &= \frac{n|B|\Gamma(n/2)}{2\Gamma((n + \alpha + s)/2)\Gamma((\alpha + s)/2 + 1)} \sum_{k=0}^\infty \frac{a_k}{k!} |x|^{2k} \end{aligned}$$

where

$$\begin{aligned} a_k &= \frac{\Gamma((n + \alpha + s)/2 + k)\Gamma((\alpha + s)/2 + 1 + k)}{\Gamma(n/2 + k)} \left\{ 2 \int_0^1 (1 - t^2)^\alpha t^{n+2k-1} dt \right\} \\ &= \Gamma(\alpha + 1) \frac{\Gamma((n + \alpha + s)/2 + k)\Gamma((\alpha + s)/2 + 1 + k)}{\Gamma(n/2 + \alpha + 1 + k)} \end{aligned}$$

for each $k \geq 0$. On the other hand, we have

$$(12) \quad \frac{1}{(1 - |x|^2)^s} = \frac{1}{\Gamma(s)} \sum_{k=0}^\infty \frac{\Gamma(s + k)}{k!} |x|^{2k}.$$

Recall that, by Sterling’s formula, to each $a \geq 0$ corresponds a constant C_a such that

$$C_a^{-1} < \frac{\Gamma(t + a)}{t^{t+a-1/2}e^{-t}} < C_a$$

for $t \geq 1$. Using this, one may verify that

$$(13) \quad C_{\alpha,s}^{-1} < \frac{a_k}{\Gamma(s + k)\Gamma(\alpha + 1)} < C_{\alpha,s}$$

for all $k \geq 1$ where $C_{\alpha,s} > 0$ is a constant continuously depending on α, s and staying bounded as $\alpha \rightarrow -1$ and $s \rightarrow 0$. Also, note that the constant factor in (11) stays bounded as $\alpha \rightarrow -1$ and $s \rightarrow 0$. Thus we conclude the lemma from (11), (12) and (13). The proof is complete. \square

THEOREM 2.2. *Given $v > 0$, there is a constant $C_v > 0$ such that*

$$C_v^{-1} \frac{p^2}{p - 1} \leq \|A_\alpha\|_p \leq C_v \frac{p^2}{(\alpha + 1)(p - 1)}$$

for $1 < p < \infty$ and $0 < \alpha + 1 < \nu$.

PROOF. Fix $\nu > 0$. Let $1 < p < \infty$ and assume $0 < \alpha + 1 < \nu$. We first establish the upper estimate. We use the well known Shur's test (see [8, Theorem 3.6]). As a test function we take the function $h(x) := (1 - |x|^2)^{-(\alpha+1)/pq}$ where q is the conjugate exponent of p . By Lemma 2.1 we have an estimate

$$\begin{aligned} \int_B \frac{h(x)^p dV_\alpha(x)}{[x, y]^{n+\alpha}} &= \lambda_\alpha J_{(\alpha+1)/p-1, (\alpha+1)/q}(y) \\ &\leq C_\nu \lambda_\alpha \Gamma\left(\frac{\alpha+1}{p}\right) \Gamma\left(\frac{\alpha+1}{q}\right) h(y)^p \end{aligned}$$

for $y \in B$. Similarly, we have

$$\int_B \frac{h(y)^q dV_\alpha(y)}{[x, y]^{n+\alpha}} \leq C_\nu \lambda_\alpha \Gamma\left(\frac{\alpha+1}{p}\right) \Gamma\left(\frac{\alpha+1}{q}\right) h(x)^q$$

for $x \in B$. Thus we obtain by Shur's test

$$\begin{aligned} \|\Lambda_\alpha\|_p &\leq C_\nu \lambda_\alpha \Gamma\left(\frac{\alpha+1}{p}\right) \Gamma\left(\frac{\alpha+1}{q}\right) \\ &\leq C_\nu \frac{1}{\Gamma(\alpha+1)} \Gamma\left(\frac{\alpha+1}{p}\right) \Gamma\left(\frac{\alpha+1}{q}\right) \\ &= C_\nu \int_0^1 t^{(\alpha+1)/p-1} (1-t)^{(\alpha+1)/q-1} dt. \end{aligned}$$

On the other hand, we have

$$\int_0^1 t^{(\alpha+1)/p-1} (1-t)^{(\alpha+1)/q-1} dt \leq C_\nu \frac{p+q}{\alpha+1} = C_\nu \frac{p^2}{(\alpha+1)(p-1)};$$

the first inequality can be easily verified by splitting the integral into two pieces $\int_0^{1/2} + \int_{1/2}^1$.

We now turn to the lower estimate. Since the norm $\|\Lambda_\alpha\|_p$ should continuously depend on parameters p and α , we may assume $\alpha < 0$ and $p \neq 2$ for the rest of the proof.

First, consider the case where $1 < p < 2$. It suffices to establish

$$(14) \quad \|\Lambda_\alpha\|_p \geq \frac{C_\nu}{p-1}.$$

To prove this, we use the test function $f(x) := (1 - |x|^2)^{-s}$ where $s = (\alpha + 1)(2 - p)$. Note $0 < s < 1$ (because $\alpha < 0$) and $\alpha - s + 1 > \alpha - ps + 1 > 0$. Since $\Gamma(\alpha - s + 1) \approx (\alpha - s + 1)^{-1} = 1/(\alpha + 1)(p - 1)$ and $\Gamma(\alpha + 1)(\alpha + 1) = \Gamma(\alpha + 2) \approx 1$, we have by Lemma 2.1

$$(15) \quad \Lambda_\alpha f(x) = \lambda_\alpha J_{\alpha-s, s}(x) \geq \frac{C_\nu \Gamma(s)}{p-1} [f(x) - 1]$$

for $x \in B$. For the norm of f , a straightforward calculation yields

$$\|f\|_{L_\alpha^p}^p = c_n \lambda_\alpha \frac{\Gamma(\alpha - ps + 1)}{\Gamma(n/2 + \alpha - ps + 1)}$$

where c_n is a dimensional constant. On the other hand, using the elementary inequality $1 - u^s \geq s(1 - u)$ for $0 \leq u \leq 1$ and $0 < s < 1$, we have

$$\begin{aligned} \|f - 1\|_{L^\alpha}^p &= \lambda_\alpha \int_B (1 - |x|^2)^{\alpha-ps} [1 - (1 - |x|^2)^s]^p dx \\ &= \frac{c_n \lambda_\alpha}{2} \int_0^1 (1 - t)^{n/2-1} t^{\alpha-ps} (1 - t^s)^p dt \\ &\geq \frac{c_n \lambda_\alpha}{2} s^p \int_0^1 (1 - t)^{n/2+p-1} t^{\alpha-ps} dt \\ &= \frac{c_n \lambda_\alpha}{2} \frac{s^p \Gamma(n/2 + p) \Gamma(\alpha - ps + 1)}{\Gamma(n/2 + \alpha + p - ps + 1)}. \end{aligned}$$

It follows that

$$\frac{\|f - 1\|_{L^\alpha}^p}{\|f\|_{L^\alpha}^p} \geq s \left\{ \frac{\Gamma(n/2 + p) \Gamma(n/2 + \alpha - ps + 1)}{2\Gamma(n/2 + \alpha + p - ps + 1)} \right\}^{1/p}.$$

This, together with (15), yields

$$(16) \quad \|\Lambda_\alpha\|_p \geq \frac{C_v}{p-1} \left\{ \frac{\Gamma(n/2 + p) \Gamma(n/2 + \alpha - ps + 1)}{\Gamma(n/2 + \alpha + p - ps + 1)} \right\}^{1/p},$$

because $s\Gamma(s) = \Gamma(s + 1) \approx 1$ and $2^{-1/p} \approx 1$. Note that the expression in the bracket of (16) stays bounded below and above as $\alpha \rightarrow -1$ and $p \rightarrow 1$. Thus we obtain the desired estimate for $1 < p < 2$.

We now consider the case where $2 < p < \infty$. Let $2 < p < \infty$ and q be the conjugate exponent of p . Then we have by duality

$$\begin{aligned} \|\Lambda_\alpha\|_p &= \sup_{\|\varphi\|_{L^\alpha}^p=1} \|\Lambda_\alpha \varphi\|_{L^\alpha} \\ &= \sup_{\|\varphi\|_{L^\alpha}^p=1} \sup_{\|\psi\|_{L^\alpha}^q=1} \left| \int_B (\Lambda_\alpha \varphi) \overline{\psi} dV_\alpha \right| \\ &= \sup_{\|\psi\|_{L^\alpha}^q=1} \sup_{\|\varphi\|_{L^\alpha}^p=1} \left| \int_B \varphi \overline{\Lambda_\alpha \psi} dV_\alpha \right| \\ &= \sup_{\|\psi\|_{L^\alpha}^q=1} \|\Lambda_\alpha \psi\|_{L^\alpha} \\ &= \|\Lambda_\alpha\|_q. \end{aligned}$$

Thus we have by (14)

$$\|\Lambda_\alpha\|_p \geq \frac{C_v}{q-1}.$$

Since $4/(q-1) \geq q^2/(q-1) = p^2/(p-1)$, this completes the proof for $2 < p < \infty$. The proof is complete. □

3. Upper estimate of $\|\Pi_\alpha\|_p$ when $\alpha \rightarrow -1$. Recall $|\Pi_\alpha \psi| \leq C_\alpha \Lambda_\alpha |\psi|$. Thus, as an immediate consequence of Theorem 2.2, we obtain

$$\|\Pi_\alpha\|_p \leq C_\alpha \frac{p^2}{p-1} \quad (\alpha \text{ fixed})$$

for $1 < p < \infty$. However, this estimate does not provide any information on how $\|\Pi_\alpha\|_p$ behaves when α varies with p fixed, especially when $\alpha \rightarrow -1$. In this section we intend to show that the constant C_α above grows at most as fast as $(\alpha + 1)^{-1}$ when $\alpha \rightarrow -1$. We do not know whether such a growth rate is sharp, but suspect probably *not*. In fact Zhu [9] conjectured the boundedness of C_α when $\alpha \rightarrow -1$ in the context of holomorphic Bergman spaces over the unit disk and we agree with him.

We begin with an elementary integral estimate which is also well known except for constant factors. While one may use series expansion and Sterling's formula (as in the proof of Lemma 2.1) to obtain a more precise upper bound, the upper bound as stated is enough for our purpose.

LEMMA 3.1. *Let $\alpha > -1$, $s > 0$ and $0 \leq \varepsilon < 1$. Then*

$$\int_0^1 \frac{(1-r)^\alpha}{r^\varepsilon (1-tr)^{\alpha+s+1}} dr \leq \varepsilon^{-\varepsilon} \left(\frac{1}{s} + \frac{1}{\alpha+1} + \frac{\varepsilon}{(1-\varepsilon)^2} \right) (1-t)^{-s}$$

for $0 \leq t < 1$. Here, $\varepsilon^{-\varepsilon}$ is understood to be 1 for $\varepsilon = 0$.

PROOF. Let

$$I_\varepsilon(t) := \int_0^1 \frac{(1-r)^\alpha}{r^\varepsilon (1-tr)^{\alpha+s+1}} dr, \quad 0 \leq t < 1.$$

For $\varepsilon = 0$, integrating by parts, we have

$$I_0(t) = \frac{1}{\alpha+1} + \frac{\alpha+s+1}{\alpha+1} \int_0^1 \frac{t(1-r)^{\alpha+1}}{(1-tr)^{\alpha+s+2}} dr.$$

Note

$$\int_0^1 \frac{t(1-r)^{\alpha+1}}{(1-tr)^{\alpha+s+2}} dr \leq \int_0^1 \frac{t dr}{(1-tr)^{s+1}} = \frac{(1-t)^{-s} - 1}{s}.$$

Accordingly, we have

$$I_0(t) \leq \frac{1}{\alpha+1} + \frac{\alpha+s+1}{s(\alpha+1)} [(1-t)^{-s} - 1] = \left(\frac{1}{s} + \frac{1}{\alpha+1} \right) (1-t)^{-s} - \frac{1}{s},$$

which implies the inequality for $\varepsilon = 0$. For $0 < \varepsilon < 1$, we have

$$\begin{aligned} I_\varepsilon(t) &= \int_0^\varepsilon + \int_\varepsilon^1 \\ &\leq \frac{(1-t)^{-s}}{1-\varepsilon} \int_0^\varepsilon \frac{dr}{r^\varepsilon} + \varepsilon^{-\varepsilon} I_0(t) \\ &= \frac{\varepsilon^{1-\varepsilon}}{(1-\varepsilon)^2} (1-t)^{-s} + \varepsilon^{-\varepsilon} I_0(t), \end{aligned}$$

which yields the inequality for $0 < \varepsilon < 1$. The proof is complete. □

A proof of the following lemma for $\alpha = 0$ can be found in [3, Proposition 2.2]. However, the proof therein does not seem to extend to general α . Here, we provide a proof for general α by modifying the argument in the proof of [4, Lemma 2.7].

LEMMA 3.2. *Let $\alpha > -1$ and $s > 0$. Then*

$$\int_0^1 \frac{(1-t)^\alpha dt}{[x, ty]^{s+\alpha+1}} \leq 4 \cdot 3^{s+\alpha+1} \left(\frac{1}{s} + \frac{1}{\alpha+1} \right) [x, y]^{-s}$$

for $x, y \in B$.

PROOF. Fix $x, y \in B$. If $|x||y| < 1/2$, then $[x, ty] \geq 1 - t|x||y| > 1 - |x||y| > 1/2$ for all $0 \leq t < 1$. Also, if $x \cdot y \leq 0$, then $[x, ty] \geq 1$ for all $0 \leq t < 1$. So, the estimate is trivial if either $|x||y| < 1/2$ or $x \cdot y \leq 0$.

Now, assume $|x||y| > 1/2$ and $x \cdot y > 0$. Let $\theta = \theta(x, y) \geq 0$ be the half of the positive angle between x and y . Note $\theta < \pi/4$. Put $c = (|y||x|)^{-1}(1 - \sin \theta)$. First, consider the case $c \geq 1$. Note

$$(17) \quad [x, y]^2 = (1 - |x||y|)^2 + 4|x||y| \sin^2 \theta.$$

Thus $[x, y] < 3(1 - |x||y|)$, because $c \geq 1$. It follows from Lemma 3.1 that

$$\begin{aligned} \int_0^1 \frac{(1-t)^\alpha dt}{[x, ty]^{s+\alpha+1}} &\leq \int_0^1 \frac{(1-t)^\alpha dt}{(1-t|x||y|)^{s+\alpha+1}} \\ &\leq \left(\frac{1}{s} + \frac{1}{\alpha+1} \right) (1 - |x||y|)^{-s} \\ &< \left(\frac{1}{s} + \frac{1}{\alpha+1} \right) 3^s [x, y]^{-s}, \end{aligned}$$

which implies the desired estimate.

Next consider the case $c < 1$. In this case we have

$$(18) \quad 0 < 1 - c = \frac{\sin \theta - (1 - |x||y|)}{|x||y|} \leq 2 \sin \theta.$$

Also, since $1 - |x||y| < \sin \theta$, we have

$$(19) \quad [x, y] \leq 3 \sin \theta$$

by (17). We split the integral into two pieces

$$\int_0^1 \frac{(1-t)^\alpha dt}{[x, ty]^{s+\alpha+1}} = \int_0^c + \int_c^1$$

and estimate each integral separately.

For the first integral, we note

$$\int_0^c \leq \int_0^c \frac{(1-t)^\alpha dt}{(1-t|x||y|)^{s+\alpha+1}} := I + II$$

where, by integration by parts,

$$I = \frac{1}{\alpha + 1} \left[1 - \frac{(1 - c)^{\alpha+1}}{(1 - c|x||y|)^{s+\alpha+1}} \right]$$

and

$$II = |x||y| \frac{s + \alpha + 1}{\alpha + 1} \int_0^c \frac{(1 - t)^{\alpha+1}}{(1 - t|x||y|)^{s+\alpha+2}} dt.$$

Since $1 - c|x||y| = \sin \theta$, we have by (18)

$$|I| = \frac{(\sin \theta)^{-s}}{\alpha + 1} \left| (\sin \theta)^s - \left(\frac{1 - c}{\sin \theta} \right)^{\alpha+1} \right| \leq \frac{1 + 2^{\alpha+1}}{\alpha + 1} (\sin \theta)^{-s}$$

and

$$II \leq \frac{s + \alpha + 1}{\alpha + 1} \int_0^c \frac{|x||y| dt}{(1 - t|x||y|)^{s+1}} \leq \left(\frac{1}{s} + \frac{1}{\alpha + 1} \right) (\sin \theta)^{-s}.$$

Combining these observations with (19), we see

$$\int_0^c \leq 3 \left(\frac{1}{s} + \frac{1}{\alpha + 1} \right) 2^{\alpha+1} (\sin \theta)^{-s} \leq 3 \left(\frac{1}{s} + \frac{1}{\alpha + 1} \right) 3^{\alpha+s+1} [x, y]^{-s},$$

which implies the desired estimate for the first integral.

To estimate the second integral, note that

$$t|x||y| \geq c|x||y| = 1 - \sin \theta > 1 - \frac{1}{\sqrt{2}} > \frac{1}{4}, \quad c \leq t < 1,$$

because $\theta < \pi/4$. So, using (17) (with ty in place of y) and (18), we have

$$\begin{aligned} \int_c^1 &\leq (\sin \theta)^{-(s+\alpha+1)} \int_c^1 (1 - t)^\alpha dt \\ &= \frac{(1 - c)^{\alpha+1}}{\alpha + 1} (\sin \theta)^{-(s+\alpha+1)} \\ &\leq \frac{2^{\alpha+1}}{\alpha + 1} (\sin \theta)^{-s}. \end{aligned}$$

Consequently, we conclude from (19) that

$$\int_c^1 \leq \frac{3^{s+\alpha+1}}{\alpha + 1} [x, y]^{-s},$$

which gives the desired estimate for the second integral. This completes the proof for the case $c < 1$ and the proof of the lemma. \square

LEMMA 3.3. *Given a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$, there is a constant $C_\gamma > 0$ such that*

$$|\partial_y^\gamma P(x, y)| \leq \frac{C_\gamma}{[x, y]^{n+|\gamma|-1}}$$

for $x, y \in B$. Here, $\partial_y^\gamma = (\partial/\partial y_1)^{\gamma_1} \dots (\partial/\partial y_n)^{\gamma_n}$ and $|\gamma| = \gamma_1 + \dots + \gamma_n$.

PROOF. The asserted inequality will hold, once we have the estimate

$$\partial_y^\gamma \left(\frac{1}{[x, y]^n} \right) \leq \frac{C_\gamma}{[x, y]^{n+|\gamma|}}.$$

This is a special case of [3, Lemma 2.1] but under the additional assumption $2|x - y| \geq \max\{1 - |x|, 1 - |y|\}$, which was used therein only for the estimate

$$[x, y] = \sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)} \gtrsim |x - y| + (1 - |x|) + (1 - |y|).$$

The estimate above is actually valid for all $x, y \in B$, because if $2|x - y| \leq 1 - |x|$ and $1 - |y| \leq 1 - |x|$, then

$$|x - y| + (1 - |x|) + (1 - |y|) \approx 1 - |x| \leq 1 - |y| + |x - y| \lesssim [x, y].$$

This completes the proof. □

The next lemma shows that the constant C_α in (2) stays bounded when α approaches -1 .

LEMMA 3.4. *Given $\nu > 0$, there is a constant $C_\nu > 0$ such that*

$$|R_\alpha(x, y)| \leq \frac{C_\nu}{[x, y]^{n+\alpha}}, \quad x, y \in B$$

for $0 < \alpha + 1 < \nu$.

PROOF. For $n = 2$ one may use (7). Since $|1 - x\bar{y}| = [x, y]$, it is clear that $|R_\alpha(x, y)| \leq C_\alpha [x, y]^{-(2+\alpha)}$ which implies the desired estimate.

Now, assume $n \geq 3$ for the rest of the proof. Fix ν . Let $0 < \alpha + 1 < \nu$ and $\alpha = m + \varepsilon$ where m is an integer and $0 \leq \varepsilon < 1$. Fix $x, y \in B$. Let $g(t) = P(x, ty)$ and $f(t) = t^{n/2+\alpha}g(t)$. Then we have by Proposition 1.1

$$(20) \quad R_\alpha(x, y) = \frac{\omega_\alpha}{\Gamma(1 - \varepsilon)} \int_0^1 \frac{f^{(m+2)}(t)}{(1 - t)^\varepsilon} dt;$$

note that ω_α stays bounded as $\alpha \rightarrow -1$.

We assume that $n/2 + \alpha$ is not an integer; the estimate below is simpler when $n/2 + \alpha$ is an integer. Note that $f^{(m+2)}(t)$ is a linear combination of functions of the form $t^{n/2+\alpha-m-2+k}g^{(k)}(t)$ where $k = 0, 1, \dots, m + 2$. Since $n \geq 3$, we have $t^{n/2+\alpha-m-2+k} \leq t^{-1/2}$ for each k . It follows from Lemma 3.3 that

$$(21) \quad |f^{(m+2)}(t)| \leq \frac{C_\nu}{t^{1/2}[x, ty]^{n+m+1}}.$$

On the other hand, Lemma 3.2 gives

$$\begin{aligned} \int_0^1 \frac{(1 - t)^{-\varepsilon}}{t^{1/2}[x, ty]^{n+m+1}} dt &\leq \int_0^{1/2} + \int_{1/2}^1 \\ &\leq \sqrt{2} \cdot 2^{n+\alpha+1} + \sqrt{2} \int_0^1 \frac{(1 - t)^{-\varepsilon}}{[x, ty]^{n+m+1}} dt \\ &\leq C_\nu \left(2^\alpha [x, y]^{n+\alpha} + \frac{4^{n+m+1}}{(n + \alpha)(1 - \varepsilon)} \right) [x, y]^{-(n+\alpha)} \end{aligned}$$

$$\leq C_v \frac{[x, y]^{-(n+\alpha)}}{1 - \varepsilon}.$$

This, together with (21) and (20), yields

$$|R_\alpha(x, y)| \leq C_v \frac{[x, y]^{-(n+\alpha)}}{\Gamma(1 - \varepsilon)(1 - \varepsilon)} = C_v \frac{[x, y]^{-(n+\alpha)}}{\Gamma(2 - \varepsilon)}.$$

Now, since $\Gamma(2 - \varepsilon) \approx 1$, we conclude the desired estimate for $n \geq 3$. This completes the proof. \square

The next proposition is now an immediate consequence of Lemma 3.4 and Theorem 2.2.

PROPOSITION 3.5. *Given $v > 0$, there is a constant $C_v > 0$ such that*

$$\| \Pi_\alpha \|_p \leq C_v \frac{p^2}{(\alpha + 1)(p - 1)}$$

for $1 < p < \infty$ and $0 < \alpha + 1 < v$.

4. Lower estimate of $\| \Pi_\alpha \|_p$. In this section we establish the lower estimate for the operator norm $\| \Pi_\alpha \|_p$. The first thing to do is to show that the order $-(n + \alpha)$ in Lemma 3.4 is best possible. Namely, we first show that the inequality there can be reversed (modulo constant factor) when (x, y) belongs to a certain region.

As an auxiliary tool, we briefly review pseudohyperbolic distance ρ on B given by

$$\rho(x, y) = \frac{|x - y|}{[x, y]}, \quad x, y \in B.$$

For $x \in B$ and $0 < r < 1$, let $E_r(x)$ denote pseudohyperbolic ball with radius r and center x . Then a straightforward calculation gives us that $E_r(x)$ is a Euclidean ball (with center $(1 - r^2)(1 - |x|^2 r^2)^{-1}x$ and radius $(1 - |x|^2)(1 - |x|^2 r^2)^{-1}r$). The following lemma is taken from [2].

LEMMA 4.1. *The inequality*

$$\frac{1 - \rho(x, y)}{1 + \rho(x, y)} \leq \frac{[x, z]}{[y, z]} \leq \frac{1 + \rho(x, y)}{1 - \rho(x, y)}$$

holds for $x, y, z \in B$.

We also need the derivative estimate

$$(22) \quad | \nabla_y R_\alpha(x, y) | \leq \frac{C_\alpha}{[x, y]^{n+\alpha+1}}, \quad x, y \in B;$$

see [4, Lemma 2.8].

In what follows we denote by $\Gamma_\beta(\zeta)$, where $\beta > 1$ and $\zeta \in \partial B$, the non-tangential approach region, with aperture β and vertex ζ , consisting of all points $x \in B$ such that

$$|x - \zeta| \leq \beta(1 - |x|).$$

Also, we denote by x' the radial projection of $x \in B, x \neq 0$, onto ∂B .

LEMMA 4.2. *Given $\alpha > -1$, there exist numbers $\beta > 1$, $r_0 \in (0, 1)$ and a constant $C_\alpha > 0$ such that*

$$R_\alpha(x, y) > \frac{C_\alpha}{[x, y]^{n+\alpha}}$$

whenever $y \in \Gamma_\beta(x')$ and $|x|, |y| \geq r_0$.

PROOF. The case $n = 2$ can be treated by utilizing the explicit formula (7). We skip the details. So, assume $n \geq 3$ for the rest of the proof.

Fix $\alpha > -1$. Let $x, y \in B$. To begin with assume $|x||y| \geq 1/2$. First, we consider the case $y' = x'$. Write $\alpha = m + \varepsilon$ where m is an integer and $0 \leq \varepsilon < 1$. Let $g(t) = P(x, ty)$ and $f(t) = t^{n/2+\alpha}g(t)$. Let $M(t) = t^{n/2+\alpha}g^{(m+2)}(t)$ and $E(t) = f^{(m+2)}(t) - M(t)$. Using these functions and Proposition 1.1, we decompose $R_\alpha(x, y)$ into the sum of the major and error terms as follows:

$$(23) \quad \omega_\alpha^{-1} \Gamma(1 - \varepsilon) R_\alpha(x, y) = \int_0^1 \frac{M(t)}{(1-t)^\varepsilon} dt + \int_0^1 \frac{E(t)}{(1-t)^\varepsilon} dt.$$

Since $x' = y'$, we have

$$g(t) = \frac{1 + t|x||y|}{(1 - t|x||y|)^{n-1}} = \frac{2}{(1 - t|x||y|)^{n-1}} - \frac{1}{(1 - t|x||y|)^{n-2}}$$

so that

$$(24) \quad g^{(k)}(t) = \frac{(n+k-3)!}{(n-2)!} \frac{(|x||y|)^k}{(1-t|x||y|)^{n+k-1}} [2(n+k-2) - (n-2)(1-t|x||y|)]$$

for integers $k \geq 1$. In particular, we have

$$g^{(m+2)}(t) \geq \frac{(n+m-1)!}{(n-2)!2^{m+1}} \frac{1}{(1-t|x||y|)^{n+m+1}}$$

and thus obtain an estimate for the major term

$$(25) \quad \begin{aligned} \int_0^1 \frac{M(t)}{(1-t)^\varepsilon} dt &\geq C_\alpha \int_0^1 \frac{t^{n/2+\alpha} dt}{(1-t)^\varepsilon (1-|x||y|t)^{n+m+1}} \\ &\geq \frac{C_\alpha}{(1-|x||y|)^{n+m+1}} \int_{|x||y|}^1 \frac{dt}{(1-t)^\varepsilon} \\ &= \frac{C_\alpha}{(1-\varepsilon)(1-|x||y|)^{n+\alpha}}. \end{aligned}$$

We now estimate the error term. To do so, we assume that $n/2 + \alpha$ is not an integer, as in the proof of Lemma 3.4. Since $n \geq 3$, we see from (24) (or as in the proof of Lemma 3.4) that

$$|E(t)| \leq \frac{C_\alpha}{t^{1/2}(1-t|x||y|)^{n+m}}.$$

So, integrating both sides of the above against the measure $(1-t)^{-\varepsilon} dt$, we obtain by Lemma 3.1

$$(26) \quad \int_0^1 \frac{E(t)}{(1-t)^\varepsilon} dt \leq \frac{C_\alpha}{(1-\varepsilon)(1-|x||y|)^{n+\alpha-1}}.$$

Note $[x, y] = 1 - |x||y|$, because $x' = y'$. Therefore, denoting the constants C_α in (25) and (26) by $C_{\alpha,1}$ and $C_{\alpha,2}$, respectively, we obtain by (23)

$$\omega_\alpha^{-1} \Gamma(1 - \varepsilon) R_\alpha(x, y) \geq \frac{C_{\alpha,1} - (1 - |x||y|)C_{\alpha,2}}{(1 - \varepsilon)[x, y]^{n+\alpha}}.$$

Thus we see that there exists a number r_0 sufficiently close to 1 such that

$$(27) \quad R_\alpha(x, y) \geq \frac{C_{\alpha,3}}{[x, y]^{n+\alpha}} \quad \text{where} \quad C_{\alpha,3} = \frac{\omega_\alpha C_{\alpha,1}}{2\Gamma(2 - \varepsilon)},$$

completing the proof for the case $x' = y'$ (with arbitrary $\beta > 1$).

We now consider the case where $x' \neq y'$. Let $\beta > 1$ be an aperture to be chosen later and let $y \in \Gamma_\beta(x')$. Also assume $|x|, |y| > r_0$. Since

$$|y - x'|^2 = (1 - |y|)^2 + |y||y' - x'|^2,$$

we have

$$|y - |y|x'|^2 = |y|^2|y' - x'|^2 \leq |y - x'|^2 - (1 - |y|)^2 < \delta^2(1 - |y|)^2$$

where $\delta = \sqrt{\beta^2 - 1}$. Thus

$$|y - |y|x'| < \delta(1 - |y|) < \delta(1 - |y|^2) < \delta[y, |y|x'],$$

i.e., $y \in E_\delta(|y|x')$. Consequently, we have

$$(28) \quad \begin{aligned} R_\alpha(x, y) &\geq R_\alpha(x, |y|x') - |R_\alpha(x, y) - R_\alpha(x, |y|x')| \\ &= R_\alpha(x, |y|x') - |y - |y|x'| \sup_{z \in E_\delta(|y|x')} |\nabla_z R_\alpha(x, z)|. \end{aligned}$$

We now estimate the second term of the above. Let $z \in E_\delta(|y|x')$. Note $[y, |y|x'] \leq [y, x] + [x, |y|x'] \leq 2[x, y]$. Thus

$$|y - |y|x'| < \delta[y, |y|x'] \leq 2\delta[x, y].$$

Meanwhile, since (assume $\delta < 1/4$)

$$\frac{[x, y]}{[x, z]} \leq \frac{1 + \rho(y, z)}{1 - \rho(y, z)} < \frac{1 + 2\delta}{1 - 2\delta} < 3$$

by Lemma 4.1, we have by (22)

$$|\nabla_z R_\alpha(x, z)| \leq \frac{C_\alpha}{[x, z]^{n+\alpha+1}} \leq \frac{C_\alpha}{[x, y]^{n+\alpha+1}}.$$

Combining these observations, we have

$$|y - |y|x'| \sup_{z \in E_\delta(|y|x')} |\nabla_z R_\alpha(x, z)| \leq \frac{\delta C_{\alpha,4}}{[x, y]^{n+\alpha}}$$

where $C_{\alpha,4}$ is a constant depending only on n and α .

Now, assuming $|x|, |y| > r_0$ and substituting (27) and the above estimate into (28), we obtain $R_\alpha(x, y) \geq (C_{\alpha,3} - \delta C_{\alpha,4})[x, y]^{-(n+\alpha)}$ and thus conclude

$$R_\alpha(x, y) > \frac{C_{\alpha,3}}{2[x, y]^{n+\alpha}}$$

for some β sufficiently close to 1, as desired. This completes the proof. □

Now we are ready to prove the lower estimate for the operator norm of $\|\Pi_\alpha\|_p$.

PROPOSITION 4.3. *Given $\alpha > -1$, there is a constant $C_\alpha > 0$ such that*

$$\|\Pi_\alpha\|_p \geq C_\alpha \frac{p^2}{p-1}$$

for $1 < p < \infty$.

PROOF. We provide a proof only for $1 < p \leq 2$; the case $2 \leq p < \infty$ then follows by duality argument as in the proof of Theorem 2.2.

Fix $\alpha > -1$. Our test function will be the characteristic function supported on a Euclidean ball. Let $\beta = \beta(n, \alpha) > 1$ and $r_0 = r_0(n, \alpha) \in (0, 1)$ be the numbers as in Lemma 4.2. Fix a number $\beta_0 \in (1, \beta)$ sufficiently close to β such that $\beta - \beta_0 < 1/2$. Increasing r_0 if necessary, we may assume $1/\beta_0 + (1 - r_0)/2 < 1$. Let $c = c(n, p, \alpha)$ be a sufficiently small positive number to be chosen later. To begin with let $c < (1 - r_0)/(2\beta_0)$ and put

$$r_1 := c(\beta - \beta_0)(1 - c).$$

Finally, let $z = (1 - c)\mathbf{e}$ where $\mathbf{e} = (1, 0, \dots, 0)$ and ψ be the characteristic function supported on the ball $B_{r_1}(z)$ of radius r_1 and center z .

Since $r_1 < c/2$, we have

$$\frac{c}{2} < c - r_1 < 1 - |y| < c + r_1 < 2c, \quad y \in B_{r_1}(z)$$

and thus

$$(29) \quad \|\psi\|_{L_\alpha^p}^p = V_\alpha[B_{r_1}(z)] = \lambda_\alpha \int_{B_{r_1}(z)} (1 - |y|^2)^\alpha dy \approx \lambda_\alpha c^{n+\alpha}.$$

We now estimate the L_α^p -norm of $\Pi_\alpha\psi$. Let

$$Q = \left\{ \zeta \in \partial B; \zeta \cdot \mathbf{e} > \frac{1}{\beta_0} + \frac{1 - r_0}{2} \right\}$$

and

$$E = \{\mathbf{e} - t\zeta; \zeta \in Q, 2\beta_0 c < t < 1 - r_0\}.$$

For $x = \mathbf{e} - t\zeta \in E$, we have

$$(30) \quad \frac{1 - |x|}{|x - \mathbf{e}|} \geq \frac{1 - |x|^2}{2t} = \zeta \cdot \mathbf{e} - \frac{t}{2} > \frac{1}{\beta_0}.$$

This implies $r_0 < |x| < 1 - 2c$ for $x \in E$.

Let $x \in E$ and $y \in B_{r_1}(z)$. Since

$$|\mathbf{e} - y'| = \frac{|z - |z|y'|}{|z|} \leq \frac{2|z - y|}{|z|} < \frac{2r_1}{1 - c} = 2c(\beta - \beta_0) < (\beta - \beta_0)(1 - |x|),$$

we have by (30)

$$|x - y'| \leq |x - \mathbf{e}| + |\mathbf{e} - y'| < \beta(1 - |x|).$$

In other word, we have

$$(31) \quad E \subset \Gamma_\beta(y') \quad \text{for each } y \in B_{r_1}(z).$$

Also, note

$$(32) \quad |y| \geq |z| - |z - y| > 1 - c - r_1 > 1 - 2c > |x| > r_0.$$

Meanwhile, since $|x| < 1 - 2c < |y|$, we have

$$[x, y]^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2) < |x - y|^2 + (1 - |x|^2)^2,$$

which implies

$$(33) \quad [x, y] \leq |x - \mathbf{e}| + |\mathbf{e} - y| + 1 - |x|^2 < 4|x - \mathbf{e}|$$

where the last inequality follows from the fact

$$|\mathbf{e} - y| \leq |\mathbf{e} - z| + |z - y| < c + r_1 < 2c < |x - \mathbf{e}|.$$

It follows from (31), (32), Lemma 4.2 and (33) that

$$R_\alpha(x, y) \geq \frac{C_\alpha}{[x, y]^{n+\alpha}} \geq \frac{C_\alpha}{|x - \mathbf{e}|^{n+\alpha}}.$$

Since this holds for all $x \in E$ and $y \in B_{r_1}(z)$, we obtain

$$\Pi_\alpha \psi(x) = \int_{B_{r_1}(z)} R_\alpha(x, y) dV_\alpha(y) \geq C_\alpha \frac{\lambda_\alpha c^{n+\alpha}}{|x - \mathbf{e}|^{n+\alpha}} \quad x \in E,$$

and thus by (30)

$$(34) \quad \int_B |\Pi_\alpha \psi|^p dV_\alpha \geq C_\alpha^p \lambda_\alpha^{p+1} c^{p(n+\alpha)} \int_E \frac{dx}{|x - \mathbf{e}|^{p(n+\alpha)-\alpha}}.$$

The integral in the right side of the above can be explicitly computed as

$$|Q| \int_{2\beta_0 c}^{1-r_0} t^{(n+\alpha)(1-p)-1} dt = \frac{C_\alpha (2\beta_0 c)^{-(n+\alpha)(p-1)}}{p-1} \left\{ 1 - \left(\frac{2\beta_0 c}{1-r_0} \right)^{(n+\alpha)(p-1)} \right\},$$

where $|Q|$ denotes the surface area of Q , which depends only on n and α . Thus, fixing c such that

$$\left(\frac{2\beta_0 c}{1-r_0} \right)^{(n+\alpha)(p-1)} < \frac{1}{2},$$

we have by (29) and (34)

$$\int_B |\Pi_\alpha \psi|^p dV_\alpha \geq C_\alpha^{p+1} \lambda_\alpha^p \frac{(2\beta_0)^{-(n+\alpha)(p-1)}}{p-1} \|\psi\|_{L_\alpha^p}^p.$$

This yields

$$(35) \quad \|\Pi_\alpha \psi\|_{L_\alpha^p} \geq \frac{C_\alpha \lambda_\alpha}{(p-1)^{1/p}} \|\psi\|_{L_\alpha^p},$$

where we used the fact $C_\alpha^{1/p} \approx C_\alpha$ for $1 < p \leq 2$. Note that

$$\frac{1}{(p-1)^{1/p}} = \frac{(p-1)^{(p-1)/p}}{p-1} \geq \frac{4A}{p-1} \geq A \frac{p^2}{p-1}$$

for some absolute constant A . Therefore we deduce from (35) that

$$(36) \quad \|\Pi_\alpha \psi\|_{L_\alpha^p} \geq \frac{C_\alpha \lambda_\alpha}{p-1} \|\psi\|_{L_\alpha^p},$$

which implies the assertion for $1 < p \leq 2$. The proof is complete. \square

Combining Propositions 3.5 and 4.3, we conclude the next theorem.

THEOREM 4.4. *Given $\alpha > -1$, there is a constant $C_\alpha > 0$ such that*

$$C_\alpha^{-1} \frac{p^2}{p-1} \leq \|\Pi_\alpha\|_p \leq C_\alpha \frac{p^2}{p-1}$$

for $1 < p < \infty$.

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